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ON STANCU TYPE GENERALIZATION OF (p,q)-BASKAKOV-KANTOROVICH OPERATORS

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Abstract. In the current paper, we introduce Stancu type generalization of Baskakov-Kantorovich operators based on (p,q)-integers and estimate the moments. We show the convergence of the new operators via the weighted Korovkin theorem. Then we investigate direct results by using Peetre's K-functional and modulus of continuity. In addition, we give pointwise estimation by the help of functions belonging to Lipschitz class. Moreover, we demonstrate the Voronovskaya-type theorem for the newly constructed operators. In the last section, we represent some illustrative graphics to show the convergence of the constructed operators to the selected function by using MATLAB.

1. Introduction

Quantum calculus, namely the (q-calculus), has an extensive research area in approximation theory. Varied generalizations of some linear positive operators based on q-calculus and their approximation properties have been discussed widely for three decades. Furthermore, quantum calculus is extended to post-quantum calculus, which is denoted by (p,q)-calculus. The new parameter p provides flexibility to the approximation. (p,q)-calculus is used effectively in many areas of mathematics, such as neural network, Lie group, field theory, hypergeometric series and differential equations. Mursaleen et al. [15] initiated the research of (p,q)-calculus in approximation theory. Further, they defined (p,q)-analogue of Bernstein-Stancu operators in [16]. The application of (p,q)-calculus to well known operators magnetizes great attention of researchers. We can refer some of the recent papers as [1], [2], [6], [8], [10], [11], [14], [18] and [19].

Let us briefly mention some notations and definitions of (p,q)-calculus. For

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$0 < q < p \leq 1$ and nonnegative integer m, the (p,q)-integers of m are described by

$$[m]_{p,q} = \frac{p^m - q^m}{p - q}. \tag{1}$$

The relation between q-calculus and (p,q)-calculus is given by $[m]_{p,q} = p^{m-1}[m]_{q/p}$. The (p,q)-factorial is defined by

$$[m]_{p,q}! = \prod_{i=1}^{m} [i]_{p,q}, \quad m \geq 1,$$

where $[0]_{p,q}! = 1$. Moreover, $0 \leq s \leq m$ nonnegative integers,

$$\left[\begin{array}{c} m \\ s \end{array}\right]_{p,q} = \frac{[m]_{p,q}!}{[s]_{p,q}![m-s]_{p,q}!}$$

are the (p,q)-binomial coefficients. The (p,q)-binomial expansion is represented by

$$(x \oplus y)^m_{p,q} = (x+y)(px+qy)(p^2x+q^2y)\dots(p^{m-1}x+q^{m-1}y).$$

The (p,q)-derivative of the function $f : \mathbb{R} \to \mathbb{R}$ is given as

$$(D_{p,q}f) = \frac{f(px) - f(qx)}{(p-q)x},\ x \neq 0, \ \ (D_{p,q}f)(0) = f'(0).$$

The (p,q)-integral of the arbitrary function f is defined by Acar et al. [3]

$$\int_a^b f(x) d_{p,q}x = (p-q)(b-a) \sum_{s=0}^{\infty} \frac{q^s}{p^{s+1}} f\left(a + (b-a)\frac{q^s}{p^{s+1}}\right), \ \left|\frac{q}{p}\right| < 1, \tag{2}$$

where a is a positive real number. More details about (p,q)-calculus can be read from [12] and [17].

In 2016, Aral and Gupta [4] introduced the (p,q)-analogue of Baskakov operators

$$B_{m,p,q}(f;x) = \sum_{s=0}^{\infty} b^{p,q}_{m,s}(x) f\left(\frac{p^{m-1}[s]_{p,q}}{q^{s-1}[m]_{p,q}}\right), \tag{3}$$

where $x \in [0,\infty)$, $0 < q < p \leq 1$ and

$$b^{p,q}_{m,s}(x) = \left[\begin{array}{c} m+s-1 \\ s \end{array}\right]_{p,q} p^{s+m(m-1)/2} q^{s(s-1)/2} \frac{x^s}{(1 \oplus x)^{m+s}_{p,q}}. \tag{4}$$

The moments of (p,q)-Baskakov operators are defined by

$$B_{m,p,q}(e_0;x) = 1, B_{m,p,q}(e_1;x) = x, B_{m,p,q}(e_2;x) = x^2 + \frac{p^{m-1}x}{[m]_{p,q}}\left(1 + \frac{p}{q}x\right), \tag{5}$$

where $e_j(x) = x^j, j = 0,1,2$. If we take $p = 1$, we obtain q-Baskakov operators [5]. The (p,q)-analogue of Baskakov-Kantorovich operators is given by Acar et al. [3]

$$B^*_{m,p,q}(f;x) = [m]_{p,q} \sum_{s=0}^{\infty} b^{p,q}_{m,s}(x) q^{-s} \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} f\left(\frac{p^{m-1}t}{q^{s-1}}\right) d_{p,q}t, \tag{6}$$

where $x \in [0,\infty)$, $0 < q < p \leq 1$ and $b^{p,q}_{m,s}(x)$ is as given by (4). We present the moments of (p,q)-Baskakov-Kantorovich operators as follows:

$$\begin{aligned} B^{*}_{m,p,q}(e_0;x) &= 1, \quad B^{*}_{m,p,q}(e_1;x) = px + \frac{qp^{m-1}}{(p+q)[m]_{p,q}}, \\ B^{*}_{m,p,q}(e_2;x) &= \left(p^2x^2 + \frac{p^{m+1}x}{[m]_{p,q}}\left(1+\frac{p}{q}x\right)\right) \\ &\quad + \frac{2qp^m x}{(p+q)[m]_{p,q}} + \frac{q^2p^{2m-2}}{(p^2+pq+q^2)[m]^2_{p,q}}, \end{aligned}$$

where $e_j(x) = x^j, j = 0,1,2$.

In the following section, we will give Stancu type generalization of the operators (6) for $0 \leq \alpha \leq \beta$, $0 < q < p \leq 1$ and each $x \in [0,\infty)$. Then we will calculate the moments of the constructed operators. Further, we will present the convergence of the operators according to weighted Korovkin theorem.

2. Construction of the Operator

Definition 1. *For any $x \in [0,\infty)$, $0 < q < p \leq 1$, $0 \leq \alpha \leq \beta$, we construct the (p,q)-analogue of Stancu type Baskakov-Kantorovich operators by*

$$K^{\alpha,\beta}_{m,p,q}(f;x) = [m]_{p,q}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} f\left(\frac{p^{m-1}q^{1-s}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}\right)d_{p,q}t, \quad (7)$$

where

$$b^{p,q}_{m,s}(x) = \left[\begin{array}{c} m+s-1 \\ s \end{array}\right]_{p,q} p^{s+m(m-1)/2}q^{s(s-1)/2}\frac{x^s}{(1\oplus x)^{m+s}_{p,q}}. \quad (8)$$

Lemma 2. [3] *For $0 < q < p \leq 1$ and each nonnegative integer m, we have the following equalities:*

$$\begin{aligned} \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} d_{p,q}t &= \frac{q^s}{[m]_{p,q}}, & (9) \\ \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} t\,d_{p,q}t &= \frac{pq^s[s]_{p,q}}{[m]^2_{p,q}} + \frac{q^{2s}}{(p+q)[m]^2_{p,q}}, & (10) \\ \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} t^2 d_{p,q}t &= \frac{p^2q^s[s]^2_{p,q}}{[m]^3_{p,q}} + \frac{2pq^{2s}[s]_{p,q}}{(p+q)[m]^3_{p,q}} + \frac{q^{3s}}{(p^2+pq+q^2)[m]^3_{p,q}}. & (11) \end{aligned}$$

Proof. By using (2) and the equality $[s+1]_{p,q} = q^s + p[s]_{p,q}$, the proof is completed. □

Lemma 3. *Let $K^{\alpha,\beta}_{m,p,q}(.;.)$ be given by (7) and (8). Then we obtain the following equalities*

$$K^{\alpha,\beta}_{m,p,q}(1;x) = 1, \quad (12)$$

$$K_{m,p,q}^{\alpha,\beta}(t;x) = \frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x + \frac{qp^{m-1}}{([m]_{p,q}+\beta)(p+q)} + \frac{\alpha}{[m]_{p,q}+\beta}, \tag{13}$$

$$\begin{aligned} K_{m,p,q}^{\alpha,\beta}(t^2;x) &= \frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)x^2 \\ &+\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^m q}{p+q}\right)x \\ &+\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right). \end{aligned} \tag{14}$$

Proof. *(i)* From the definition of the operators (7), we can obviously show the first moment (12) by using (9) and $q^s = [s+1]_{p,q} - p[s]_{p,q}$ as follows:

$$\begin{aligned} K_{m,p,q}^{\alpha,\beta}(1;x) &= [m]_{p,q}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} d_{p,q}t \\ &= [m]_{p,q}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\left(\frac{[s+1]_{p,q}-p[s]_{p,q}}{[m]_{p,q}}\right) = B_{m,p,q}(e_0;x) = 1. \end{aligned}$$

(ii) We have the following equality for the second moment (13) by the help of (9) and (10).

$$\begin{aligned} K_{m,p,q}^{\alpha,\beta}(t;x) &= [m]_{p,q}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} \frac{p^{m-1}q^{1-s}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}d_{p,q}t \\ &= \frac{[m]_{p,q}^2}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}p^{m-1}q^{1-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} t d_{p,q}t \\ &+\frac{\alpha[m]_{p,q}}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} 1 d_{p,q}t \\ &= \frac{[m]_{p,q}^2}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\frac{p^{m-1}}{q^{s-1}}\left(\frac{pq^s[s]_{p,q}}{[m]_{p,q}^2}+\frac{q^{2s}}{(p+q)[m]_{p,q}^2}\right) \\ &+\frac{\alpha[m]_{p,q}}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)q^{-s}\frac{q^s}{[m]_{p,q}} \\ &= \frac{p[m]_{p,q}}{[m]_{p,q}+\beta}\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x)\frac{p^{m-1}[s]_{p,q}}{q^{s-1}[m]_{p,q}} \\ &+\left(\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}\right)\sum_{s=0}^{\infty} b_{m,s}^{p,q}(x). \end{aligned}$$

As we use the moments of (p,q)-Baskakov operators given by (5), we get

$$K_{m,p,q}^{\alpha,\beta}(t;x) = \frac{p[m]_{p,q}}{[m]_{p,q}+\beta}B_{m,p,q}(e_1;x)$$

$$
\begin{aligned}
&+\left(\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}\right)B_{m,p,q}(e_0;x)\\
=\quad&\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}.
\end{aligned}
$$

(iii) Similarly, we will calculate the third moment $K^{\alpha,\beta}_{m,p,q}(t^2;x)$ as follows:

$$
\begin{aligned}
K^{\alpha,\beta}_{m,p,q}(t^2;x) \quad = \quad & [m]_{p,q}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}}\left(\frac{p^{m-1}q^{1-s}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}\right)^2 d_{p,q}t\\
= \quad & \frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\left\{[m]^2_{p,q}\frac{p^{2m-2}}{q^{2s-2}}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} t^2 d_{p,q}t\right.\\
& \left.+2\alpha[m]_{p,q}\frac{p^{m-1}}{q^{s-1}}\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} t d_{p,q}t+\alpha^2\int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} 1 d_{p,q}t\right\}\\
= \quad & \frac{[m]^3_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\frac{p^{2m-2}}{q^{2s-2}}\\
& \qquad\times\left\{\frac{p^2q^s[s]^2_{p,q}}{[m]^3_{p,q}}+\frac{2pq^{2s}[s]_{p,q}}{(p+q)[m]^3_{p,q}}+\frac{q^{3s}}{(p^2+pq+q^2)[m]^3_{p,q}}\right\}\\
& +\frac{2\alpha[m]^2_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\frac{p^{m-1}}{q^{s-1}}\left\{\frac{pq^s[s]_{p,q}}{[m]^2_{p,q}}+\frac{q^{2s}}{(p+q)[m]^2_{p,q}}\right\}\\
& +\frac{\alpha^2[m]_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)q^{-s}\frac{q^s}{[m]_{p,q}}\\
= \quad & \frac{p^2[m]^2_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\frac{p^{2m-2}}{q^{2s-2}}\frac{[s]^2_{p,q}}{[m]^2_{p,q}}\\
& +\frac{2}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\frac{p^{2m-1}}{q^{s-2}}\frac{[s]_{p,q}}{p+q}\\
& +\frac{1}{([m]_{p,q}+\beta)^2(p^{+}pq+q^2)}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)p^{2m-2}q^2\\
& +\frac{2\alpha p[m]_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\frac{p^{m-1}}{q^{s-1}}\frac{[s]_{p,q}}{[m]_{p,q}}\\
& +\frac{2\alpha}{([m]_{p,q}+\beta)^2(p+q)}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)qp^{m-1}+\frac{\alpha^2}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\\
= \quad & \frac{p^2[m]^2_{p,q}}{([m]_{p,q}+\beta)^2}\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\frac{p^{2m-2}}{q^{2s-2}}\frac{[s]^2_{p,q}}{[m]^2_{p,q}}\\
& +\left(\frac{2p^m q[m]_{p,q}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{2\alpha p[m]_{p,q}}{([m]_{p,q}+\beta)^2}\right)\sum_{s=0}^{\infty} b^{p,q}_{m,s}(x)\frac{p^{m-1}}{q^{s-1}}\frac{[s]_{p,q}}{[m]_{p,q}}
\end{aligned}
$$

$$+\left(\frac{p^{2m-2}q^2}{([m]_{p,q}+\beta)^2(p^+pq+q^2)}+\frac{2\alpha qp^{m-1}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{\alpha^2}{([m]_{p,q}+\beta)^2}\right)$$
$$\times\sum_{s=0}^{\infty}b_{m,s}^{p,q}(x).$$

After that, we take into consideration the moments of (p,q)-Baskakov operators (5) and we obtain

$$\begin{aligned}
K_{m,p,q}^{\alpha,\beta}(t^2;x) &= \frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}B_{m,p,q}(e_2;x)\\
&\quad+\left(\frac{2p^m q[m]_{p,q}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{2\alpha p[m]_{p,q}}{([m]_{p,q}+\beta)^2}\right)B_{m,p,q}(e_1;x)\\
&\quad+\left(\frac{p^{2m-2}q^2}{([m]_{p,q}+\beta)^2(p^+pq+q^2)}+\frac{2\alpha qp^{m-1}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{\alpha^2}{([m]_{p,q}+\beta)^2}\right)\\
&\quad\times B_{m,p,q}(e_0;x)\\
&= \frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(x^2+\frac{p^{m-1}x}{[m]_{p,q}}\left(1+\frac{p}{q}x\right)\right)\\
&\quad+\left(\frac{2p^m q[m]_{p,q}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{2\alpha p[m]_{p,q}}{([m]_{p,q}+\beta)^2}\right)x\\
&\quad+\left(\frac{p^{2m-2}q^2}{([m]_{p,q}+\beta)^2(p^+pq+q^2)}+\frac{2\alpha qp^{m-1}}{([m]_{p,q}+\beta)^2(p+q)}+\frac{\alpha^2}{([m]_{p,q}+\beta)^2}\right)\\
&= \frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)x^2\\
&\quad+\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^m q}{p+q}\right)x\\
&\quad+\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right).
\end{aligned}\tag{15}$$

□

Corollary 4. *Central moments* $\phi_n^{\alpha,\beta}(x)=K_{m,p,q}^{\alpha,\beta}((t-x)^n;x)$ *for* $n=1,2$ *are given by*

$$\phi_1^{\alpha,\beta}(x)=\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}-1\right)x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}\tag{16}$$

and

$$\begin{aligned}
\phi_2^{\alpha,\beta}(x) &= \left(\frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)-2\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}+1\right)x^2\\
&\quad+\left(\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^m q}{p+q}\right)\right.\\
&\qquad\left.-\frac{2qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-\frac{2\alpha}{[m]_{p,q}+\beta}\right)x
\end{aligned}$$

$$+\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right). \tag{17}$$

Proof. It can be seen evidently that the operators $K^{\alpha,\beta}_{m,p,q}(f;x)$ are linear and positive. We will use the linearity of the operators over again to show the equality of the first central moment $\phi^{\alpha,\beta}_1(x)=K^{\alpha,\beta}_{m,p,q}(t-x;x)$.

$$\begin{aligned}\phi^{\alpha,\beta}_1(x) &= K^{\alpha,\beta}_{m,p,q}(t;x)-xK^{\alpha,\beta}_{m,p,q}(1;x)\\ &= \left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}-1\right)x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}.\end{aligned}$$

For the second central moment $\phi^{\alpha,\beta}_2(x)=K^{\alpha,\beta}_{m,p,q}((t-x)^2;x)$, we will again use the linearity of the operators $K^{\alpha,\beta}_{m,p,q}$

$$\begin{aligned}\phi^{\alpha,\beta}_2(x) &= K^{\alpha,\beta}_{m,p,q}(t^2;x)-2xK^{\alpha,\beta}_{m,p,q}(t;x)+x^2K^{\alpha,\beta}_{m,p,q}(1;x)\\ &= \mu_1(m)x^2+\mu_2(m)x+\mu_3(m), \end{aligned} \tag{18}$$

where we briefly denote

$$\begin{aligned}\mu_1(m) &= \frac{p^2[m]^2_{p,q}}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)-\frac{2p[m]_{p,q}}{[m]_{p,q}+\beta}+1,\\ \mu_2(m) &= \frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^mq}{p+q}\right)\\ &\quad -\frac{2qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-\frac{2\alpha}{[m]_{p,q}+\beta},\\ \mu_3(m) &= \frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right).\end{aligned}$$

Let us choose, $\tilde{\mu}(m):=max\left\{\mu_1(m),\frac{\mu_2(m)}{2},\mu_3(m)\right\}$. Finally, we write

$$\phi^{\alpha,\beta}_2(x)\leq\tilde{\mu}(m)(1+x)^2 \tag{19}$$

as desired. □

Remark 5. *For $0<q<p\leq 1$, $\lim\limits_{m\to\infty}[m]_{p,q}=\frac{1}{p-q}$. To get the convergence results of our operators $K^{\alpha,\beta}_{m,p,q}(f;x)$, we take the sequences $0<q_m<p_m\leq 1$ such that $\lim\limits_{m\to\infty}p_m=1$, $\lim\limits_{m\to\infty}q_m=1$, $\lim\limits_{m\to\infty}p_m^m=1$ and $\lim\limits_{m\to\infty}q_m^m=1$. Thus, we have $\lim\limits_{m\to\infty}\frac{1}{[m]_{p_m,q_m}}=0$. Such a sequence can be defined by taking $p_m=1-\frac{1}{2m^2}$, $q_m=1-\frac{1}{m^2}$. It can be clearly seen that $\lim\limits_{m\to\infty}p_m^m=1$, $\lim\limits_{m\to\infty}q_m^m=1$ and $\lim\limits_{m\to\infty}\frac{1}{[m]_{p_m,q_m}}=0$. Additionally, here we say that $\mu_1(m)\to 0$, $\mu_2(m)\to 0$, $\mu_3(m)\to 0$ as $m\to\infty$, hence $\tilde{\mu}(m)\to 0$ as $m\to\infty$.*

3. Weighted Approximation

First of all, we recall the definitions of the weighted spaces: $C[0,\infty)$ indicates the set of all continuous functions f defined on $[0,\infty)$. $B_2[0,\infty)$ is the set of all functions f defined on $[0,\infty)$ satisfying the condition $|f(x)| \leq M(1+x^2)$, where $M > 0$ is constant. Then $B_2[0,\infty)$ is a linear normed space with the norm $||f||_2 = \underset{x\geq 0}{sup}\frac{|f(x)|}{1+x^2}$. In addition, $C_2[0,\infty)$ signifies the subspace of all continuous functions in $B_2[0,\infty)$. Furthermore, $C_2^*[0,\infty)$ denotes the subspace of all continuous functions in $B_2[0,\infty)$ for which $\underset{x\to\infty}{lim}\frac{f(x)}{1+x^2}$ is finite. The class of all real valued continuous and bounded functions f on $[0,\infty)$ is denoted by $C_B[0,\infty)$. The norm is defined as $\|f\|_{C_B} = \underset{x\geq 0}{sup}|f(x)|$. Also, $C_B^2[0,\infty)$ denotes the space of the functions f, for which f, f' and f'' are continuous on $[0,\infty)$. The norm of function f in the space $C_B^2[0,\infty)$ is denoted by $\|f\|_{C_B^2[0,\infty)} = \|f\|_{C_B[0,\infty)} + \|f'\|_{C_B[0,\infty)} + \|f''\|_{C_B[0,\infty)}$.

Theorem 6. *Let $K_{m,p,q}^{\alpha,\beta}(f;x)$ satisfy the conditions in Remark 5 for $0 < q_m < p_m \leq 1$ and $0 \leq \alpha \leq \beta$. Then for each function $f \in C_2^*[0,\infty)$, $K_{m,p,q}^{\alpha,\beta}(f;x)$ converge uniformly to f on $[0,\infty)$.*

Proof. It is sufficient from the weighted Korovkin theorem [9] to make clear that

$$\lim_{m\to\infty}\left\|K_{m,p_m,q_m}^{\alpha,\beta}e_i - e_i\right\|_2 = 0, \quad i = 0,1,2$$

to prove the theorem, where $e_i(x) = x^i$, $i = 0,1,2$.
(i) By using (12), it is clear that

$$\lim_{m\to\infty}\left\|K_{m,p_m,q_m}^{\alpha,\beta}e_0 - e_0\right\|_2 = \lim_{m\to\infty}\sup_{x\geq 0}\frac{|K_{m,p_m,q_m}^{\alpha,\beta}(1;x) - 1|}{1+x^2} = 0.$$

(ii) Using (13), we obtain

$$\begin{aligned}
&\lim_{m\to\infty}\left\|K_{m,p_m,q_m}^{\alpha,\beta}e_1 - e_1\right\|_2 \\
&= \lim_{m\to\infty}\sup_{x\geq 0}\frac{\left|K_{m,p_m,q_m}^{\alpha,\beta}(t;x) - x\right|}{1+x^2} \\
&= \lim_{m\to\infty}\sup_{x\geq 0}\frac{\left|\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta} - 1\right)x + \frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)} + \frac{\alpha}{[m]_{p,q}+\beta}\right|}{1+x^2} \\
&\leq \lim_{m\to\infty}\left|\frac{p[m]_{p,q}}{[m]_{p,q}+\beta} - 1\right|\sup_{x\geq 0}\frac{x}{1+x^2} \\
&+ \lim_{m\to\infty}\left(\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)} + \frac{\alpha}{[m]_{p,q}+\beta}\right)\sup_{x\geq 0}\frac{1}{1+x^2} \\
&\leq \lim_{m\to\infty}\frac{1}{2}\left|\frac{p[m]_{p,q}}{[m]_{p,q}+\beta} - 1\right| + \lim_{m\to\infty}\left(\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)} + \frac{\alpha}{[m]_{p,q}+\beta}\right)
\end{aligned}$$

$$= 0.$$

(iii) By the help of (14), we can write

$$\begin{aligned}
&\lim_{m\to\infty}\left\|K_{m,p,q}^{(\alpha,\beta)}e_2-e_2\right\|_2=\lim_{m\to\infty}\sup_{x\geq0}\frac{|K_{m,p_m,q_m}^{(\alpha,\beta)}(t^2;x)-x^2|}{1+x^2}\\
=\ &\lim_{m\to\infty}\sup_{x\geq0}\left|\left(\frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)-1\right)x^2\right.\\
&\quad+\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^mq}{p+q}\right)x\\
&\quad\left.+\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right)\right|\Big/(1+x^2)\\
\leq\ &\lim_{m\to\infty}\left|\frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)-1\right|\sup_{x\geq0}\frac{x^2}{1+x^2}\\
&+\lim_{m\to\infty}\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^mq}{p+q}\right)\sup_{x\geq0}\frac{x}{1+x^2}\\
&+\lim_{m\to\infty}\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right)\sup_{x\geq0}\frac{1}{1+x^2}\\
\leq\ &\lim_{m\to\infty}\left(\left|\frac{p^2[m]_{p,q}^2}{([m]_{p,q}+\beta)^2}\left(1+\frac{p^m}{q[m]_{p,q}}\right)-1\right|\right.\\
&\quad+\frac{1}{2}\frac{[m]_{p,q}}{([m]_{p,q}+\beta)^2}\left(p^{m+1}+2\alpha p+\frac{2p^mq}{p+q}\right)\\
&\quad\left.+\frac{1}{([m]_{p,q}+\beta)^2}\left(\frac{p^{2m-2}q^2}{p^2+pq+q^2}+\frac{2\alpha qp^{m-1}}{p+q}+\alpha^2\right)\right)\\
=\ &0.
\end{aligned}$$

Thus, the proof is completed as desired. □

4. Direct Results

In this section, we will give an auxiliary lemma to prove the main results and then discuss the local approximation properties in terms of Peetre's K-functionals and modulus of continuities. Peetre's K-functionals are defined as follows:

$$K_2(f,\delta):=\inf_{g\in C_B^2[0,\infty)}\{\|f-g\|_{C_B[0,\infty)}+\delta\,\|g\|_{C_B^2[0,\infty)}\}.$$

The modulus of continuity of the function $f\in C_B[0,\infty)$ is given by

$$\omega(f,\delta):=\sup_{0<h<\delta}\ \sup_{x,x+h\in[0,\infty)}|f(x+h)-f(x)|.$$

The second order modulus of smoothness of the function $f \in C_B[0,\infty)$ is defined by

$$\omega_2(f,\delta) := \sup_{0<h<\sqrt{\delta}} \sup_{x,x+h\in[0,\infty)} |f(x+2h) - 2f(x+h) + f(x)|,$$

where $\delta > 0$. By [7], it is known that for $M > 0$

$$K_2(f,\delta) \leq M\omega_2(f,\sqrt{\delta}).$$

Before we mention the local approximation properties, we will give the following lemmas.

Lemma 7. *For $f \in C_B[0,\infty)$, we have*

$$|K^{\alpha,\beta}_{m,p,q}(f;x)| \leq ||f||.$$

Proof.

$$\begin{aligned}
|K^{\alpha,\beta}_{m,p,q}(f;x)| &= \left| [m]_{p,q} \sum_{s=0}^{\infty} b^{p,q}_{m,s}(x) q^{-s} \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} f\left(\frac{p^{m-1}q^{1-s}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}\right) d_{p,q}t \right| \\
&\leq [m]_{p,q} \sum_{s=0}^{\infty} b^{p,q}_{m,s}(x) q^{-s} \int_{\frac{p[s]_{p,q}}{[m]_{p,q}}}^{\frac{[s+1]_{p,q}}{[m]_{p,q}}} \left| f\left(\frac{p^{m-1}q^{1-s}[m]_{p,q}t+\alpha}{[m]_{p,q}+\beta}\right) \right| d_{p,q}t \\
&\leq ||f|| K^{\alpha,\beta}_{m,p,q}(1;x) \\
&= ||f||.
\end{aligned}$$

□

Lemma 8. *Let $f \in C_B[0,\infty)$ and $g \in C^2_B[0,\infty)$. We define the auxiliary operators $K^*_{m,p,q}$ such as*

$$\begin{aligned}
K^*_{m,p,q}(g;x) &= K^{\alpha,\beta}_{m,p,q}(g;x) + g(x) \\
&\quad - g\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x + \frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)} + \frac{\alpha}{[m]_{p,q}+\beta}\right). \quad (20)
\end{aligned}$$

And then, for all $g \in C^2_B[0,\infty)$, we obtain

$$|K^*_{m,p,q}(g;x) - g(x)| \leq \|g''\|_{C_B} \left(\tilde{\mu}(m)(1+x)^2 + \eta^2_m(\alpha,\beta,x)\right),$$

where

$$\eta_m(\alpha,\beta,x) = \frac{((p-1)[m]_{p,q}-\beta)\,x+\alpha}{[m]_{p,q}+\beta} + \frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}.$$

Proof. By using the auxiliary operators $K^*_{m,p,q}$, we have

$$\begin{aligned}
K^*_{m,p,q}(g;x) &= K^{\alpha,\beta}_{m,p,q}(g;x) + g(x) \\
&\quad - g\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x + \frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)} + \frac{\alpha}{[m]_{p,q}+\beta}\right). \quad (21)
\end{aligned}$$

It is apparent from Lemma 3 that

$$\begin{aligned}
K^*_{m,p,q}(1;x) &= 1,\\
K^*_{m,p,q}(t-x;x) &= K^{(\alpha,\beta)}_{m,p,q}((t-x);x)+(x-x)\\
&\quad -\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}-x\right)\\
&= \left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}-1\right)x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}\\
&\quad -\left(\frac{p[m]_{p,q}}{[m]_{p,q}+\beta}x+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}+\frac{\alpha}{[m]_{p,q}+\beta}-x\right)\\
&= 0. \qquad (22)
\end{aligned}$$

So, we can say that the operators $K^*_{m,p,q}(f;x)$ are linear. For a given function $g\in C^2[0,\infty]$, we write the Taylor expansion as follows:

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du,\ \ t\in[0,\infty). \qquad (23)$$

If we apply $K^*_{m,p,q}$ operators to both sides of the equality (23), we obtain

$$\begin{aligned}
K^*_{m,p,q}(g;x) &= K^*_{m,p,q}\left(g(x)+(t-x)g'(x)+\int_x^t (t-u)g''(u)du;x\right)\\
&= g(x)+K^*_{m,p,q}\left((t-x)g'(x);x\right)+K^*_{m,p,q}\left(\int_x^t (t-u)g''(u)du;x\right).
\end{aligned}$$

Thus,

$$K^*_{m,p,q}(g;x)-g(x)=g'(x)K^*_{m,p,q}\left((t-x);x\right)+K^*_{m,p,q}\left(\int_x^t (t-u)g''(u)du;x\right).$$

Using (21) and (22), we get

$$\begin{aligned}
K^*_{m,p,q}(g;x)-g(x) &= K^*_{m,p,q}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&= K^{\alpha,\beta}_{m,p,q}\left(\int_x^t (t-u)g''(u)du;x\right)\\
&\quad -\int_x^{\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}}\\
&\quad \times\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-u\right)g''(u)du. \qquad (24)
\end{aligned}$$

Furthermore,

$$\left|\int_x^t (t-u)g''(u)du\right| \leq \int_x^t |t-u||g''(u)|du \leq ||g''||_{C_B}\int_x^t |t-u|du \leq (t-x)^2||g''||_{C_B} \tag{25}$$

and

$$\begin{aligned}
&\left|\int_x^{\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}}\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-u\right)g''(u)du\right| \\
&\leq \left|\,||g''||_{C_B}\int_x^{\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}}\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-u\right)du\right| \\
&\leq ||g''||_{C_B}\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}-x\right)^2 \\
&= ||g''||_{C_B}\left(\frac{((p-1)[m]_{p,q}-\beta)x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}\right)^2. \qquad (26)
\end{aligned}$$

Here, we rewrite (25) and (26) in the absolute value of (24). So, by the help of (19) we obtain

$$|K^*_{m,p,q}(g;x)-g(x)| \leq \|g''\|_{C_B}\left(\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right). \tag{27}$$

It completes the proof. □

Now, we will calculate the rate of convergence of our operators $K^{\alpha,\beta}_{m,p,q}(f;x)$ by means of Peetre's K-functionals.

Theorem 9. *Let $f \in C_B[0,\infty)$, $0 < q_m < p_m \leq 1$ and $0 \leq \alpha \leq \beta$. Then we have for all $x \in [0,\infty)$, there exists a positive constant M such that,*

$$\begin{aligned}
|K^{\alpha,\beta}_{m,p,q}(f;x)-f(x)| \quad &\leq \quad M\omega_2(f,\sqrt{\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)}) \\
&\quad +\omega(f,\eta_m(\alpha,\beta,x)). \qquad (28)
\end{aligned}$$

Proof. From the auxiliary operator (20), for every $g \in C_B^2[0,\infty)$

$$\begin{aligned}
|K^{\alpha,\beta}_{m,p,q}(f;x)-f(x)| \quad &= \quad \Big|\, K^*_{m,p,q}(f;x)-f(x) \\
&\quad +f\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}\right)-f(x) \\
&\quad +K^*_{m,p,q}(g;x)-K^*_{m,p,q}(g;x)+g(x)-g(x)\,\Big| \\
&\leq \quad |K^*_{m,p,q}(f-g;x)-(f-g)(x)|+|K^*_{m,p,q}(g;x)-g(x)| \\
&\quad +\left|f\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}\right)-f(x)\right|.
\end{aligned}$$

By using Lemma 7 and Lemma 8 we have

$$\begin{aligned}|K_{m,p,q}^{\alpha,\beta}(f;x)-f(x)| &\leq 4||f-g||_{C_B}\\ &+\left|f\left(\frac{p[m]_{p,q}x+\alpha}{[m]_{p,q}+\beta}+\frac{qp^{m-1}}{(p+q)([m]_{p,q}+\beta)}\right)-f(x)\right|\\ &+||g''||_{C_B}\left(\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)\right). \qquad (29)\end{aligned}$$

Then we take the infimum on the right-hand side. Finally, using the property of Peetre's K-functionals, we get

$$\begin{aligned}|K_{m,p,q}^{\alpha,\beta}(f;x)-f(x)| &\leq 4K_2(f,\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x))+\omega(f,\eta_m(\alpha,\beta,x))\\ &\leq M\omega_2(f,\sqrt{\tilde{\mu}(m)(1+x)^2+\eta_m^2(\alpha,\beta,x)})\\ &\quad+\omega(f,\eta_m(\alpha,\beta,x)).\end{aligned}$$

Thus, the proof is completed. □

We will estimate the rate of convergence of the operators $K_{m,p,q}^{\alpha,\beta}(f;x)$ by means of the modulus of continuity on the finite interval.

Theorem 10. *Let $f\in C_2[0,\infty)$, $0<q_m<p_m\leq 1$, $0\leq\alpha\leq\beta$ and $\omega_{a+1}(f,\delta)$ be the modulus of continuity on the finite interval $[0,a+1]\subset[0,\infty)$, where $a>0$. Then we have the following inequality for all $x\in[0,\infty)$,*

$$|K_{m,p,q}^{\alpha,\beta}(f;x)-f(x)|\leq 4M_f(1+a^2)\tilde{\mu}(m)(1+x)^2+2\omega_{a+1}(f,(1+x)\sqrt{\tilde{\mu}(m)}).$$

There exists a positive constant M_f, which is independent of m and $\tilde{\mu}(m)$.

Proof. We have known that $\omega_{a+1}(.,\delta)$ satisfies the following inequality

$$|f(t)-f(x)|\leq 4M_f(1+a^2)(t-x)^2+\left(1+\frac{|t-x|}{\delta}\right)\omega_{a+1}(f,\delta),\quad \delta>0. \qquad (30)$$

We apply $K_{m,p,q}^{\alpha,\beta}$ operators to both sides of the last inequality. After that, by choosing $\delta=(1+x)\sqrt{\tilde{\mu}(m)}$, using (19) and applying Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|K_{m,p,q}^{\alpha,\beta}(f;x)-f(x)| &\leq 4M_f(1+a^2)\phi_2^{\alpha,\beta}(x)+\left(1+\frac{\sqrt{\phi_2^{\alpha,\beta}(x)}}{\delta}\right)\omega_{a+1}(f,\delta)\\ &\leq 4M_f(1+a^2)\tilde{\mu}(m)(1+x)^2+2\omega_{a+1}(f,(1+x)\sqrt{\tilde{\mu}(m)})\end{aligned}$$

and this completes the proof as desired. □

5. Pointwise Estimates

In the current part, we compute the rate of convergence locally by using functions, which belong to Lipschitz class.

Definition 11. *Let $0 < a \leq 1$ and $E \subset [0,\infty)$. Then if $f \in C_B[0,\infty)$ is locally in $Lip(a)$,*

$$|f(y) - f(x)| \leq M|y-x|^a, y \in E, x \in [0,\infty) \tag{31}$$

is hold.

Theorem 12. *For every $x \in [0,\infty)$ and $0 \leq \alpha \leq \beta$, we have*

$$|K^{\alpha,\beta}_{m,p_m,q_m}(f;x) - f(x)| \leq M(\tilde{\mu}(m)^{a/2}(1+x)^a + 2(d(x,E))^a), \tag{32}$$

where M is a constant depending on a and f. Here, $d(x,E) = inf\{|t-x| : t \in E\}$ defines the distance between x and E.

Proof. Suppose that x_0 is in the closure of E such that $|x - x_0| = d(x,E)$. We write by the help of the triangle inequality

$$|f(t) - f(x)| \leq |f(t) - f(x_0)| + |f(x) - f(x_0)|. \tag{33}$$

If we use the inequality (31), we get

$$\begin{aligned}
|K^{\alpha,\beta}_{m,p_m,q_m}(f;x) - f(x)| &\leq K^{\alpha,\beta}_{m,p_m,q_m}(|f(t) - f(x_0)|;x) \\
&\quad + K^{\alpha,\beta}_{m,p_m,q_m}(|f(x) - f(x_0)|;x) \\
&\leq M\{K^{\alpha,\beta}_{m,p_m,q_m}(|t-x_0|^a;x) + |x-x_0|^a\} \\
&\leq M\{K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^a + |x-x_0|^a;x) + |x-x_0|^a\} \\
&= M\{K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^a;x) + 2|x-x_0|^a\}.
\end{aligned}$$

After that, we use Hölder inequality with $p = 2/a$ and $q = 2/(2-a)$

$$\begin{aligned}
|K^{\alpha,\beta}_{m,p_m,q_m}(f;x) - f(x)| &\leq M\{[K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^{ap};x)]^{1/p} + 2(d(x,E))^a\} \\
&= M\{[K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^2;x)]^{a/2} + 2(d(x,E))^a\} \\
&\leq M\{(\tilde{\mu}(m)(1+x)^2)^{a/2} + 2(d(x,E))^a\} \\
&= M\{(\tilde{\mu}(m)^{a/2}(1+x)^a) + 2(d(x,E))^a\}.
\end{aligned}$$

□

Now, we will obtain local direct estimation of the constructed operators by using the Lipschitz type maximal function of order a. Lenze [13] gives the definition of Lipschitz type maximal function $\tilde{\omega}_a$ as follows:

$$\tilde{\omega}_a(f,x) = \sup_{t \neq x,\, t \in [0,\infty)} \frac{|f(t) - f(x)|}{|t-x|^a}, x \in [0,\infty) \text{ and } a \in (0,1].$$

Theorem 13. *Let $0 < a \leq 1$ and $f \in C_B[0,\infty)$. Then for all $x \in [0,\infty)$,*

$$|K^{\alpha,\beta}_{m,p_m,q_m}(f;x) - f(x)| \leq \tilde{\omega}_a(f,x)\tilde{\mu}(m)^{a/2}(1+x)^a. \tag{34}$$

Proof. By using the inequality (34), we see

$$|K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)| \leq \tilde{\omega}_a(f,x)K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^a;x).$$

Using Hölder inequality with $p=2/a$ and $q=2/(2-a)$

$$\begin{aligned}|K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)| &\leq \tilde{\omega}_a(f,x)[K^{\alpha,\beta}_{m,p_m,q_m}(|t-x|^2;x)]^{a/2}\\ &\leq \tilde{\omega}_a(f,x)\tilde{\mu}(m)^{a/2}(1+x)^a.\end{aligned}$$

Thus, the proof is successfully completed. □

6. Voronovskaya Type Theorem

Theorem 14. *Let $K^{\alpha,\beta}_{m,p,q}(f;x)$ satisfy the conditions in Remark 5 for $0<q_m<p_m\leq 1$ and $0\leq\alpha\leq\beta$. Then for each function $f\in C_2^*[0,\infty)$ such that $f',f''\in C_2^*[0,\infty)$, we have*

$$\begin{aligned}\lim_{m\to\infty}[m]_{p_m,q_m}\left(K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)\right) &= \left(\eta x+\alpha+\frac{1}{2}\right)f'(x)\\ &\quad+\left((\mu+1)x^2+x\right)\frac{f''(x)}{2} \quad (35)\end{aligned}$$

uniformly on $[0,A]$ for any $A>0$.

Here $\eta=\lim\limits_{m\to\infty}[m]_{p_m,q_m}\left(\frac{p[m]_{p_m,q_m}}{[m]_{p_m,q_m}+\beta}-1\right)$, $\mu=\lim\limits_{m\to\infty}[m]_{p_m,q_m}\left(\frac{p[m]_{p_m,q_m}}{[m]_{p_m,q_m}+\beta}-1\right)^2$.

Proof. For a given function $f,f',f''\in C_2^*[0,\infty]$, we can write by the Taylor expansion

$$f(t)=f(x)+(t-x)f'(x)+\frac{(t-x)^2}{2}f''(x)+r(t,x)(t-x)^2 \quad t,x\in[0,\infty), \quad (36)$$

where $r(t,x)$ is Peano form of the remainder term and $\lim\limits_{t\to x}r(t,x)=0$. Applying $K^{\alpha,\beta}_{m,p_m,q_m}$ operators to (36), we obtain

$$\begin{aligned}K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)&=f'(x)K^{\alpha,\beta}_{m,p_m,q_m}((t-x);x)\\ &+\frac{f''(x)}{2}K^{\alpha,\beta}_{m,p_m,q_m}((t-x)^2;x)+K^{\alpha,\beta}_{m,p_m,q_m}(r(t,x)(t-x)^2;x).\end{aligned}$$

Using the Cauchy-Schwarz inequality for the remainder term, we obtain

$$K^{\alpha,\beta}_{m,p_m,q_m}(r(t,x)(t-x)^2;x)\leq\sqrt{K^{\alpha,\beta}_{m,p_m,q_m}(r^2(t,x);x)}\sqrt{K^{\alpha,\beta}_{m,p_m,q_m}((t-x)^4;x)}. \quad (37)$$

By using $r(t,x)\in C_2^*[0,\infty)$ and $\lim\limits_{t\to x}r(t,x)=0$, we observe that

$$\lim_{m\to\infty}K^{\alpha,\beta}_{m,p_m,q_m}(r^2(t,x);x)=r^2(x,x)=0 \quad (38)$$

uniformly for each $x\in[0,A]$. Hence, by using (37), (38) and positivity of the linear operators $K^{\alpha,\beta}_{m,p_m,q_m}$, we have

$$\lim_{m\to\infty}[m]_{p_m,q_m}K^{\alpha,\beta}_{m,p_m,q_m}(r(t,x)(t-x)^2;x)=0. \quad (39)$$

Therefore,

$$\lim_{m\to\infty}[m]_{p_m,q_m}\left(K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)\right)=\lim_{m\to\infty}[m]_{p_m,q_m}f'(x)K^{\alpha,\beta}_{m,p_m,q_m}((t-x);x)$$
$$+\lim_{m\to\infty}[m]_{p_m,q_m}\frac{f''(x)}{2}K^{\alpha,\beta}_{m,p_m,q_m}((t-x)^2;x). \tag{40}$$

Consider

$$\begin{aligned}\lim_{m\to\infty}[m]_{p_m,q_m}f'(x)K^{\alpha,\beta}_{m,p_m,q_m}((t-x);x) &= f'(x)\lim_{m\to\infty}[m]_{p_m,q_m}\phi_1^{\alpha,\beta}(x)\\ &= f'(x)\left(\eta x+\alpha+\frac{1}{2}\right)\end{aligned} \tag{41}$$

and

$$\begin{aligned}\lim_{m\to\infty}[m]_{p_m,q_m}\frac{f''(x)}{2}K^{\alpha,\beta}_{m,p_m,q_m}((t-x)^2;x) &= \frac{f''(x)}{2}\lim_{m\to\infty}[m]_{p_m,q_m}\phi_2^{\alpha,\beta}(x)\\ &= \frac{f''(x)}{2}\left((\mu+1)\,x^2+x\right),\end{aligned} \tag{42}$$

where

$$\eta=\lim_{m\to\infty}[m]_{p_m,q_m}\left(\frac{p[m]_{p_m,q_m}}{[m]_{p_m,q_m}+\beta}-1\right),$$

$$\mu=\lim_{m\to\infty}[m]_{p_m,q_m}\left(\frac{p[m]_{p_m,q_m}}{[m]_{p_m,q_m}+\beta}-1\right)^2.$$

Then by using (40), (41) and (42), we obtain

$$\lim_{m\to\infty}[m]_{p_m,q_m}\left(K^{\alpha,\beta}_{m,p_m,q_m}(f;x)-f(x)\right)=\left(\eta x+\alpha+\frac{1}{2}\right)f'(x)+\left((\mu+1)\,x^2+x\right)\frac{f''(x)}{2}$$

as desired. □

7. Graphical Analysis

In this part, we work on the convergence of Stancu type (p,q)-Baskakov-Kantorovich operators to the selected function $f(x)=20x^2-30x+4$. Just now, we give Matlab algorithms. Algorithm 1 is given to define (p,q)-integers of m.

Algorithm 1.

```
function y=pqinteger(m,p,q)
    y=(p^(m)-q^(m))/(p-q);
end
```

Algorithm 2 is presented to show the convergence of the constructed operators $K^{\alpha,\beta}_{m,p,q}(f;x)$ to the function $f(x)=20x^2-30x+4$.

Algorithm 2.

```
m=100; alpha1=[ 2.7, 3.5, 4.5]; beta1=[ 8.5, 8.7, 9.1];
inf=100; b=inf; syms t
for j=1:3
    alpha=alpha1(j);       beta=beta1(j);
    p1=[0.97, 0.98, 0.99]; q1=[0.9, 0.91, 0.92];
    subplot(1,3,j)
for i=1:3
    p=p1(i); q=q1(i); u=1;
    for x=0:1:b
        ts=0;
 for s=0:inf
            z=1;
            for j=1:m+s
                z=z*((p^(j-1))+(q^(j-1)).*x);
            end
  h1=1;
  for a1=0:m+s-2
  h1=h1*pqinteger(m+s-1-a1,p,q);
  end
   if (m==1)
       h2=1;
   end
   if (m~=1)
        h2=1;
        for a2=0:m-2
        h2=h2*pqinteger(m-1-a2,p,q);
        end
   end
   h3=1;
   for a3=0:s-1
   h3=h3*pqinteger(s-a3,p,q);
   end
   fact=h1/(h2*h3); f12=0;
      for m=0:b
      x1=(p^(m-1)*q^(1-s)*pqinteger(s,p,q)+alpha+(p^(-2))...
          *(q^(m+1)))/(pqinteger(m,p,q)+beta);
      f1=20*x1^2-30*x1+4; f11=(q^m)/(p^(m+1))*f1;  f12=f12+f11;
      end
      ft=(p-q)*f12; z;
      B=fact*(p^(s+m*(m-1)/2))*(q^(s*(s-1)/2))*(x^s)*ft;
      ts=ts+B/z;
   end
      a(u)=ts; u=u+1;
     end
      x=0:1:b;
     if (i==1)
         c=plot(x,a,'g'); hold on
    elseif (i==2)
         c=plot(x,a,'r'); hold on
     else (i==3)
         c=plot(x,a,'m'); end
end
 x=0:1:b;
 y=20*x.^2-30.*x+4;
plot(x,y,'--p')
     legend('p=0.97, q=0.9','p=0.98, q=0.91','p=0.99, q=0.92','function')
end
```

Additionally, in Figure 1 we have plotted illustrative graphics of the operators $K_{m,p,q}^{\alpha,\beta}(f;x)$ for different values of parameters p, q, α and β.

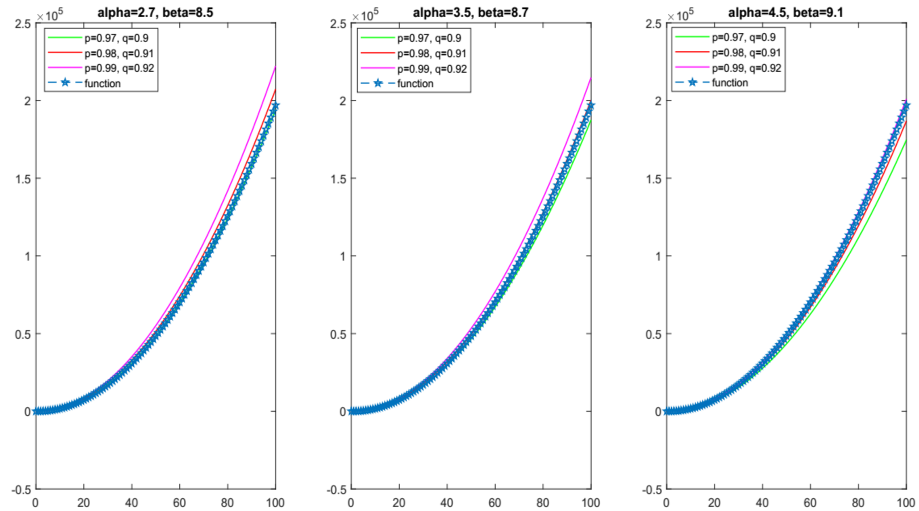

FIGURE 1. Convergence of $K_{m,p,q}^{\alpha,\beta}(f;x)$: Stancu type (p,q)-Baskakov-Kantorovich operators for fixed $m = 100$.

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