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Recurrence Relations for the Moments of Minimum Order Statistics from Exponential Distribution

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Abstract. In this paper, certain recurrence relations for the moments of minimum order statistics of a random sample of size *n* arising from exponential distribution are obtained. The usefulness of these relations in evaluating the moments of exponential minimum order statistics is also discussed.

***Keywords*:** Exponential distribution, Minimum order statistics, Moments, Recurrence relations.

Üstel Dağılımdan Minimum Sıra İstatistiğinin Momentleri İçin Yinelenme İlişkileri

Özet. Bu çalışmada, üstel dağılımdan elde edilen *n* birimlik bir rasgele örneklemin minimum sıra istatistiğinin momentleri için belli yinelenme ilişkileri elde edilmiştir. Üstel minimum sıra istatistiğinin momentleri değerlendirilmesinde bu ilişkilerin yararlılığı ayrıca tartışılmıştır.

***Anahtar Kelimeler*:** Üstel dağılım, Minimum sıra istatistiği, Momentler, Yinelenme ilişkileri.

1. INTRODUCTION

The use of recurrence relations for the moments of order statistics is quite well-known in the statistical literature (see, for example [1] and [2]). For improved forms of these results, it can be seen in [3] and [4]. Balakrishnan et al. have reviewed many recurrence relations and identities for the moments of order statistics arising from several specific continuous distributions such as normal, Cauchy, logistic, gamma and exponential [5]. For some recent results on moments of the order statistics arising from some other important specific distributions [6] and [7]. David and Nagaraja have given an account of the recurrence relations for the moments of order statistics arising from arbitrary as well as some specific distributions [8]. There are other studies on this subject (see, for example, [9], [10], [11], [12], [13] and [14]). Hence the aim of this paper is to consider minimum order statistics of a random sample of size *n* drawn from exponential distribution and derive some recurrence relations for the moments of the minimum order statistics.

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2. MATERIALS AND METHODS

2.1. Minimum Order Statistics from Exponential Distribution

Let X be a random variable whose distribution function (d.f.) and probability density function (p.d.f.) are $F(x)$ and $f(x)$ respectively. Let $X_1, X_2, ..., X_n$ have independent and identical d.f. $F(x)$ and p.d.f. $f(x)$. $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denotes the order statistics of these random variables. For $1 \leq r \leq n$, the p.d.f. of $X_{r:n}$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!}\left[F(x)\right]^{r-1}\left[1-F(x)\right]^{n-r} f(x)$$

For $r=1$, the p.d.f. of $X_{1:n}$ is given by

$$f_{1:n}(x) = n\left[1-F(x)\right]^{n-1} f(x)$$

Let $X_1, X_2, ..., X_n$ have independent and identically distributed exponential distribution with θ mean parameter. Then, d.f. and p.d.f. are $F(x)$ and $f(x)$ respectively

$$\begin{aligned} F(x) &= 1-e^{-x/\theta} \ , \ x>0, \ \theta>0 \\ f(x) &= \frac{1}{\theta}e^{-x/\theta} \ , \ x>0, \ \theta>0 \end{aligned} \tag{1}$$

The p.d.f. of $X_{1:n}$ is given by

$$\begin{aligned} f_{1:n}(x) &= n\left[e^{-x/\theta}\right]^{n-1}\frac{1}{\theta}e^{-x/\theta} = \frac{n}{\theta}e^{-nx/\theta} \\ X_{1:n} &\sim \text{Exp}(\theta/n) \end{aligned} \tag{2}$$

2.2. Recurrence Relations for the Moments of Exponential Minimum Order Statistics

Theorem 1. Let $X_1, X_2, ..., X_n$ have independent and identically distributed exponential distribution with θ mean parameter (i.e., if Eq. (1) then). Then, the k^{th} $(k=1,2,3,...)$ moments of minimum order statistics ($X_{1:n}$) is given by

$$\mu_{1:n}^{(k)} = E(X_{1:n}^k) = k!\left(\frac{\theta}{n}\right)^k \tag{3}$$

Proof.

$$\mu_{1:n}^{(k)} = E(X_{1:n}^k) = \int_0^\infty x^k f_{1:n}(x)dx = \int_0^\infty x^k \frac{n}{\theta}e^{-nx/\theta}dx = \frac{n}{\theta}\int_0^\infty x^k e^{-nx/\theta}dx$$

Let's remember that $\int_0^\infty x^{\alpha-1}e^{-x/\beta}dx = \Gamma(\alpha)\beta^\alpha$ and $\Gamma(\alpha) = (\alpha-1)!$, $\alpha \in Z^+$. Then,

$$\mu_{1:n}^{(k)} = E(X_{1:n}^k) = \frac{n}{\theta}\int_0^\infty x^k e^{-nx/\theta}dx = \frac{n}{\theta}\Gamma(k+1)\left(\frac{\theta}{n}\right)^{k+1} = k!\left(\frac{\theta}{n}\right)^k$$

Corollary 1. By Theorem 1, the expected value (1^{th} moment), 2^{th} moment and the variance of $X_{1:n}$ are given by

$$\mu_{1:n}^{(1)} = \mu_{1:n} = E(X_{1:n}) = \frac{\theta}{n}$$

$$\mu_{1:n}^{(2)} = E(X_{1:n}^2) = 2\left(\frac{\theta}{n}\right)^2$$

$$Var(X_{1:n}) = E(X_{1:n}^2) - E(X_{1:n})^2 = \mu_{1:n}^{(2)} - (\mu_{1:n})^2 = 2\left(\frac{\theta}{n}\right)^2 - \left(\frac{\theta}{n}\right)^2 = \left(\frac{\theta}{n}\right)^2$$

Theorem 2. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+1)^{th}$ moments of $X_{1:n}$ is given by

$$\mu_{1:n}^{(k+1)} = (k+1)\frac{\theta}{n}\mu_{1:n}^{(k)} \tag{4}$$

Proof.

$$\mu_{1:n}^{(k+1)} = E(X_{1:n}^{k+1}) = (k+1)!\left(\frac{\theta}{n}\right)^{k+1} = (k+1)\frac{\theta}{n}\mu_{1:n}^{(k)}$$

Theorem 3. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n}$ is given by

$$\mu_{1:n}^{(k+\ell)} = (k+\ell)\ldots(k+1)\left(\frac{\theta}{n}\right)^\ell \mu_{1:n}^{(k)} \tag{5}$$

Proof.

$$\mu_{1:n}^{(k+\ell)} = E(X_{1:n}^{k+\ell}) = (k+\ell)!\left(\frac{\theta}{n}\right)^{k+\ell} = (k+\ell)\ldots(k+1)\left(\frac{\theta}{n}\right)^\ell \mu_{1:n}^{(k)}$$

Theorem 4. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the k^{th} moments of $X_{1:n+1}$ is given by

$$\mu_{1:n+1}^{(k)} = k!\left(\frac{\theta}{n+1}\right)^{k} \tag{6}$$

Proof.

$$\mu_{1:n+1}^{(k)} = E(X_{1:n+1}^{k}) = k!\left(\frac{\theta}{n+1}\right)^{k}$$

Theorem 5. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+1)^{th}$ moments of $X_{1:n+1}$ is given by

$$\mu_{1:n+1}^{(k+1)} = (k+1)\left(\frac{\theta}{n+1}\right)\mu_{1:n+1}^{(k)} \tag{7}$$

Proof.

$$\mu_{1:n+1}^{(k+1)} = E(X_{1:n+1}^{k+1}) = (k+1)!\left(\frac{\theta}{n+1}\right)^{k+1} = (k+1)\left(\frac{\theta}{n+1}\right)\mu_{1:n+1}^{(k)}$$

Theorem 6. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n+1}$ is given by

$$\mu_{1:n+1}^{(k+\ell)} = (k+\ell)\ldots(k+1)\left(\frac{\theta}{n+1}\right)^{\ell}\mu_{1:n+1}^{(k)} \tag{8}$$

Proof.

$$\mu_{1:n+1}^{(k+\ell)} = E(X_{1:n+1}^{k+\ell}) = (k+\ell)!\left(\frac{\theta}{n+1}\right)^{k+\ell} = (k+\ell)\ldots(k+1)\left(\frac{\theta}{n+1}\right)^{\ell}\mu_{1:n+1}^{(k)}$$

Theorem 7. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the k^{th} moments of $X_{1:n+m}$ is given by

$$\mu_{1:n+m}^{(k)} = k!\left(\frac{\theta}{n+m}\right)^{k} \tag{9}$$

Proof.

$$\mu_{1:n+m}^{(k)} = E(X_{1:n+m}^{k}) = k!\left(\frac{\theta}{n+m}\right)^{k}$$

Theorem 8. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n+m}$ is given by

$$\mu_{1:n+m}^{(k+\ell)} = (k+\ell)...(k+1)\left(\frac{\theta}{n+m}\right)^{\ell} \mu_{1:n+m}^{(k)} \tag{10}$$

Proof.

$$\mu_{1:n+m}^{(k+\ell)} = E(X_{1:n+m}^{k+\ell}) = (k+\ell)!\left(\frac{\theta}{n+m}\right)^{k+\ell} = (k+\ell)...(k+1)\left(\frac{\theta}{n+m}\right)^{\ell} \mu_{1:n+m}^{(k)}$$

Theorem 9. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the k^{th} moments of $X_{1:n+1}$ is given by

$$\mu_{1:n+1}^{(k)} = \left(\frac{n}{n+1}\right)^{k} \mu_{1:n}^{(k)} \tag{11}$$

Proof.

$$\mu_{1:n+1}^{(k)} = k!\left(\frac{\theta}{n+1}\right)^{k} \text{ and } \mu_{1:n}^{(k)} = k!\left(\frac{\theta}{n}\right)^{k}$$

$$\mu_{1:n+1}^{(k)} = c_1 \mu_{1:n}^{(k)}$$

$$k!\left(\frac{\theta}{n+1}\right)^{k} = c_1 k!\left(\frac{\theta}{n}\right)^{k}$$

$$c_1 = \left(\frac{n}{n+1}\right)^{k}$$

Theorem 10. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the k^{th} moments of $X_{1:n+m}$ is given by

$$\mu_{1:n+m}^{(k)} = \left(\frac{n}{n+m}\right)^{k} \mu_{1:n}^{(k)} \tag{12}$$

Proof.

$$\mu_{1:n+m}^{(k)} = k!\left(\frac{\theta}{n+m}\right)^{k} \text{ and } \mu_{1:n}^{(k)} = k!\left(\frac{\theta}{n}\right)^{k}$$

$$\mu_{1:n+m}^{(k)} = c_2 \mu_{1:n}^{(k)}$$

$$k!\left(\frac{\theta}{n+m}\right)^{k}=c_2 k!\left(\frac{\theta}{n}\right)^{k}$$

$$c_2=\left(\frac{n}{n+m}\right)^{k}$$

Theorem 11. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n+1}$ is given by

$$\mu_{1:n+1}^{(k+\ell)}=\left(\frac{n}{n+1}\right)^{k+\ell}\mu_{1:n}^{(k+\ell)} \tag{13}$$

Proof.

$$\mu_{1:n+1}^{(k+\ell)}=(k+\ell)!\left(\frac{\theta}{n+1}\right)^{k+\ell} \text{ and } \mu_{1:n}^{(k+\ell)}=(k+\ell)!\left(\frac{\theta}{n}\right)^{k+\ell}$$

$$\mu_{1:n+1}^{(k+\ell)}=c_3\mu_{1:n}^{(k+\ell)}$$

$$(k+\ell)!\left(\frac{\theta}{n+1}\right)^{k+\ell}=c_3(k+\ell)!\left(\frac{\theta}{n}\right)^{k+\ell}$$

$$c_3=\left(\frac{n}{n+1}\right)^{k+\ell}$$

Theorem 12. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n+m}$ is given by

$$\mu_{1:n+m}^{(k+\ell)}=\left(\frac{n}{n+m}\right)^{k+\ell}\mu_{1:n}^{(k+\ell)} \tag{14}$$

Proof.

$$\mu_{1:n+m}^{(k+\ell)}=(k+\ell)!\left(\frac{\theta}{n+m}\right)^{k+\ell} \text{ and } \mu_{1:n}^{(k+\ell)}=(k+\ell)!\left(\frac{\theta}{n}\right)^{k+\ell}$$

$$\mu_{1:n+m}^{(k+\ell)}=c_4\mu_{1:n}^{(k+\ell)}$$

$$(k+\ell)!\left(\frac{\theta}{n+m}\right)^{k+\ell}=c_4(k+\ell)!\left(\frac{\theta}{n}\right)^{k+\ell}$$

$$c_4=\left(\frac{n}{n+m}\right)^{k+\ell}$$

Theorem 13. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n+m}$ is given by

$$\mu_{1:n+m}^{(k+\ell)} = \frac{(k+\ell)...(k+1)}{(n+m)^{\ell}} \theta^{\ell} \left(\frac{n}{n+m} \right)^{k} \mu_{1:n}^{(k)} \tag{15}$$

Proof.

$$\mu_{1:n+m}^{(k+\ell)} = (k+\ell)! \left(\frac{\theta}{n+m} \right)^{k+\ell} \text{ and } \mu_{1:n}^{(k)} = k! \left(\frac{\theta}{n} \right)^{k}$$

$$\mu_{1:n+m}^{(k+\ell)} = c_5 \mu_{1:n}^{(k)}$$

$$(k+\ell)! \left(\frac{\theta}{n+m} \right)^{k+\ell} = c_5 k! \left(\frac{\theta}{n} \right)^{k}$$

$$c_5 = \frac{(k+\ell)...(k+1)}{(n+m)^{\ell}} \theta^{\ell} \left(\frac{n}{n+m} \right)^{k}$$

Theorem 14. If X random variables have exponential distribution with θ mean parameter (i.e., if Eq. (1) then), the $(k+\ell)^{th}$ moments of $X_{1:n}$ is given by

$$\mu_{1:n}^{(k+\ell)} = (k+\ell)...(k+1)\theta^{\ell} \frac{(n+m)^{k}}{n^{k+\ell}} \mu_{1:n+m}^{(k)} \tag{16}$$

Proof.

$$\mu_{1:n}^{(k+\ell)} = (k+\ell)! \left(\frac{\theta}{n} \right)^{k+\ell} \text{ and } \mu_{1:n+m}^{(k)} = k! \left(\frac{\theta}{n+m} \right)^{k}$$

$$\mu_{1:n}^{(k+\ell)} = c_6 \mu_{1:n+m}^{(k)}$$

$$(k+\ell)! \left(\frac{\theta}{n} \right)^{k+\ell} = c_6 k! \left(\frac{\theta}{n+m} \right)^{k}$$

$$c_6 = (k+\ell)...(k+1)\theta^{\ell} \frac{(n+m)^{k}}{n^{k+\ell}}$$

3. CONCLUSION

In this study, the k^{th} moments of $X_{1:n}$ and some recurrence relations related to this k^{th} moments are presented. The recurrence relations for the moments of minimum order statistics are important in the theory of order statistics. The moments of minimum order statistics can be obtained by some other moments of minimum order statistics.

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