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The Regularized Trace Formula Of A Second Order Differentıal Equation Given With Anti-Perıodic Boundary Conditions

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Abstract. In this study, we examined the formula of the regularized trace of the self-adjoint operator which is formed by

$\ell(y) = -y'' + p(x)y$

differential expression and

$y(0) + y(\pi) = 0$

$y'(0) + y'(\pi) = 0$

anti-periodic boundary condition.

***Keywords*:** Regularized trace, Eigenvalues, Eigen functions.

Ters Periyodik Sınır Koşulları İle Verilmiş İkinci Mertebeden Diferansiyel Denklemin Düzenli İz Formülü

Özet. Bu çalışmada,

$\ell(y) = -y'' + p(x)y$

diferansiyel ifadesi ve

$y(0) + y(\pi) = 0$

$y'(0) + y'(\pi) = 0$

ters periyodik sınır koşulları ile oluşturulmuş kendine eş operatörün düzenli iz formülü incelenmiştir.

Anahtar Kelimeler: Düzenli iz, Öz değer, Öz fonksiyon.

1. INTRODUCTION

$p(x)$ is a real valued, continuous function in$[0, \pi]$, L_0 and L get two self-adjoint operators generated by the following expressions

$$\ell_0(y) = -y''$$

and

$$\ell(y) = -y'' + p(x)y \tag{1}$$

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with the same boundary conditions

$$y(0) + y(\pi) = 0$$

$$y'(0) + y'(\pi) = 0 \qquad (2)$$

respectively, in the space $L_2[0,\pi]$. The spectrum of operator L_0 coincides with the set $\{(2n+1)^2\}_{n=0}^{\infty}$. Every point of the spectrum is an eigenvalue with multiplicity two.

Let

$$\mu_k = \begin{cases} k^2, & if\ k\ is\ odd \\ (k-1)^2, & if\ k\ is\ even \end{cases} \qquad (k = 1,2,\dots)$$

is the eigenvalues of operator L_0 and

$$\psi_1 = \sqrt{\frac{2}{\pi}}\sin x\,, \psi_2 = \sqrt{\frac{2}{\pi}}\cos x\,, \psi_3 = \sqrt{\frac{2}{\pi}}\sin 3x, \psi_4 = \sqrt{\frac{2}{\pi}}\cos 3x, \dots$$

are the orthonormal eigenfunctions corresponding to this eigenvalues.

Also we showed the eigenvalues of operator L by $\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \le \lambda_k \le \cdots$ and corresponding orthonormal eigenfunctions by $\varphi_0, \varphi_1, \varphi_2, \dots, \varphi_k, \dots$

In this study, we obtained a formula for the sum of series by Dikii's method,

$$\sum_{n=1}^{\infty}(\lambda_n - \mu_n)$$

which is called the formula of regularized trace of operator L.

The regularized trace theory, which was first examined by Gelfand and Levitan and they derived the formula of regularized trace for the Sturm-Liouville operator [1], attracted the attention of many authors. Dikii [2] provided and developed Gelfand and Levitan's formulas by their own method. Later, Levitan [6] suggested one more method for computing the traces of the Sturm–Liouville operator. There are numerous investigations on the calculation of the regularized trace of differential operator equations [3-17].

2. CALCULATION

Let us show the following equation

$$\lim_{N\to\infty}\sum_{n=1}^{N}[(\varphi_n, L\varphi_n) - (\psi_n, L\psi_n)] = 0 \qquad (3)$$

which will be used later. For this we consider the transfer matrix $(u_{ik})_{i,k=1}^{\infty}$ from the orthonormal basis $\{\varphi_k\}$ to orthonormal basis $\{\psi_k\}$ as in [2] :

$$\psi_k = \sum_{i=1}^{\infty} u_{ik}\varphi_i \qquad (k = 1,2,\dots)$$

where $u_{ik} = (\varphi_i, \psi_k)$ and $(u_{ik})_{i,k=1}^{\infty}$ are the unitary matrix, that is

$$\sum_{i=1}^{\infty} u_{ik}^2 = 1 \qquad (k = 1,2, \dots)$$

Let us give some limitations for u_{ik} . It is clear that

$$L\psi_k = \mu_k \psi_k + p\psi_k \tag{4}$$

If we multiply both side of equality (4) by φ_i we obtain

$$(L\psi_k, \varphi_i) = (\mu_k \psi_k, \varphi_i) + (p\psi_k, \varphi_i)$$

Or

$$\lambda_i(\psi_k, \varphi_i) = \mu_k(\psi_k, \varphi_i) + (p\psi_k, \varphi_i)$$

and

$$(\lambda_i - \mu_k)(\psi_k, \varphi_i) = (p\psi_k, \varphi_i)$$

.

With respect to [2] taking the square of both sides of the last equality and summing from 1 to ∞ respect to i we obtain

$$\sum_{i=1}^{\infty} (\lambda_i - \mu_k)^2 (\psi_k, \varphi_i)^2 = \sum_{i=1}^{\infty} (p\psi_k, \varphi_i) = \|p\psi_k\|^2 = \int_0^{\pi} [p(x)\psi_k(x)]^2 dx \le p_0^2 \tag{5}$$

where $p_0 = max_{0 \le x \le \pi} |p(x)|$.

Suppose that the following conditions hold:

1. For the eigenvalues and the eigenfunctions of the L operator holds the asymptotic formulas

$$\lambda_k = \mu_k + O\left(\frac{1}{k}\right) \quad , \quad \varphi_k = \psi_k + O\left(\frac{1}{k}\right) \qquad [10].$$

2. $\int_0^{\pi} p(x)\, dx = 0$.

Hence

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)^2 u_{ik}^2 < C \qquad (C = const.)\ (k < N) \ . \tag{6}$$

We will use condition 1 in the inequalities we will obtain.

Obviously,

$$\sum_{i=N+1}^{\infty} (\lambda_i - \mu_k) u_{ik}^2 < C \ \Rightarrow \ \sum_{i=N+1}^{\infty} (\lambda_i - \mu_k)(\lambda_i - \lambda_k) u_{ik}^2 < C$$

$$\Rightarrow \sum_{i=N+1}^{\infty} (\lambda_i - \lambda_k)^2 u_{ik}^2 < C$$

is obtained for all integer N from equation (6)

And we obtain

$$\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2 \leq \frac{C}{\lambda_{N+1}-\mu_k} \qquad (k<N). \tag{7}$$

Now let us prove the equation (3).

$$(\psi_k, L\psi_k)=\left(\sum_{i=1}^{\infty}u_{ik}\varphi_i\, , \sum_{i=1}^{\infty}\lambda_i u_{ik}\varphi_i\right)=\sum_{i=1}^{\infty}\lambda_i u_{ik}^2$$

If we take the sum on k from 1 to N on both sides of this equation we get

$$\sum_{k=1}^{N}(\psi_k, L\psi_k)=\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_i u_{ik}^2 .$$

Since $\sum_{i=1}^{\infty}u_{ki}^2=1$ we get

$$\sum_{k=1}^{N}(\varphi_k, L\varphi_k)=\sum_{k=1}^{N}\lambda_k=\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_k u_{ki}^2$$

So now we need to prove

$$\lim_{N\to\infty}\left(\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_i u_{ik}^2-\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_k u_{ki}^2\right)=0. \tag{8}$$

$$\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_i u_{ik}^2-\sum_{k=1}^{N}\sum_{i=1}^{\infty}\lambda_k u_{ki}^2=\sum_{k=1}^{N}\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2+\sum_{k=1}^{N}\sum_{i=N+1}^{\infty}\lambda_k\left(u_{ik}^2-u_{ki}^2\right). \tag{9}$$

Let us calculate first sum on the right side of equality (9). For convenience while let $N+1$ be even number then we have

$$\sum_{k=1}^{N}\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2=\sum_{k=1}^{N-1}\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2+(\lambda_{N+1}-\lambda_N)u_{(N+1)N}^2+\sum_{i=N+2}^{\infty}(\lambda_i-\lambda_N)u_{iN}^2 \tag{10}$$

Let us calculate first and third sum on the right side of equality (10) by inequality (7), for $N\to\infty$

$$\sum_{k=1}^{N}\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2<\frac{1}{4N}+\frac{1}{2(N+1)}\left[\ln\frac{N^2+N}{N-1}\right]\to 0 \tag{11}$$

and

$$\sum_{i=N+2}^{\infty}(\lambda_i-\lambda_N)u_{iN}^2 \le \frac{C}{\lambda_{N+2}-\mu_N} \le \frac{C}{4N+4} \to 0 \quad (12)$$

Now we shall calculate the second term on the right side of equality (10) when $N \to \infty$. Suppose that $N+1$ is even , we have

$$(\lambda_{N+1}-\lambda_N)u_{(N+1)N}^2 \le N^2 + O\left(\frac{1}{N+1}\right) - N^2 - O\left(\frac{1}{N}\right) \to 0 \qquad (N \to \infty) \quad (13)$$

In this way, for even number $N+1$ from the expressions (10), (11), (12) and (13) we have

$$\lim_{N\to\infty}\sum_{k=1}^{N}\sum_{i=N+1}^{\infty}(\lambda_i-\lambda_k)u_{ik}^2 = 0. \quad (14)$$

Formula (14) can also calculated for odd number $N+1$.

Now we shall calculate second sum on the right side of equality (9).

$$u_{ik}+u_{ki} = (\varphi_i,\psi_k)+(\varphi_k,\psi_i) = -(\varphi_i-\psi_i,\varphi_k-\psi_k) \quad (15)$$

By equality (15) and condition 1., we have

$$|u_{ik}+u_{ki}| \le \|\varphi_i-\psi_i\|\,\|\varphi_k-\psi_k\| < \frac{C}{ik}\,. \quad (16)$$

According to Cauchy-Schwarz inequality we have

$$\sum_{i=N+1}^{\infty}(\lambda_i-\mu_k)\left|u_{ik}^2-u_{ki}^2\right| = \sum_{i=N+1}^{\infty}(\lambda_i-\mu_k)\left|u_{ik}-u_{ki}\right|\left|u_{ik}+u_{ki}\right|$$

$$\le \sqrt{\sum_{i=N+1}^{\infty}\left|u_{ik}-u_{ki}\right|^2}\;\sqrt{\sum_{i=N+1}^{\infty}(\lambda_i-\mu_k)^2\left|u_{ik}-u_{ki}\right|^2}$$

$$< \frac{C}{(k-1)\sqrt{N+1}}\,. \quad (17)$$

Hence

$$\sum_{i=N+1}^{\infty}\left|u_{ik}^2-u_{ki}^2\right| < \frac{C}{(k-1)\sqrt{N+1}[N^2-(k-1)^2]} \quad (18)$$

Now we shall evaluate the second sum on the right side of equality (9),

$$\sum_{k=1}^{N}\lambda_k\sum_{i=N+1}^{\infty}\left|u_{ik}^2-u_{ki}^2\right| = \lambda_N\sum_{i=N+1}^{\infty}\left|u_{iN}^2-u_{Ni}^2\right| + \sum_{k=1}^{N-1}\lambda_k\sum_{i=N+1}^{\infty}\left|u_{ik}^2-u_{ki}^2\right|$$

$$= \lambda_N \left| u_{N+1N}^2 - u_{NN+1}^2 \right| + \lambda_N \sum_{i=N+2}^{\infty} \left| u_{iN}^2 - u_{Ni}^2 \right| + \sum_{k=1}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} \left| u_{ik}^2 - u_{ki}^2 \right| \qquad (19)$$

By inequality (16) we have

$$\lambda_N \left| u_{N+1N}^2 - u_{NN+1}^2 \right| = \lambda_N \left| u_{N+1N} - u_{NN+1} \right| \left| u_{N+1N} + u_{NN+1} \right|$$

$$< \frac{CN^2}{N^2(N+1)^2} \left| u_{N+1N} - u_{NN+1} \right| \to 0 \quad (N \to \infty) \qquad (20)$$

By the expression (18) we evaluate the second and third sum on the right side of equality (19)

$$\lambda_N \sum_{i=N+2}^{\infty} \left| u_{iN}^2 - u_{Ni}^2 \right| < \frac{CN^2}{(N-1)\sqrt{N+2}[(N+2)^2 - (N+1)^2]} \to \infty \ (N \to \infty) \qquad (21)$$

and

$$\sum_{k=1}^{N-1} \lambda_k \sum_{i=N+1}^{\infty} \left| u_{ik}^2 - u_{ki}^2 \right| < \frac{CN}{\sqrt{N+1}} \sum_{k=2}^{N} \frac{1}{N^2 - (k-1)^2} \sim C \frac{\ln N}{\sqrt{N}} \to 0 \ (N \to \infty). \qquad (22)$$

From the expressions (19), (20),(21) and (22) we have

$$\lim_{N \to \infty} \sum_{k=1}^{N} \sum_{i=N+1}^{\infty} \lambda_k \left(u_{ik}^2 - u_{ki}^2 \right) = 0 \qquad (23)$$

Thus from the expressions (9), (14), and (23) we obtain formula (8). Therefore formula (3) have proved.

3. CONCLUSION

$$(\varphi_k, L\varphi_k) = \lambda_k \qquad \text{and} \qquad (\psi_k, L\psi_k) = \mu_k + (\psi_k, p\psi_k).$$

If we use these into formula (3) then we obtain

$$\sum_{k=1}^{N} [(\psi_k, L\psi_k) - (\varphi_k, L\varphi_k)] = \sum_{k=1}^{N} (\mu_k - \lambda_k) + \sum_{k=1}^{N} (\psi_k, p\psi_k) \to 0\,, \qquad (N \to \infty). \qquad (24)$$

Now we shall calculate

$$\lim_{N \to \infty} \sum_{k=1}^{N} (\psi_k, p\psi_k).$$

According to condition 2. we have for even number N

$$\sum_{k=1}^{N}(\psi_k, p\psi_k) = \frac{1}{\pi}\int_0^{\pi} p(x)\ dx + \frac{N}{\pi}\int_0^{\pi} p(x)\ dx = 0 \qquad (25)$$

Similarly we have for odd number N

$$\sum_{k=1}^{N}(\psi_k, p\psi_k) = -\frac{1}{\pi}\int_0^{\pi} p(x)\cos 2Nx\ dx \to 0\,, \qquad (N \to \infty)\,. \qquad (26)$$

From the expressions (25) and (26) we have

$$\lim_{N\to\infty}\sum_{k=1}^{N}(\psi_k, p\psi_k) = 0$$

Hence from the expressions (24) and (26) we have

$$\lim_{N\to\infty}\sum_{k=1}^{N}(\lambda_k - \mu_k) = 0.$$

So we have proved the following theorem.

THEOREM : The following formula is true when we considered $p(x)$ is a continuous function and conditions 1.,2. are fulfilled

$$\sum_{n=1}^{\infty}(\lambda_n - \mu_n) = 0\ . \qquad (27)$$

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