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Blow-Up and Global Solutions of a Wave Equation with Initial-Boundary Conditions

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ABSTRACT

In this paper, we study a wave equation with interior source function and linear damping term. We obtain that the solutions of this equation are global in time and blow-up in finite time under suitable conditions.

Keywords: Global Solution, Blow-up solution, damping term

1. INTRODUCTION

In this paper, we consider the following initial-boundary value problem

$$\begin{cases} u_{tt} - ku_{xx} - \left(a(x)u_x\right)_x + bu_t = f(u), & x \in [0,1] \times \left(0,T\right), \\ u(0,t) = u_x(0,t) = u(1,t) = u_x(1,t) = 0, & t \in \left(0,T\right), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & x \in \left[0,1\right], \end{cases}$$

(1.1)

where $a(x) \in C^1[0,1]$ and $a(x) > 0$, k and b nonnegative constant, $f(s) \in C(\mathbb{R})$.

Models of this type are of interest in applications in various areas in mathematical physics [1, 2, 3] as well as in geophysics and ocean acoustics, where for example, the coefficient $a(x)$ represents the "effective tension" [6].

In [1], Bayrak and Can considered the following a nonlinear wave equation with initial-boundary conditions

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$$\begin{cases} u_{tt}+\alpha u_t+2\beta u_{xxxx}-2[(a(x)+b)u_x]_x+\dfrac{\beta}{3}(u_x^3)_{xxx} & \\ \qquad -[(a(x)+b)u_x^3]_x-\beta(u_{xx}^2u_x)_x = f(u), & (x,t)\in[0,1]\times(0,T), \\ u(0,t)=u(1,t)=u_{xx}(0,t)=u_{xx}(1,t)=0, & t\in(0,T), \\ u(x,0)=u_0(x), \quad u_t(x,0)=u_1(x), & x\in[0,1]. \end{cases} \tag{1.2}$$

They gave nonexistence of the global solution in time for the equation (1.2). In [2], Hao et al proved the blow-up solution in finite time and global solution in time for same equation under different conditions. In [3], Wu and Li studied a nonlinear damped system with boundary input and output. They proved that under some conditions the system has global solutions and blow up solutions. In [4], Feng et al considered the wave equation with nonlinear damping and source terms. They bounded up with the interaction between the boundary damping $-|y_t(L,t)|^{m-1}y_t(L,t)$ and the interior source $|y(t)|^{p-1}y(t)$. Then they found a sufficient condition for obtaining the blow up solution of their problem. In [5], Dinlemez and Aktaş studied a nonlinear string equation with initial and boundary conditions. They proved that the solution is global in time and the solution with a negative initial energy blow up in finite time for their problems. In [7], Takamura and Wakasa were interested in the “almost” global-in-time existence of classical solutions in the general theory for nonlinear wave equations. Several interesting works about blow up and global solutions for nonlinear wave equations given in [8-15].

First of all we will estimate the energy of problem (1.2).

Multiplying (1.2) with u_t and integrating over (0.1), then we get

$$\frac{d}{dt}\left[\frac{1}{2}\|u_t\|_2^2+\frac{k}{2}\|u_x\|_2^2+\frac{1}{2}\int_0^1 a(x)u_x^2dx-\int_0^1 F(u)dx\right]=-\|u_t\|_2^2$$

(1.3) where

$$F(u)=\int_0^u f(\xi)d\xi.$$

So the energy equation of the initial-boundary problem (1.2) is defined by

$$E(t)=\frac{1}{2}\|u_t\|_2^2+\frac{k}{2}\|u_x\|_2^2+\frac{1}{2}\int_0^1 a(x)u_x^2dx-\int_0^1 F(u)dx\,. \tag{1.4}$$

Therefore we obtain

$$\frac{d}{dt}E(t)=-b\|u_t\|_2^2\,. \tag{1.5}$$

2. MAIN RESULTS

Now we give the following theorem for global solutions.

Theorem1. Let $u(x,t)$ be a solution of the initial-boundary problem (1.2) with $a(x)>0$. There exists a positive constant A such that the function $f(s)$ satisfies

$$f^2(s)\le AF(s)\quad \text{for } s\in\mathbb{R}\,. \tag{2.1}$$

Then the solution $u(x,t)$ is the global solution of the initial-boundary problem (1.2).
Proof: Let

$$G(t)=E(t)+2\int_0^1 F(u)dx\,. \tag{2.2}$$

Taking a derivative of $G(t)$ and using (1.5), we obtain

$$G'(t)=-b\|u_t\|_2^2+2\int_0^1 f(u)u_t dx\,, \tag{2.3}$$

$$G'(t)\le 2\int_0^1 f(u)u_t dx\,. \tag{2.4}$$

Using Cauchy-Schwartz's inequality and Young's inequality in (2.4) respectively, we get

$$|G'(t)|\le 2\int_0^1 |f(u)u_t|dx,$$
$$\le 2\|f(u)\|_2\|u_t\|_2\,.$$

Then we obtain

$$G'(t)\le \frac{1}{2\eta}\int_0^1 f^2(u)dx+2\eta\|u_t\|_2^2\,, \tag{2.5}$$

where η is positive constant. Now we use (2.1) in (2.5), we yield

$$|G'(t)|\le \frac{A}{2\eta}\int_0^1 F(u)dx+2\eta\|u_t\|_2^2\,. \tag{2.6}$$

From defining of $G(t)$ we have

$$2G(t)=\|u_t\|_2^2+k\|u_x\|^2+\int_0^1 a(x)u_x^2dx+2\int_0^1 F(u)dx.$$

Therefore we get

$$\|u_t\|_2^2+2\int_0^1 F(u)dx\le 2G(\mathrm{t})\,. \tag{2.7}$$

Using (2.6), we obtain

$$|G'(t)|\le \beta\left\{\|u_t\|_2^2+2\int_0^1 F(u)dx\right\}, \tag{2.8}$$

where $\beta=\max\left\{\frac{A}{4\eta},2\eta\right\}$.

Thanks to (2.7) and (2.8), we have

$$G'(t) \le 2\beta G(t).$$

Then from the Gronwall's inequality, we have

$$G(t) \le G(0)e^{2\beta t}.$$

Thus, together with the continuous principle and the definition of $G(t)$, we complete the proof of the Theorem 1.

Theorem 2. Let $u(x,t)$ be a solution of the initial-boundary problem (1.2). Assume that

(i) $f(s)$ satisfies the following condition

$$sf(s) \ge 4F(s), \qquad \text{for } s \in \mathbb{R}, \tag{2.9}$$

(ii) The initial values satisfy

$$E(0) \le 0, \qquad 0 < \int_0^1 u_0(x)u_1(x)dx, \tag{2.10}$$

(iii) $u(x,t)$ satisfies $1 > \|u\|.$

(2.11)

Then the solution $u(x,t)$ blows up in finite time $T_{\max}$ and

$$T_{\max} \le \frac{1-\gamma}{\alpha\gamma L^{\frac{\gamma}{1-\gamma}}(0)}. \tag{2.12}$$

where α is a positive constant and γ is a positive constant such that $0 < \gamma \le \frac{1}{4}$.

Proof: Let

$$H(t) := -E(t) \tag{2.13}$$

and

$$L(t) := H^{1-\gamma}(t) + \int_0^1 uu_t dx. \tag{2.14}$$

Combining (1.5), (2.10) and (2.13), we obtain

$$\frac{d}{dt}H(t) = b\|u_t\|_2^2 \ge 0, \tag{2.15}$$

therefore we get

$$H(t) \ge H(0) \ge 0, \quad \text{for } t \ge 0. \tag{2.16}$$

Taking a derivative of (2.14) and using (2.15), we have

$$\frac{d}{dt}L(t) = b(1-\gamma)H^{-\gamma}(t)\|u_t\|_2^2 + \|u_t\|_2^2 + \int_0^1 uu_{tt}dx. \tag{2.17}$$

From the initial-boundary problem (1.2), we write

$$u_{tt} = ku_{xx} + \left(a(x)u_x\right)_x - bu_t + f(u). \tag{2.18}$$

Then, multiplying (2.18) with u and integrating over $[0,1]$, using integration by parts and boundary conditions when necessary, we obtain

$$\int_0^1 u u_{tt} dx = -k\|u_x\|_2^2 - \int_0^1 a(x)u_x^2 dx - b\int_0^1 u u_t dx + \int_0^1 f(u)u dx. \tag{2.19}$$

And then using (2.19) in (2.17), we get

$$\frac{d}{dt}L(t) = (1-\gamma)H^{-\gamma}(t)b\|u_t\|_2^2 + \|u_t\|_2^2 - k\|u_x\|_2^2 - \int_0^1 a(x)u_x^2 dx - b\int_0^1 u u_x dx + \int_0^1 f(u)u dx. \tag{2.20}$$

By recalling the definitions of $E(t)$ and $H(t)$ we have

$$4H(t) = -2\|u_t\|_2^2 - 2k\|u_x\|_2^2 - 2\int_0^1 a(x)u_x^2 dx + 4\int_0^1 F(u)dx \ . \tag{2.21}$$

Hence applying (2.21) in (2.20) we write

$$\begin{aligned}\frac{d}{dt}L(t) = b(1-\gamma)H^{-\gamma}(t)\|u_t\|_2^2 + \|u_t\|_2^2 - k\|u_x\|_2^2 - \int_0^1 a(x)u_x^2 dx - b\int_0^1 u u_x dx + \int_0^1 f(u)u dx \\ + 4H(t) + 2\|u_t\|_2^2 + 2k\|u_x\|_2^2 + 2\int_0^1 a(x)u_x^2 dx - 4\int_0^1 F(u)dx\end{aligned} \tag{2.22}$$

then we yield

$$\frac{d}{dt}L(t) \ge b(1-\gamma)H^{-\gamma}(t)\|u_t\|_2^2 + 3\|u_t\|_2^2 + k\|u_x\|_2^2 - b\int_0^1 u u_t dx + \int_0^1 \left(f(u)u - 4F(u)\right)dx + 4H(t). \tag{2.23}$$

By using Cauchy-Schwartz's inequaity and Young's inequality respectively, we have

$$\int_0^1 u u_t dx \le \int_0^1 |u||u_t|\, dx \le \|u\|\|u_t\| \le \frac{b}{2}\|u\|_2^2 + \frac{1}{2b}\|u_t\|_2^2 \ . \tag{2.24}$$

From (2.23) and (2.24), we obtain

$$\begin{aligned}\frac{d}{dt}L(t) \ge b(1-\gamma)H^{-\gamma}(t)\|u_t\|_2^2 + 3\|u_t\|_2^2 + k\|u_x\|_2^2 - \frac{b^2}{2}\|u\|_2^2 - \frac{1}{2}\|u_t\|_2^2 \\ + \int_0^1 \left(f(u)u - 4F(u)\right)dx + 4H(t).\end{aligned} \tag{2.25}$$

Using (2.9) and Poincare inequality for $k\|u_x\|_2^2$ respectively in the equation (2.25), we get

$$\frac{d}{dt}L(t) \ge \frac{5}{2}\|u_t\|_2^2 + 4H(t) + \left(\lambda k - \frac{b^2}{2}\right)\|u\|_2^2 \ge 0 \tag{2.26}$$

where k is positive constant such that $k > \frac{b^2}{\lambda}$. Thanks to (2.26) and the definition of $L(t)$, we have

$$L(t) \ge L(0) > 0 \ .$$

Next, we will estimate $L^{\frac{1}{1-\gamma}}(t)$. Using Hölder inequality, we obtain

$$\left|\int_0^1 uu_t dx\right| \le \int_0^1 |uu_t|\,dx \le \|u\|_2\,\|u_t\|_2 ,$$

then we get

$$\left|\int_0^1 uu_t dx\right|^{\frac{1}{1-\gamma}} \le \left(\int_0^1 |uu_t|\,dx\right)^{\frac{1}{1-\gamma}} \le \|u\|_2^{\frac{1}{1-\gamma}}\,\|u_t\|_2^{\frac{1}{1-\gamma}} .$$

By Young's inequality

$$XY \le \frac{\delta^{\zeta}}{\zeta}X^{\zeta} + \frac{\delta^{-\omega}}{\omega}Y^{\omega}, \qquad X,Y \ge 0 \text{ for all } \delta > 0,\ \frac{1}{\zeta}+\frac{1}{\omega}=1,$$

with $\omega = 2(1-\gamma)$ and $\zeta = \dfrac{2(1-\gamma)}{1-2\gamma}$, we yield

$$\left|\int_0^1 uu_t dx\right|^{\frac{1}{1-\gamma}} \le C\left[\|u\|_2^{\frac{2}{1-2\gamma}} + \|u_t\|_2^2\right],$$

where C depends on γ. Using (2.11) and considering the relation $2 < \dfrac{2}{1-2\gamma} \le 4$, we get

$$\left|\int_0^1 uu_t dx\right|^{\frac{1}{1-\gamma}} \le C\left[\|u\|_2^{\frac{2}{1-2\gamma}} + \|u_t\|_2^2\right] \le C\left[\|u\|_2^4 + \|u_t\|_2^2\right] \le C\left[\|u\|_2^2 + \|u_t\|_2^2\right]. \tag{2.27}$$

Now we estimate $L^{\frac{1}{1-\gamma}}(t)$.

It follows from the definition of $L(t)$ for all $t>0$ and using (2.27) and the following inequality for $p \ge 1,\ a,b>0$ $(a+b)^p \le 2^{p-1}(a^p+b^p)$, we obtain

$$L^{\frac{1}{1-\gamma}}(t) = \left(H^{1-\gamma}(t) + \int_0^1 uu_t dx\right)^{\frac{1}{1-\gamma}},$$

$$\le 2^{\frac{\gamma}{1-\gamma}}\left(H(t) + \left|\int_0^1 uu_t dx\right|^{\frac{1}{1-\gamma}}\right),$$

$$\le \kappa\left[H(t) + \|u\|_2^2 + \|u_t\|_2^2\right], \tag{2.28}$$

where $\kappa = 2^{\frac{\gamma}{1-\gamma}}C$.

From (2.26) and (2.28) we get

$$\frac{dL(t)}{dt} \ge \alpha L^{\frac{1}{1-\gamma}}(t), \tag{2.29}$$

where $\alpha = \dfrac{\mu}{\kappa}$, $\mu = \min\left\{\dfrac{5}{2}, \left(\lambda k - \dfrac{b^2}{2}\right)\right\}$ and using Gronwall's inequality in (2.29), we obtain

$$L^{\frac{\gamma}{1-\gamma}}(t) \geq \frac{1}{L^{\frac{-\gamma}{1-\gamma}}(0) - \alpha t \dfrac{\gamma}{1-\gamma}}. \tag{2.30}$$

Then, (2.30) shows that $L(t)$ blows-up in time

$$T^* \leq \frac{1-\gamma}{\alpha\gamma L^{\frac{\gamma}{1-\gamma}}(0)}.$$

Therefore the proof is completed.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

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