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On a Fuzzy Boundary Value Problem with an Eigenvalue Parameter Contained in the Boundary Condition

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Abstract. A fuzzy boundary value problem with an eigenvalue parameter contained in the boundary condition is investigated in this paper. The examination is made under the approach of Hukuhara differentiability. The effect on the eigenvalue and the eigenfunction of the problem of the eigenvalue in the boundary condition is shown.

Keywords: Fuzzy Boundary Value Problem, Hukuhara Differentiability, Eigenvalue, Eigenfunction.

Sınır Koşulunda bir Özdeğer Parametre Bulunan bir Fuzzy Sınır Değer Problemi Üzerine

Özet. Bu çalışmada sınır koşulunda bir özdeğer parametre içeren bir fuzzy sınır değer problemi araştırıldı. Bu araştırma Hukuhara diferansiyellenebilirlik yaklaşımı altında yapıldı. Sınır koşulundaki özdeğer parametrenin problemin özdeğer ve özfonksiyonu üzerindeki etkisi gösterildi.

Anahtar Kelimeler: Fuzzy Sınır Değer Problem, Hukuhara Diferansiyellenebilirlik, Özdeğer, Özfonksiyon.

1. INTRODUCTION

The fuzzy differential equation can be examined several approaches. The first approach is Hukuhara differentiability and for this, firstly the existence and uniqueness of the solution of a fuzzy differential equation are examined [1, 2]. Gültekin and Altınışık [3] have investigated the existence and uniqueness of solutions of two-point fuzzy boundary value problems using the Hukuhara differentiability. Gültekin Çitil and Altınışık [4] have defined the fuzzy Sturm-Liouville equation and they have examined eigenvalues and eigenfunctions of the problem under the approach of the Hukuhara differentiability.

The second approach is generalized derivative. The generalized derivative was presented in [5] and examined in [6-10]. Khastan and Nieto [11] have studied the fuzzy boundary value problem using the generalized derivative and introduced a new notion of the solution.

The third approach is to generate the fuzzy solution from the crips solution. In this approach offered by Gasilov at al. [12], a differential equation with fuzzy boundary values is investigated and the problem is interpreted as a set of crips problems.

This paper is on the eigenvalues and the eigenfunctions of the fuzzy boundary value problem with an eigenvalue parameter contained in the boundary condition under the approach Hukuhara differentiability. The drawn figures are plotted using the mathematica program.

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Definition 1.1

A function $u: \mathbb{R} \to [0,1]$ satisfying the following properties is a fuzzy number:

u is normal, u is convex fuzzy set, u is upper semi-continuous on $\mathbb{R}$ and $cl\{x \in \mathbb{R} | u(x) > 0\}$ is compact, where cl denotes the closure of a subset [9].
Let $\mathbb{R}_F$ be the space of fuzzy numbers.

Definition 1.2

Let $u \in \mathbb{R}_F$. The α-level set of u is $[u]^\alpha = \{x \in \mathbb{R} | u(x) \geq \alpha\}$, $0 < \alpha \leq 1$. If $\alpha = 0$, the support of u is $[u]^0 = cl\{x \in \mathbb{R} | u(x) > 0\}$. $[u]^\alpha = [\underline{u}_\alpha, \overline{u}_\alpha]$ shows the α-level set of u. $\underline{u}_\alpha$ and $\overline{u}_\alpha$ denote the lower and upper branches of $[u]^\alpha$, respectively [11].

Remark 1.1

$[\underline{u}_\alpha, \overline{u}_\alpha]$ satisfying the following properties is the valid α-level set:

1) $\underline{u}_\alpha$ is bounded monotonic increasing (nondecreasing) left-continuous function on $(0,1]$ and right-continuous for $\alpha = 0$,

2) $\overline{u}_\alpha$ is bounded monotonic decreasing (nonincreasing) left-continuous function on $(0,1]$and right-continuous for $\alpha = 0$,

3) $\underline{u}_\alpha \leq \overline{u}_\alpha$, $0 \leq \alpha \leq 1$ [9].

Definition 1.3

The α-level set of A which is a symmetric triangular number with support $[\underline{a}, \overline{a}]$ is

$$[A]^\alpha = \left[\underline{a} + \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha, \overline{a} - \left(\frac{\overline{a} - \underline{a}}{2}\right)\alpha\right] \text{ [14].}$$

Definition 1.4

The sum $u + v$ and the product λu are defined by $[u + v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda [u]^\alpha$, $\forall \alpha \in [0,1]$, for $u, v \in \mathbb{R}_F$ and $\lambda \in \mathbb{R}$ [2].

The metric structure is given by the Hausdorff distance

$$D: \mathbb{R}_F \times \mathbb{R}_F \to \mathbb{R}_+ \cup \{0\},$$

by

$$D(u,v) = \sup_{\alpha \in [0,1]} \max\left\{ \left|\underline{u}_\alpha - \underline{v}_\alpha\right|, \left|\overline{u}_\alpha - \overline{v}_\alpha\right| \right\} \text{ [9].}$$

Definition 1.5

Let $u, v \in \mathbb{R}_F$. w which is $u = v + w$, $w \in \mathbb{R}_F$ is called the H-difference of u and v and it is denoted $u \underset{H}{-} v$ [14].

Definition 1.6

Let $I=(a,b)$, for $a,b\in\mathbb{R}_F$, and $F: I\to\mathbb{R}_F$ be a fuzzy function. If there exist an element $F'(t_0)\in\mathbb{R}_F$ and the limits

$$\lim_{h\to 0^+}\frac{F(t_0+h)-F(t_0)}{h}=\lim_{h\to 0^+}\frac{F(t_0)-F(t_0+h)}{h}=F'(t_0),$$

then F is Hukuhara differentiable at $t_0\in I$ [9].

Theorem 1.1

Let $f: I\to\mathbb{R}_F$ be a function and show $[f(t)]^\alpha=\left[\underline{f}_\alpha(t),\overline{f}_\alpha(t)\right]$, for each $\alpha\in[0,1]$. If f is Hukuhara differentiable, $\underline{f}_\alpha$ and $\overline{f}_\alpha$ are differentiable functions and $[f'(t)]^\alpha=\left[\underline{f}_\alpha{}'(t),\overline{f}_\alpha{}'(t)\right]$ [13].

2. RESULTS AND DISCUSSION

Consider the fuzzy Sturm-Liouville problem

$$y''+\lambda y=0,\ \ x\epsilon(a,b) \quad \textbf{(2.1)}$$

$$\beta_1 y(a)+\beta_2\lambda y'(a)=0, \quad \textbf{(2.2)}$$

$$\beta_3 y(b)+\beta_4 y'(b)=0, \quad \textbf{(2.3)}$$

where $\lambda>0, \beta_1,\beta_2,\beta_3,\beta_4\geq 0, \beta_1^2+\beta_2^2\neq 0$ and $\beta_3^2+\beta_4^2\neq 0$. Let $\lambda=k^2, k>0$,

$$\phi(x,\lambda)=\left(k^2\beta_2 Cos(ka)+\frac{\beta_1}{k}Sin(ka)\right)Cos(kx)+$$

$$+\left(-\frac{\beta_1}{k}Cos(ka)+k^2\beta_2 Sin(ka)\right)Sin(kx)$$

and

$$\psi(x,\lambda)=\left(\beta_4 Cos(kb)+\frac{\beta_3}{k}Sin(kb)\right)Cos(kx)+\left(-\frac{\beta_3}{k}Cos(kb)+\beta_4 Sin(kb)\right)Sin(kx)$$

be the solutions of the classical differential equation $y''+\lambda y=0$ satisfying the conditions $y(a)=\lambda\beta_2, y'(a)=-\beta_1$ and $y(b)=\beta_4, y'(b)=-\beta_3$, respectively. Then,

$$[\phi(x,\lambda)]^\alpha=\left[\underline{\phi}_\alpha(x,\lambda),\overline{\phi}_\alpha(x,\lambda)\right]=[\alpha,2-\alpha]\phi(x,\lambda) \quad \textbf{(2.4)}$$

and

$$[\psi(x,\lambda)]^\alpha=\left[\underline{\psi}_\alpha(x,\lambda),\overline{\psi}_\alpha(x,\lambda)\right]=[\alpha,2-\alpha]\psi(x,\lambda) \quad \textbf{(2.5)}$$

are the solutions of the fuzzy differential equation (2.1) satisfying the conditions (2.2) and (2.3), respectively. The eigenvalues of the fuzzy Sturm-Liouville problem (2.1)-(2.3) if and only if are consist of the zeros of Wronskian functions

$$\underline{W}_\alpha(\lambda)=W\left(\underline{\phi}_\alpha,\underline{\psi}_\alpha\right)(x,\lambda)=\underline{\phi}_\alpha(x,\lambda)\underline{\psi}'_\alpha(x,\lambda)-\underline{\psi}_\alpha(x,\lambda)\underline{\phi}'_\alpha(x,\lambda), \quad \textbf{(2.6)}$$

$$\overline{W}_\alpha(\lambda) = W\left(\overline{\phi}_\alpha, \overline{\psi}_\alpha\right)(x,\lambda) = \overline{\phi}_\alpha(x,\lambda)\overline{\psi}_\alpha'(x,\lambda) - \overline{\psi}_\alpha(x,\lambda)\overline{\phi}_\alpha'(x,\lambda) \text{ [4].} \quad \textbf{(2.7)}$$

Then,

$$\underline{\phi}_\alpha(x,\lambda) = \alpha\left\{\left(k^2\beta_2 Cos(ka) + \frac{\beta_1}{k}Sin(ka)\right)Cos(kx) + \right.$$

$$\left. + \left(-\frac{\beta_1}{k}Cos(ka) + k^2\beta_2 Sin(ka)\right)Sin(kx)\right\} \quad \textbf{(2.8)}$$

$$\underline{\phi}_\alpha'(x,\lambda) = \alpha\{(-k^3\beta_2 Cos(ka) - \beta_1 Sin(ka))Sin(kx) +$$

$$+(-\beta_1 Cos(ka) + k^3\beta_2 Sin(ka))Cos(kx)\} \quad \textbf{(2.9)}$$

$$\underline{\psi}_\alpha(x,\lambda) = \alpha\left\{\left(\beta_4 Cos(kb) + \frac{\beta_3}{k}Sin(kb)\right)Cos(kx) + \right.$$

$$\left. + \left(-\frac{\beta_3}{k}Cos(kb) + \beta_4 Sin(kb)\right)Sin(kx)\right\} \quad \textbf{(2.10)}$$

$$\underline{\psi}_\alpha'(x,\lambda) = \alpha\{(-\beta_4 k Cos(kb) - \beta_3 Sin(kb))Sin(kx) +$$

$$+(-\beta_3 Cos(kb) + \beta_4 k Sin(kb))Cos(kx)\} \quad \textbf{(2.11)}$$

substituing to in (2.6) and making the necessary operations yields

$$\underline{W}_\alpha(\lambda) = \alpha^2\left((\beta_1\beta_4 - k^2\beta_2\beta_3)Cos(k(a-b))\right) + \left(\frac{\beta_1\beta_3}{k} + k^3\beta_2\beta_4\right)Sin\left(k(b-a)\right)$$

and

$$\overline{W}_\alpha(\lambda) = (2-\alpha)^2\left((\beta_1\beta_4 - k^2\beta_2\beta_3)Cos(k(a-b))\right) + \left(\frac{\beta_1\beta_3}{k} + k^3\beta_2\beta_4\right)Sin\left(k(b-a)\right).$$

Then,

$$\underline{W}_\alpha(\lambda) = \overline{W}_\alpha(\lambda) = 0 \Longrightarrow \left((\beta_1\beta_4 - k^2\beta_2\beta_3)Cos(k(a-b))\right) +$$

$$+ \left(\frac{\beta_1\beta_3}{k} + k^3\beta_2\beta_4\right)Sin\left(k(b-a)\right) = 0.$$

Showing $k = k_n$ the above equation satisfying the values k and substituing to in (2.4), (2.5),

$$[\phi(x,\lambda)]^\alpha = [\alpha, 2-\alpha]\left({k_n}^2\beta_2 Cos(k_n a) + \frac{\beta_1}{k_n}Sin(k_n a)\right)Cos(k_n x) +$$

$$+\left(-\frac{\beta_1}{k_n}Cos(k_n a) + {k_n}^2\beta_2 Sin(k_n a)\right)Sin(k_n x)$$

$$[\psi(x,\lambda)]^\alpha = [\alpha, 2-\alpha]\left(\beta_4 Cos(k_n b) + \frac{\beta_3}{k_n}Sin(k_n b)\right)Cos(k_n x) +$$

$$+\left(-\frac{\beta_3}{k_n}Cos(k_n b)+\beta_4 Sin(k_n b)\right)Sin(k_n x)$$

is obtained. If $[\phi(x,\lambda)]^\alpha$ and $[\psi(x,\lambda)]^\alpha$ are valid α-level sets, they are eigenfunctions. Therefore, $[\phi(x,\lambda)]^\alpha$ and $[\psi(x,\lambda)]^\alpha$ must be valid α-level sets. Consequently, for k_n making $[\phi(x,\lambda)]^\alpha$ is a valid α-level set, $[\phi(x,\lambda)]^\alpha$ is eigenfunction with associated the eigenvalues $\lambda_n = k_n^2$. Similarly, for k_n making $[\psi(x,\lambda)]^\alpha$ is a valid α-level set, $[\psi(x,\lambda)]^\alpha$ is eigenfunction with associated the eigenvalues $\lambda_n = k_n^2$.

Example 2.1

Consider the fuzzy boundary value problem

$$y''+\lambda y=0,\ \ y(0)+\lambda y'(0)=0,\ \ y(1)=0\,. \tag{2.12}$$

Let $\lambda = k^2, k>0$ and

$$\phi(x,\lambda)=k^2 Cos(kx)-\frac{1}{k}Sin(kx),$$

$$\psi(x,\lambda)=SinkCos(kx)-CoskSin(kx)$$

be the solutions of the classical differential equation $y''+\lambda y=0$ satisfying the conditions $y(0)=k^2$, $y'(0)=1$ and $y(1)=1$, respectively. Then,

$$[\phi(x,\lambda)]^\alpha=\left[\underline{\phi}_\alpha(x,\lambda),\overline{\phi}_\alpha(x,\lambda)\right]$$

$$=[\alpha,2-\alpha]\left(k^2 Cos(kx)-\frac{1}{k}Sin(kx)\right) \tag{2.13}$$

and

$$[\psi(x,\lambda)]^\alpha=\left[\underline{\psi}_\alpha(x,\lambda),\overline{\psi}_\alpha(x,\lambda)\right]$$

$$=[\alpha,2-\alpha]\big(SinkCos(kx)-CoskSin(kx)\big) \tag{2.14}$$

are the solutions of the fuzzy differential equation $y''+\lambda y=0$ satisfying the first and the second boundary conditions, respectively. Because of the eigenvalues of the fuzzy boundary value problem (2.12) are zeros of the functions $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$, $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$ are obtained as

$$\underline{W}_\alpha(\lambda)=\alpha^2\left(-k^2 Cos(k)+\frac{1}{k}Sin(kx)\right),$$

$$\overline{W}_\alpha(\lambda)=(2-\alpha)^2\left(-k^2 Cos(k)+\frac{1}{k}Sin(kx)\right).$$

From this we get

$$-k^3 Cos(kx)+Sin(kx)=0.$$

Computing the values k satisfying the above equation , we have

$$k_1=4.70277, k_2=7.85192, k_3=10.9948, k_4=14.1368, k_5=17.2786, \ldots$$

Let show this values $k_n, n = 1,2, ...$ and substitute in (2.13), (2.14). Then, we obtain

$$[\phi_n(x)]^\alpha = \left[\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(k_n{}^2 Cos(k_n x) - \frac{1}{k_n} Sin(k_n x)\right)$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\big(Sink_n Cos(k_n x) - Cosk_n Sin(k_n x)\big)$$

When $\left(k_n{}^2 Cos(k_n x) - \frac{1}{k_n} Sin(k_n x)\right) > 0$ and $\big(Sink_n Cos(k_n x) - Cosk_n Sin(k_n x)\big) > 0$, $[\phi_n(x)]^\alpha$ and $[\psi_n(x)]^\alpha$ are valid α-level sets. Let be $k_n x \in [(n-1)\pi, n\pi]$, $n = 1,2, \ldots$.

For $[\phi_n(x)]^\alpha$,

i) If n is odd, $Sink_n x \geq 0$. From here,

$$k_n{}^3 Cot(k_n x) \geq 1 \Rightarrow Cot(k_n x) \geq \frac{1}{k_n{}^3}$$

$$\Rightarrow k_n x \leq cot^{-1}\left(\frac{1}{k_n{}^3}\right) \Rightarrow x \leq \frac{1}{k_n} cot^{-1}\left(\frac{1}{k_n{}^3}\right).$$

Since the above inequality must be for all $0 \leq x \leq 1$, it must be

$$\frac{1}{k_n} cot^{-1}\left(\frac{1}{k_n{}^3}\right) - 1 \geq 0.$$

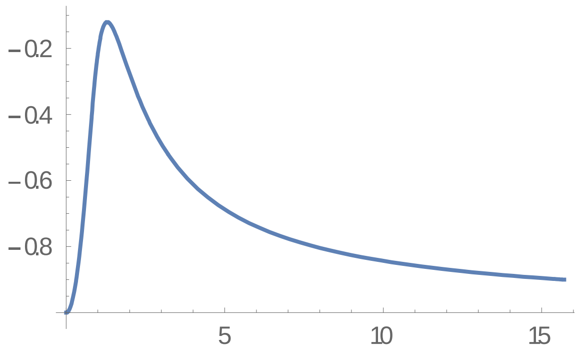

Figure 1. The graphic of the function $\frac{1}{k} cot^{-1}\left(\frac{1}{k^3}\right) - 1$

ii) If n is even, $Sink_n x \leq 0$. Then,

$$k_n{}^3 Cot(k_n x) \leq 1 \Rightarrow Cot(k_n x) \leq \frac{1}{k_n{}^3}$$

$$\Rightarrow k_n x \geq cot^{-1}\left(\frac{1}{k_n{}^3}\right) \Rightarrow x \geq \frac{1}{k_n} cot^{-1}\left(\frac{1}{k_n{}^3}\right).$$

Since the above inequality must be for all $0 \leq x \leq 1$, it must be

$$\frac{1}{k_n} cot^{-1}\left(\frac{1}{k_n{}^3}\right) \leq 0$$

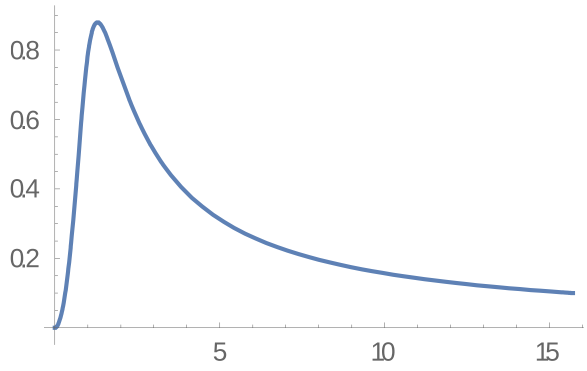

Figure 2. The graphic of the function $\frac{1}{k}cot^{-1}\left(\frac{1}{k^3}\right)$

According to Figure 1 and Figure 2, $[\phi_n(x)]^\alpha$ is not a valid α-level set.

For $[\psi_n(x)]^\alpha$,

i) If n is odd, $Sink_n x \geq 0$. Then,

$$Sink_n Cos(k_n x) - Cosk_n Sin(k_n x) \geq 0$$

$$Cot(k_n x) \geq Cotk_n \Rightarrow k_n x \leq k_n \Rightarrow x \leq 1.$$

ii) If n is even, $Sink_n x \leq 0$ and $Sink_n \leq 0$. Then,

$$Sink_n Cos(k_n x) - Cosk_n Sin(k_n x) \geq 0$$

$$Cot(k_n x) \geq Cotk_n \Rightarrow k_n x \leq k_n \Rightarrow x \leq 1.$$

Consequently; the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = [\alpha, 2-\alpha]\big(Sink_n Cos(k_n x) - Cosk_n Sin(k_n x)\big) .$$

Example 2.2

Consider the fuzzy problem

$$y'' + \lambda y = 0, \quad y(0) + \lambda y'(0) = 0, \quad y'(1) = 0 \tag{2.15}$$

Similar to example 2.1,

$$[\phi(x,\lambda)]^\alpha = \left[\underline{\phi}_\alpha(x,\lambda), \overline{\phi}_\alpha(x,\lambda)\right]$$

$$= [\alpha, 2-\alpha]\left(k^2 Cos(kx) - \frac{1}{k} Sin(kx)\right) \tag{2.16}$$

and

$$[\psi(x,\lambda)]^\alpha = \left[\underline{\psi}_\alpha(x,\lambda), \overline{\psi}_\alpha(x,\lambda)\right]$$

$$= [\alpha, 2-\alpha]\big(CoskCos(kx) + SinkSin(kx)\big) \tag{2.17}$$

are the solutions of the fuzzy differential equation $y'' + \lambda y = 0$ satisfying the first condition and the second condition, respectively. $\underline{W}_\alpha(\lambda)$ and $\overline{W}_\alpha(\lambda)$ are obtained as

$$\underline{W_\alpha}(\lambda) = \alpha^2\big(Cos(k) + k^3 Sin(kx)\big),$$

$$\overline{W}_\alpha(\lambda) = (2-\alpha)^2\big(Cos(k) + k^3 Sin(k)\big).$$

From this we get

$$Cos(kx) + k^3 Sin(kx) = 0.$$

Computing the values k satisfying the above equation , we have

$$k_1 = 0.505503, k_2 = 3.10831, k_3 = 6.27915, k_4 = 9.42358, k_5 = 12.5659, \ldots$$

Substituting this values in (2.16), (2.17), we obtain

$$[\phi_n(x)]^\alpha = \left[\underline{\phi_n}_\alpha(x), \overline{\phi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\left(k_n{}^2 Cos(k_n x) - \frac{1}{k_n} Sin(k_n x)\right),$$

$$[\psi_n(x)]^\alpha = \left[\underline{\psi_n}_\alpha(x), \overline{\psi_n}_\alpha(x)\right] = [\alpha, 2-\alpha]\big(Cosk_n Cos(k_n x) + Sink_n Sin(k_n x)\big).$$

When $\left(k_n{}^2 Cos(k_n x) - \frac{1}{k_n} Sin(k_n x)\right) > 0$ and $\big(Cosk_n Cos(k_n x) + Sink_n Sin(k_n x)\big) > 0$, $[\phi_n(x)]^\alpha$ and $[\psi_n(x)]^\alpha$ are valid α-level sets. From the example 2.1, $[\phi_n(x)]^\alpha$ is not a valid α-level set.

For $[\psi_n(x)]^\alpha$,

Let be $k_n x \in \left(\frac{2(n-1)-1}{2}\pi, \frac{2(n-1)+1}{2}\pi\right), n = 1,3,5, \ldots$ Then, $Cos(k_n x) \geq 0$ and

$$Cosk_n Cos(k_n x) + Sink_n Sin(k_n x) \geq 0 \Rightarrow tan(k_n) tan(k_n x) \geq -1.$$

i) If $k_n x \in \left(\frac{2(n-1)-1}{2}\pi, (n-1)\pi\right), n = 1,3,5, \ldots,\ tan(k_n x) \leq 0$.

$$tan(k_n x) \leq -\frac{1}{tan(k_n)} \Rightarrow k_n x \leq tan^{-1}\left(-\frac{1}{tan(k_n)}\right),$$

$$x \leq \frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right).$$

Since the above inequality must be for all $0 \leq x \leq 1$, it must be

$$\frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right) - 1 \geq 0.$$

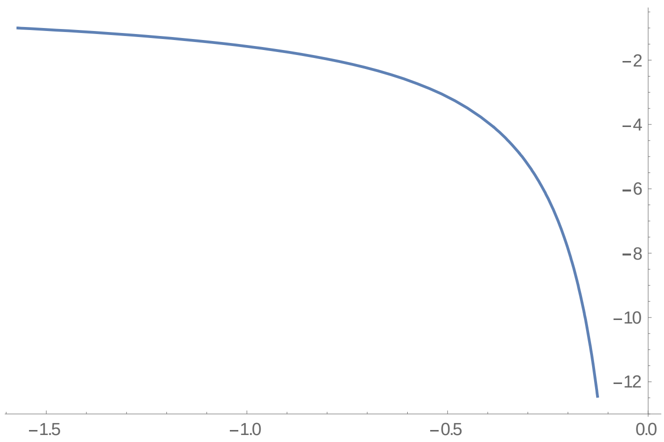

Figure 3. The graphic of the function $\frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right) - 1$

ii) If $k_n x \in \left((n-1)\pi, \frac{2(n-1)+1}{2}\pi\right), n = 1,3,5,\ldots,\ tan(k_n x) \geq 0$.

$$tan(k_n x) \geq -\frac{1}{tan(k_n)} \Longrightarrow k_n x \geq tan^{-1}\left(-\frac{1}{tan(k_n)}\right),$$

$$x \geq \frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right).$$

Since the above inequality must be for all $0 \leq x \leq 1$, it must be

$$\frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right) \leq 0.$$

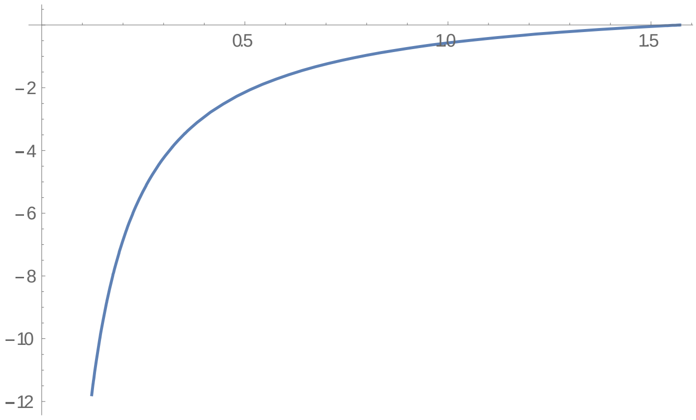

Figure 4. The graphic of the function $\frac{1}{k_n} tan^{-1}\left(-\frac{1}{tan(k_n)}\right)$.

Consequently;

$k_n x \in \left((n-1)\pi, \frac{(2n-1)\pi}{2}\right)$, n=1,3,5,…. the eigenvalues are $\lambda_n = k_n^2$, with associated eigenfunctions

$$[y_n(x)]^\alpha = [\alpha, 2-\alpha]\left(Sink_n Cos(k_n x) - Cosk_n Sin(k_n x)\right).$$

3. CONCLUSION

A fuzzy Sturm-Liouville problem with an eigenvalue parameter contained in the boundary condition is investigated in this paper. The examination is made using the Hukuhara differentiability. The effect on the eigenvalue and the eigenfunction of the problem of the eigenvalue in the boundary condition is shown. Two examples are solved. It is shown that function that provides boundary condition with eigenvalue parameter of the differential equation does not define eigenfunction. But examples can be multiplied. Also, different studies can be made. This is a new area and these results will be useful for other mathematicians.

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