Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.39-4(2018) 882-899

The Forward Kinematics of Rolling Contact of Spacelike Surfaces in Lorentzian 3-Space

Mehmet AYDINALP[1], Mustafa KAZAZ[1*], Hasan Hüseyin UĞURLU[2]

[1]*Manisa Celal Bayar University, Faculty of Arts and Sciences, Department of Mathematics, Manisa, TURKEY*

[2]*Gazi University, Faculty of Education, Department of Secondary Education Science and Mathematics Teaching, Mathematics Teaching Program, Ankara, TURKEY*

Received: 03.02.2018; Accepted: 15.10.2018 *http://dx.doi.org/10.17776/csj.389649*

Abstract. In this paper, we investigate the forward kinematics of spin-rolling motion without sliding of one spacelike surface on another spacelike surface along the spacelike contact trajectory curves of the surfaces in Lorentzian 3-space. A Darboux frame method is adopted to develop instantaneous kinematics of spin-rolling motion, which occurs in a nonholonomic system. Then new kinematic formulations of spin-rolling motion of spacelike moving surface with respect to contravariant vectors, rolling velocity, and geometric invariants are obtained. The translational velocity formulation of an arbitrary point and the equation of the angular velocity formulation on the spacelike moving surface are derived. The equation, which is represented with geometric invariants, can be easily applied to arbitrary spacelike parametric surface and spacelike contact trajectory curve and can be differentiated to any order. The influence of the relative curvatures and torsion on spin-rolling kinematics is clearly presented.

2010 AMS Subject Classification: 70B10, 53A17, 53A25, 53A35.

***Keywords*:** Lorentzian 3-Space, Darboux Frame, Forward Kinematics, Rolling Contact, Pure-rolling, Spin-rolling.

Lorentziyen 3-Uzayda Spacelike Yüzeylerin Kontak Yuvarlanmasinin İleri Kinematiği

Özet. Bu makalede, 3-boyutlu Lorentziyen uzayda bir spacelike yüzeyin bir diğer spacelike yüzey üzerinde yüzeylerin kontak spacelike yörünge eğrileri boyunca kaymaksızın dönme-yuvarlanma hareketinin ileri kinematiğini inceledik. Holonomik olmayan bir sistemde ortaya çıkan dönme-yuvarlanma hareketinin anlık kinematiğini kurmak için bir Darboux çatı metodu kullanılmıştır. Ardından, kontravaryant vektörler, yuvarlanma hızı ve geometrik değişmezlere göre hareketli spacelike yüzeyin dönme-yuvarlanma hareketinin yeni kinematik formülasyonları elde edilmiştir. Keyfi bir noktanın öteleme hızı formülasyonu ve hareketli spacelike yüzey üzerindeki açısal hız formülasyonunun denklemi elde edilmiştir. Geometrik değişmezlerle ifade edilen denklem, her mertebeden türevlenebilirdir ve keyfi spacelike parametrik yüzeye ve spacelike kontak yörünge eğrisine kolayca uygulanabilir. Dönme-yuvarlanma kinematiğinin bağıl eğrilikleri ve burulmasının etkisi açık olarak gösterilmektedir.

***Anahtar Kelimeler*:** Lorentziyen 3-Uzay, Darboux Çatısı, İleri Kinematik, Kontak Yuvarlanma, Has yuvarlanma, Dönme-yuvarlanma.

* Corresponding author. *Email address: mustafa.kazaz@cbu.edu.tr*
http://dergipark.gov.tr/csj ©2016 Faculty of Science, Sivas Cumhuriyet University

1. INTRODUCTION

Rolling contact is used in many areas of robotics and engineering such as spherical robots, single wheel robots, and multi-fingered robotic hands to drive from one configuration (position and orientation) to another. In mechanical systems, rolling contact without sliding engenders a non-integrable kinematic constraints on the system's velocity which are called non-holonomic constraints. This non-holonomy calls for the two contact loci have equal arc lengths in a given time interval [1].

There are two categories of kinematics of the rolling contact. The first is pure-rolling motion and the second is spin-rolling motion [2]. On the other hand, in the rolling contact, there are two geometric constraints. The first is that the unit normal vectors of the two surfaces are made to coincide at the contact point. The second is that the contact points have the same velocity. To put it another way, the two contact trajectory curves are tangent to each other and have the same rolling rate. Thus, a moving surface has spin-rolling motion or pure-rolling motion under these two geometric constraints. Further, there is another constraint for a surface to have pure-rolling motion. This constraint is explicitly demonstrated as that the two contact trajectories have the same geodesic curvature, that is, the angular velocity ω_3 in the direction of the unit normal vector $\boldsymbol{n}$ to the surface is zero. Thus, the contact trajectories are not arbitrary [3]. Pure-rolling motion has 2 degrees of freedom (DOFs). It has instantaneous rotation axis passes through the contact point in all cases and this axis is parallel to the common tangent plane of two surfaces. Spin-rolling motion, which is also called twist-rolling motion, has 3 degrees of freedom (DOFs) consisting of three angular velocity components: ω_1, ω_2 about the axes $\boldsymbol{T}$ and $\boldsymbol{g}$ on the tangent plane, respectively, and ω_3 about the common normal axis $\boldsymbol{n}$ at the contact point. Its instantaneous rotation axis can be in any arbitrary direction which is the characteristic difference from pure-rolling motion [2].

The contact kinematics is given in two classifications. The first is forward kinematics and the second is inverse kinematics. The forward kinematics includes the problem of using kinematic equations as the inputs of the geometry of the two surfaces and the contact locus on each surface to compute the motion of the moving surface as the output. The inverse kinematics includes the problem of determining the control parameters that give the desired motion the moving surface as the inputs of the geometry of two surfaces and the desired angular velocity of the moving surface. These inputs are the angular velocity components ω_1, ω_2 and ω_3 [1, 3].

Many researchers have extensively studied kinematics of a point contact between rigid bodies. Neimark and Fufaev [4] were the first to adopt the moving frame along the lines of curvature to derive the velocity equation of spin-rolling motion. Cai and Roth [5, 6] investigated instantaneous time-based kinematics of rigid objects in point contact, both in planar and spatial cases, and focused on two special motions, including sliding and pure-rolling motion, and they aimed to measure the relative motion at the point of contact. Montana [7] studied the kinematics of sliding-spin-rolling motion and derived a differential-geometric model of the rolling constraint between general bodies. Li and Canny [8] used Montana's contact equations to investigate the existence of an admissible path between two configurations in the case of pure rolling, and if it does, then how to find it. Sarkar et al. [9] extended Montana's definition but with a different approach by obtaining the acceleration equations and they demonstrated the obvious dependence on Christoffel symbols and they simplified the derivative of the metric tensor. Marigo and Bicchi [10] obtained similar equations with Montana's contact equations using a different approach that allowed an analysis of admissibility of a pure-rolling contact. Agrachev and Sachkov [11] solved the controllability problem of a pair of pure rolling rigid bodies and they showed the condition of the rolling without sliding nor spinning of a two-dimensional Riemannian manifold onto another. Chelouah and Chitour [12] gave two procedures to analyze the motion-planning problem when one manifold was a plane and the other was a convex surface. Chitour et al. [13] investigated the pure-rolling of a pair of smooth convex objects, with one being over another under quantized control. Tchon [14] identified the property of repeatability of inverse-kinematics algorithms for mobile manipulators and formulated a necessary and sufficient condition under repeatability. Tchon and Jakubiak [15] designed an extended Jacobian repeatable inverse kinematics algorithm for doubly nonholonomic mobile manipulators based on the concept of endogenous configuration space. Cui and Dai [2] investigated the forward kinematics of spin-rolling motion without sliding by applying the moving-frame method and then Cui [3] studied the kinematics of sliding-rolling motion of two contact surfaces. Cui and Dai [1] also investigated the inverse kinematics of rolling contact by using polynomial formulation when the desired angular velocity and the coordinates of the contact point on each surface were given in Euclidean 3-space. Then they obtained admissible rolling motion between two contact surfaces. For the fundamental concepts of kinematics, see [16, 17, 18, 19].

This paper is organized as follows:

In Section 2, we give basic concepts in Lorentzian 3-space.

In Section 3, we study the forward kinematics of spin-rolling without sliding of one spacelike surface on another spacelike surface by applying the moving-frame method. Initially, we give the Darboux-frame-based translational velocity formulation of an arbitrary point in Lorentzian 3-space. Then we obtain a new equation of angular velocity with respect to the rolling speed and two sets of geometric invariants containing the geodesic curvature, the normal curvature, and the geodesic torsion, namely $\{k_g, k_n, \tau_g\}$, $\{\bar{k}_g, \bar{k}_n, \bar{\tau}_g\}$. We determine the instantaneous kinematics of a spacelike moving surface by applying the translational velocity formulation and the angular velocity equation. Then we give two examples that present spin-rolling motion and pure-rolling motion of two spacelike surfaces, respectively.

In Section 4, we give a conclusion. Finally in Appendix, we give a code of an animation that shows the moving Darboux frame along a hyperbolic unit circle lying on a hyperbolic unit sphere. Then we present four codes that give the calculations of invariants of the examples in Section 3.3.

2. PRELIMINARIES

In this section, we give a brief summary of basic concepts for the reader who is not familiar with Lorentzian 3-space [20, 21, 22, 23].

Lorentzian space IR_1^3 is the real vector space IR_1^3 endowed with the Lorentzian inner product given by

$$\langle \boldsymbol{a}, \boldsymbol{b} \rangle = a_1 b_1 + a_2 b_2 - a_3 b_3,$$

where $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3) \in IR_1^3$.

According to this metric, an arbitrary vector $\boldsymbol{a} = (a_1, a_2, a_3)$ in IR_1^3 can have one of three Lorentzian causal characters: if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle > 0$ or $\boldsymbol{a} = 0$ then $\boldsymbol{a}$ is called a spacelike vector; if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle < 0$ then $\boldsymbol{a}$ is called a timelike vector; if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0$ and $\boldsymbol{a} \neq 0$ then $\boldsymbol{a}$ is called a null (lightlike) vector [21]. We note that a timelike vector is future pointing or past pointing if the first compound of vector is positive or negative, respectively.

The norm of a vector $\boldsymbol{a} \in IR_1^3$ is given by $\|\boldsymbol{a}\| = \sqrt{|\langle \boldsymbol{a}, \boldsymbol{a} \rangle|}$. If the vector $\boldsymbol{a}$ is a spacelike vector, then $\|\boldsymbol{a}\|^2 = \langle \boldsymbol{a}, \boldsymbol{a} \rangle$; if $\boldsymbol{a}$ is a timelike vector, then $\|\boldsymbol{a}\|^2 = -\langle \boldsymbol{a}, \boldsymbol{a} \rangle$ [23].

Let $\boldsymbol{a}=(a_1,a_2,a_3)$ and $\boldsymbol{b}=(b_1,b_2,b_3)$ be two vectors in IR_1^3. Then Lorentzian vector product of $\boldsymbol{a}$ and $\boldsymbol{b}$ can be defined by

$$\boldsymbol{a}\times\boldsymbol{b}=(a_3b_2-a_2b_3,a_1b_3-a_3b_1,a_1b_2-a_2b_1)\text{ [23].}$$

Definition 2.1 — [20, 22]

(*i*) Hyperbolic angle: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be future pointing (or past pointing) timelike vectors in IR_1^3. Then there is a unique real number $\theta\geq 0$ such that $\langle\boldsymbol{a},\boldsymbol{b}\rangle=-\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cosh\theta$, and this number is called the hyperbolic angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(*ii*) Central angle: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be spacelike vectors in IR_1^3 and they span a timelike vector subspace. Then there is a unique real number $\theta\geq 0$ such that $\langle\boldsymbol{a},\boldsymbol{b}\rangle=\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cosh\theta$, and this number is called the central angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(*iii*) Spacelike angle: Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be spacelike vectors in IR_1^3 and they span a spacelike vector subspace. Then there is a unique real number $\theta\geq 0$ such that $|\langle\boldsymbol{a},\boldsymbol{b}\rangle|=\|\boldsymbol{a}\|\|\boldsymbol{b}\|\cos\theta$, and this number is called the spacelike angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

(*iv*) Lorentzian timelike angle: Let $\boldsymbol{a}$ be a spacelike vector and $\boldsymbol{b}$ be a timelike vectors in IR_1^3. Then there is a unique real number $\theta\geq 0$ such that $|\langle\boldsymbol{a},\boldsymbol{b}\rangle|=\|\boldsymbol{a}\|\|\boldsymbol{b}\|\sinh\theta$, and this number is called the Lorentzian timelike angle between the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

An arbitrary curve $\alpha=\alpha(s)$ in IR_1^3 can be locally spacelike, timelike, or null (lightlike), if all of its velocity vectors $d\boldsymbol{\alpha}/ds$ are spacelike, timelike, or null (lightlike), respectively. A surface in Lorentzian space IR_1^3 is called a spacelike (timelike) surface if the normal vector of the surface is a timelike (spacelike) vector [21].

The Lorentzian and hyperbolic unit spheres are given by

$$S_1^2=\{a=(a_1,a_2,a_3)\in IR_1^3:\langle a,a\rangle=1\}\text{ and }H_0^2=\{\boldsymbol{a}=(a_1,a_2,a_3)\in IR_1^3:\langle\boldsymbol{a},\boldsymbol{a}\rangle=-1\},$$

respectively. It is easy to show that the hyperbolic unit sphere is a spacelike surface and Lorentzian unit sphere is a timelike surface.

Let S be a spacelike surface and $\alpha=\alpha(s)$ be any curve lying on the surface S. Then the curve α has to be a spacelike curve. Darboux frame $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ of α is a solid perpendicular trihedron in IR_1^3 associated with each point $M\in\alpha$, where $\boldsymbol{T}$ is the unit tangent spacelike vector to the curve α, $\boldsymbol{n}$ is the unit timelike normal vector to the spacelike surface S and $\boldsymbol{g}=-\boldsymbol{n}\times\boldsymbol{T}$ (that is, $\boldsymbol{g}$ is tangential to S which is also a spacelike vector) at the point M. We should note that

$$\boldsymbol{T}\times\boldsymbol{g}=\boldsymbol{n},\quad \boldsymbol{g}\times\boldsymbol{n}=-\boldsymbol{T},\quad \boldsymbol{n}\times\boldsymbol{T}=-\boldsymbol{g}\text{ and }\langle\boldsymbol{T},\boldsymbol{T}\rangle=1,\ \langle\boldsymbol{g},\boldsymbol{g}\rangle=1,\ \langle\boldsymbol{n},\boldsymbol{n}\rangle=-1.$$

The animation of the Darboux frame of a hyperbolic unit circle on a hyperbolic unit sphere given with Wolfram Mathematica 9 (See Appendix A).

Then the derivative formulae (the equations of motion) of the Darboux frame (trihedron) is given by

$$\frac{d\boldsymbol{m}}{ds} = \boldsymbol{T}, \quad \frac{d}{ds}\begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{g} \\ \boldsymbol{n} \end{bmatrix} = \begin{bmatrix} 0 & k_g & k_n \\ -k_g & 0 & \tau_g \\ k_n & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{g} \\ \boldsymbol{n} \end{bmatrix},$$

where $\boldsymbol{m}$ is the position vector of the point M that depends on the choice of the coordinate system. Furthermore, the position vector corresponding to an arbitrary trajectory curve on a surface in IR_1^3 may have three causal characters. Hence, we can express that $\boldsymbol{m}$ is either spacelike, timelike or lightlike position vector. The components of the vector $\boldsymbol{m}$ are obtained from the measurement along the axes of the coordinate system. In these formulae, k_g, k_n and τ_g are called the geodesic curvature, the normal curvature and the geodesic torsion, respectively. It is easy to see that the geodesic curvature k_g, the normal curvature k_n and the geodesic torsion τ_g of the spacelike curve α can be given by

$$k_g = \left\langle \frac{d\boldsymbol{T}}{ds}, \boldsymbol{g} \right\rangle, \; k_n = -\left\langle \frac{d\boldsymbol{T}}{ds}, \boldsymbol{n} \right\rangle, \; \tau_g = -\left\langle \frac{d\boldsymbol{g}}{ds}, \boldsymbol{n} \right\rangle.$$

The Darboux instantaneous rotation vector of the Darboux trihedron is defined by

$$\boldsymbol{\omega} = \tau_g \boldsymbol{T} - k_n \boldsymbol{g} - k_g \boldsymbol{n} \text{ [24].}$$

Then, for a spacelike curve $\alpha(s)$ lying on a spacelike surface S, we have the following characterizations [24]: $\alpha(s)$ is

(i) geodesic $\Leftrightarrow k_g = 0$,

(ii) asymptotic $\Leftrightarrow k_n = 0$,

(iii) principal $\Leftrightarrow \tau_g = 0$.

3. THE FORWARD KINEMATICS OF ROLLING CONTACT OF SPACELIKE SURFACES

In this section, we study the forward kinematics of rolling contact of spacelike surfaces by applying the Darboux frame method in Lorentzian 3-space. The main contribution of this section is that a new equation of the angular velocity of the spin-rolling motion of a spacelike moving surface is formed. The new formulation is specified with respect to three contravariant vectors and geometric invariants, which are arc lengths of the spacelike contact trajectory curves and the induced curvatures of the two spacelike surfaces.

3.1. The Kinematics of Spin-Rolling Motion

Firstly, we give the geometric kinematics of spin-rolling motion of two contact spacelike surfaces. We note that during the rolling motion, both of the two spacelike surfaces have the same unit timelike normal vectors at the contact point. When a spacelike fixed surface and a spacelike moving surface relative to fixed surface undergo spin-rolling motion without sliding as in Fig. 1., the fixed surface maintains its spacelike surface character at every moment, but the moving surface may not maintain its spacelike surface character entirely.

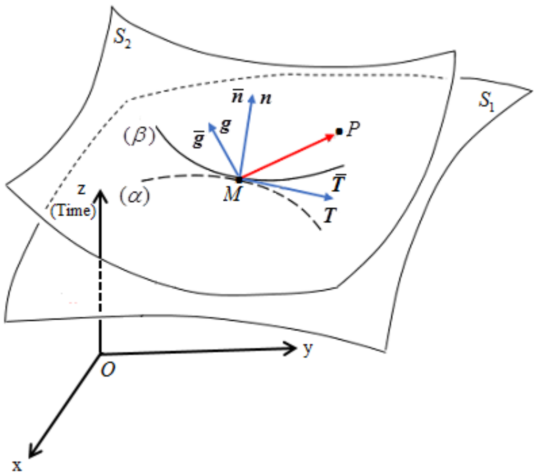

Figure 1. Spacelike surface S_2 spin-rolling on spacelike surface S_1 along spacelike curves β and α.

Now, let α and β be spacelike contact-trajectory curves on spacelike surfaces S_1 and S_2, respectively. Let us denote the Darboux frames (the right-handed orthonormal frames) attached to the contact point M of spacelike curves α and β as $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$, respectively. The vectors $\boldsymbol{T}$, $\boldsymbol{g}$, $\boldsymbol{n}$ and $\bar{\boldsymbol{T}}$, $\bar{\boldsymbol{g}}$, $\bar{\boldsymbol{n}}$ are the contravariant vectors of the spacelike fixed and spacelike moving surfaces, respectively and there is not any intrinsic coordinate systems for these contravariant vectors. By rolling constraints, the contravariant vectors $\boldsymbol{T}$ and $\bar{\boldsymbol{T}}$ are always collinear and, consequently, are $\boldsymbol{n}$ and $\bar{\boldsymbol{n}}$. Therefore, the two frames can always be made to coincide, as shown in Fig. 1, where $\boldsymbol{n}$ points outward of the surface S_1, and $\bar{\boldsymbol{n}}$ points inward of the surface S_2. Let s and $\bar{s}$ be the arc lengths of spacelike curve α and spacelike curve β, respectively. Then the derivative formulas of the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ are

$$\frac{d\boldsymbol{m}}{ds}=\boldsymbol{T},\quad \frac{d}{ds}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{g}\\ \boldsymbol{n}\end{bmatrix}=\begin{bmatrix}0 & k_g & k_n\\ -k_g & 0 & \tau_g\\ k_n & \tau_g & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{g}\\ \boldsymbol{n}\end{bmatrix}$$

and

$$\frac{d\bar{\boldsymbol{m}}}{d\bar{s}}=\bar{\boldsymbol{T}},\quad \frac{d}{d\bar{s}}\begin{bmatrix}\bar{\boldsymbol{T}}\\ \bar{\boldsymbol{g}}\\ \bar{\boldsymbol{n}}\end{bmatrix}=\begin{bmatrix}0 & \bar{k}_g & \bar{k}_n\\ -\bar{k}_g & 0 & \bar{\tau}_g\\ \bar{k}_n & \bar{\tau}_g & 0\end{bmatrix}\begin{bmatrix}\bar{\boldsymbol{T}}\\ \bar{\boldsymbol{g}}\\ \bar{\boldsymbol{n}}\end{bmatrix},$$

where $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ are the position vectors of the point M with respect to the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$, respectively. Both $\boldsymbol{m}$ and $\bar{\boldsymbol{m}}$ have three causal characters. Now, let P denote an arbitrary point on S_2. Then we can write the (spacelike, timelike or lightlike) position vector, denoted by $\bar{\boldsymbol{p}}$, of the point P in the frame $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ as $\bar{\boldsymbol{p}}=\bar{\boldsymbol{m}}+\bar{\lambda}_1\bar{\boldsymbol{T}}+\bar{\lambda}_2\bar{\boldsymbol{g}}+\bar{\lambda}_3\bar{\boldsymbol{n}}$.

Differentiating $\bar{\boldsymbol{p}}$ with respect to $\bar{s}$ gives

$$\frac{d\bar{\boldsymbol{p}}}{d\bar{s}} = (1 + \frac{d\bar{\lambda}_1}{d\bar{s}} - \bar{\lambda}_2\bar{k}_g + \bar{\lambda}_3\bar{k}_n)\bar{\boldsymbol{T}} + (\frac{d\bar{\lambda}_2}{d\bar{s}} + \bar{\lambda}_1\bar{k}_g + \bar{\lambda}_3\bar{\tau}_g)\bar{\boldsymbol{g}} + (\frac{d\bar{\lambda}_3}{d\bar{s}} + \bar{\lambda}_1\bar{k}_n + \bar{\lambda}_2\bar{\tau}_g)\bar{\boldsymbol{n}}, \tag{1}$$

where $\bar{k}_g$, $\bar{k}_n$ and $\bar{\tau}_g$ are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of β, respectively. Since P is a fixed point of S_2, then $\frac{d\bar{\boldsymbol{p}}}{d\bar{s}} = 0$.

Note that $\frac{d\bar{\boldsymbol{p}}}{d\bar{s}}$ is a spacelike vector. Putting this into (1) gives

$$\frac{d\bar{\lambda}_1}{d\bar{s}} = \bar{\lambda}_2\bar{k}_g - \bar{\lambda}_3\bar{k}_n - 1, \quad \frac{d\bar{\lambda}_2}{d\bar{s}} = -\bar{\lambda}_1\bar{k}_g - \bar{\lambda}_3\bar{\tau}_g, \quad \frac{d\bar{\lambda}_3}{d\bar{s}} = -\bar{\lambda}_1\bar{k}_n - \bar{\lambda}_2\bar{\tau}_g.$$

We can also write the (spacelike, timelike or lightlike) position vector, denoted by $\boldsymbol{p}$, of the point P in the frame $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ as $\boldsymbol{p} = \boldsymbol{m} + \lambda_1\boldsymbol{T} + \lambda_2\boldsymbol{g} + \lambda_3\boldsymbol{n}$.

Differentiating $\boldsymbol{p}$ with respect to s gives

$$\frac{d\boldsymbol{p}}{ds} = (1 + \frac{d\lambda_1}{ds} - \lambda_2 k_g + \lambda_3 k_n)\boldsymbol{T} + (\frac{d\lambda_2}{ds} + \lambda_1 k_g + \lambda_3\tau_g)\boldsymbol{g} + (\frac{d\lambda_3}{ds} + \lambda_1 k_n + \lambda_2\tau_g)\boldsymbol{n}, \tag{2}$$

where k_g, k_n and τ_g are the geodesic curvature, the normal curvature, and the geodesic torsion at point M of α, respectively. The vector $\boldsymbol{p}$ has three causal characters and, therefore, $\frac{d\boldsymbol{p}}{ds}$ has three causal characters. By the constraints for rolling contact, two spacelike contact trajectory curves have the same arc lengths at the contact point. Since the Darboux frames $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$ are made to coincide at any moment, it follows that $\lambda_1 = \bar{\lambda}_1$, $\lambda_2 = \bar{\lambda}_2$, $\lambda_3 = \bar{\lambda}_3$, and consequently

$$\frac{d\lambda_1}{ds} = \frac{d\bar{\lambda}_1}{d\bar{s}}, \quad \frac{d\lambda_2}{ds} = \frac{d\bar{\lambda}_2}{d\bar{s}}, \quad \frac{d\lambda_3}{ds} = \frac{d\bar{\lambda}_3}{d\bar{s}}. \tag{3}$$

Substituting (1) and (3) into (2) gives

$$\frac{d\boldsymbol{p}}{ds} = (\lambda_2 k_g^* - \lambda_3 k_n^*)\boldsymbol{T} + (-\lambda_1 k_g^* - \lambda_3\tau_g^*)\boldsymbol{g} + (-\lambda_1 k_n^* - \lambda_2\tau_g^*)\boldsymbol{n}, \tag{4}$$

where $k_g^* = \bar{k}_g - k_g$, $k_n^* = \bar{k}_n - k_n$, $\tau_g^* = \bar{\tau}_g - \tau_g$. The scalars k_g^*, k_n^* and τ_g^* are called induced geodesic curvature, induced normal curvature, and induced geodesic torsion, respectively. For more information on the Darboux trihedron and the induced curvatures in Euclidean 3-space, see [3, 24].

3.2. Darboux-Frame-Based Velocity Formulation of Spin-Rolling Motion

The velocity of an arbitrary point P on the spacelike moving surface S_2 in terms of time t can be obtained from (4) as follows:

$$\boldsymbol{v}_P = \frac{d\boldsymbol{p}}{ds}\frac{ds}{dt} = \sigma(\lambda_2 k_g^* - \lambda_3 k_n^*)\boldsymbol{T} + \sigma(-\lambda_1 k_g^* - \lambda_3\tau_g^*)\boldsymbol{g} + \sigma(-\lambda_1 k_n^* - \lambda_2\tau_g^*)\boldsymbol{n}, \tag{5}$$

where $\sigma = ds/dt$ is the magnitude of rolling velocity. Note that $\boldsymbol{v}_P$ has three causal characters.

This equation gives the Darboux-frame-based translational velocity formulation of an arbitrary point. Let the angular velocity of S_2 relative to spacelike fixed surface S_1 be

$$\boldsymbol{\omega} = \omega_x \boldsymbol{T} + \omega_y \boldsymbol{g} + \omega_z \boldsymbol{n}. \tag{6}$$

If $\boldsymbol{r}_{MP} = \lambda_1 \boldsymbol{T} + \lambda_2 \boldsymbol{g} + \lambda_3 \boldsymbol{n}$ is also given, then the velocity of the point P can be obtained as

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{r}_{MP} = (\lambda_2\omega_z - \lambda_3\omega_y)\boldsymbol{T} + (-\lambda_1\omega_z + \lambda_3\omega_x)\boldsymbol{g} + (-\lambda_1\omega_y + \lambda_2\omega_x)\boldsymbol{n}. \tag{7}$$

We note that $\boldsymbol{r}_{MP}$ is a spacelike secant vector, since $\boldsymbol{r}_{MP}$ is parallel to a spacelike tangent vector at a certain point on the spacelike arc MP which is formed by intersection of the spacelike moving surface and a plane passing from the points M and P. When the Eq. (5) is compared with the Eq. (7), we obtain that

$$\omega_x = -\sigma\tau_g^*, \quad \omega_y = \sigma k_n^*, \quad \omega_z = \sigma k_g^*. \tag{8}$$

From (6) and (8), the angular velocity of S_2 can be obtained as

$$\boldsymbol{\omega} = \sigma(-\tau_g^* \boldsymbol{T} + k_n^* \boldsymbol{g} + k_g^* \boldsymbol{n}). \tag{9}$$

The equation (9) contains three terms. The first two terms give the pure-rolling velocity about an axis in the tangent plane at the contact point and the third term gives the velocity of spin motion about the unit timelike normal direction at the contact point in Lorentzian 3-space. Therefore, the pure-rolling velocity can be given by $-\sigma\tau_g^* \boldsymbol{T} + \sigma k_n^* \boldsymbol{g}$ and the velocity of spin motion can be given by $\sigma k_g^* \boldsymbol{n}$. As a result, the spacelike moving surface can follow the desired trajectory spacelike curve on the fixed spacelike surface by the help of these three terms. We note that a pure-rolling motion does not have spin-rolling motion in the direction of the unit timelike normal of the spacelike surfaces. Then we give the following results:

i) Let two spacelike surfaces undergo pure-rolling motion in Lorentzian 3-space. Then the geodesic curvatures of the two corresponding contact-trajectory spacelike curves have to be equal, that is, $k_g = \overline{k}_g$.

ii) Let contact-trajectory spacelike curves α and β be geodesics on spacelike surfaces S_1 and S_2, respectively. Then the rolling motion consists of a pure-rolling motion in Lorentzian 3-space.

3.3. Examples

In this section, two examples are presented. The first example demonstrates the spin-rolling motion of a hyperbolic unit cylinder on a spacelike plane. The second example demonstrates the pure-rolling motion of a hyperbolic unit sphere on a hyperbolic unit cylinder.

3.3.1. Spin-Rolling Motion of a Hyperbolic Unit Cylinder on a Spacelike Plane

Suppose a moving hyperbolic unit cylinder (surface S_2) is rolling without sliding on a fixed spacelike plane (surface S_1) at a contact point M along spacelike curves α and β as in Fig. 2. Assume that spacelike curves α and β are parameterized by arc lengths s and $\overline{s}$, respectively. Denote the Darboux frames (the right-handed orthonormal frames) attached to the contact point M of the curves α and β as $(\boldsymbol{T}, \boldsymbol{g}, \boldsymbol{n})$ and $(\overline{\boldsymbol{T}}, \overline{\boldsymbol{g}}, \overline{\boldsymbol{n}})$, respectively. Suppose the parametric equation of spacelike plane is given by

$$x(u,v) = (u, v, 1),$$

and let α be a spacelike curve lying on the spacelike plane parameterized as

$$\alpha(t)=x\big(u(t),v(t)\big)=(\cosh t-1,\sinh t,1).$$

The derivative of s with respect to t is

$$\frac{ds}{dt}=\sqrt{\left\langle \frac{d\boldsymbol{\alpha}}{dt},\frac{d\boldsymbol{\alpha}}{dt}\right\rangle}=\sqrt{\cosh 2t}\,.$$

Using Wolfram Mathematica 9, we obtain the invariants as follows (See Appendix B.1. for the complete code for these invariants):

The unit spacelike tangent vector $\boldsymbol{T}$ is obtained as

$$\boldsymbol{T}=\frac{d\boldsymbol{\alpha}}{ds}=\frac{d\boldsymbol{\alpha}}{dt}\Big/\frac{ds}{dt}=\frac{1}{\sqrt{\cosh 2t}}(\sinh t,\cosh t,0). \tag{10}$$

We assume that the unit timelike normal vector $\boldsymbol{n}$ is pointing outwards and it is obtained as

$$\boldsymbol{n}=\frac{\boldsymbol{x}_u\times\boldsymbol{x}_v}{\|\boldsymbol{x}_u\times\boldsymbol{x}_v\|}=(0,0,1). \tag{11}$$

The unit vector $\boldsymbol{g}$, which is tangential to S_1, is obtained as

$$\boldsymbol{g}=-\boldsymbol{n}\times\boldsymbol{T}=\frac{1}{\sqrt{\cosh 2t}}(-\cosh t,\sinh t,0). \tag{12}$$

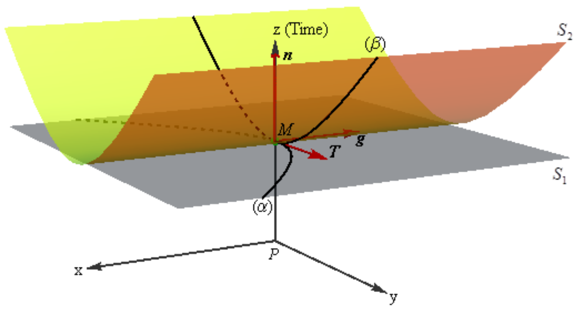

Figure 2. A hyperbolic unit cylinder spin-rolling on a spacelike plane along spacelike curves β and α.

When the algebraic operation is applied, the geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve α are obtained as

$$\left.\begin{aligned}
k_g&=\langle d\boldsymbol{T}/dt,\boldsymbol{g}\rangle\Big/\frac{ds}{dt}=\frac{-1}{(\cosh 2t)^{3/2}},\\
k_n&=-\langle d\boldsymbol{T}/dt,\boldsymbol{n}\rangle\Big/\frac{ds}{dt}=0,\\
\tau_g&=-\langle d\boldsymbol{g}/dt,\boldsymbol{n}\rangle\Big/\frac{ds}{dt}=0,
\end{aligned}\right\} \tag{13}$$

respectively (See Appendix B.1). It is obvious the curve α is both principal and asymptotic.

Now, if we parameterize the hyperbolic unit cylinder as $y(\bar{u},\bar{v})=(\bar{v},\sinh\bar{u},\cosh\bar{u})$, then it will be a spacelike surface, (See Fig. 2). Let β be a spacelike $\bar{u}$-parametric curve (namely, a hyperbolic unit circle) lying on the hyperbolic unit cylinder parameterized as $\beta(\bar{u})=y(\bar{u},0)=(0,\sinh\bar{u},\cosh\bar{u})$, where

$\bar{v} = v_0 = 0$. Since the differentiation of arc length $\bar{s}$ of the curve β with respect to $\bar{u}$ is $\frac{d\bar{s}}{d\bar{u}} = \left\| \frac{d\boldsymbol{\beta}}{d\bar{u}} \right\| = 1$, it is clear that β is a unit-speed curve. Using Wolfram Mathematica 9, we obtain the invariants as follows (See Appendix B.2. for the complete code for these invariants):

The unit spacelike tangent vector $\bar{\boldsymbol{T}}$ is given by

$$\bar{\boldsymbol{T}} = (0, \cosh \bar{u}, \sinh \bar{u}). \tag{14}$$

Let the unit timelike normal vector $\bar{\boldsymbol{n}}$ be inward and it is obtained as

$$\bar{\boldsymbol{n}} = -\frac{\boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}}}{\left\| \boldsymbol{y}_{\bar{u}} \times \boldsymbol{y}_{\bar{v}} \right\|} = (0, \sinh \bar{u}, \cosh \bar{u}). \tag{15}$$

The unit vector $\bar{\boldsymbol{g}}$, which is tangential to S_2, is obtained as

$$\bar{\boldsymbol{g}} = -\bar{\boldsymbol{n}} \times \bar{\boldsymbol{T}} = (-1, 0, 0). \tag{16}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve β are obtained as

$$\left.\begin{aligned} \bar{k}_g &= \left\langle d\bar{\boldsymbol{T}}/d\bar{u}, \bar{\boldsymbol{g}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = 0, \\ \bar{k}_n &= -\left\langle d\bar{\boldsymbol{T}}/d\bar{u}, \bar{\boldsymbol{n}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = 1, \\ \bar{\tau}_g &= -\left\langle d\bar{\boldsymbol{g}}/d\bar{u}, \bar{\boldsymbol{n}} \right\rangle \Big/ \frac{d\bar{s}}{d\bar{u}} = 0, \end{aligned}\right\} \tag{17}$$

respectively (See Appendix B.2). It is clear that the curve β is both principal and geodesic. From (9), the angular velocity of the hyperbolic unit cylinder is obtained as

$$\boldsymbol{\omega} = \sigma \left(\boldsymbol{g} + \frac{1}{(\cosh 2t)^{3/2}} \boldsymbol{n} \right). \tag{18}$$

The coordinate of the center point P of the hyperbolic unit cylinder in the frame $(\boldsymbol{T}, \boldsymbol{g}, \boldsymbol{n})$ at point M is origin. From Darboux-frame-based translation formulation (5) and (7), the velocity of point P is

$$\boldsymbol{v}_P = \boldsymbol{\omega} \times \boldsymbol{r}_{MP} = \sigma \left(\boldsymbol{g} + \frac{1}{(\cosh 2t)^{3/2}} \boldsymbol{n} \right) \times (-\boldsymbol{n}) = \sigma \boldsymbol{T}.$$

The angular velocity formulation given by the equation (18) is used to control the moving hyperbolic unit circle following the desired trajectory spacelike curve α lying on the fixed spacelike plane. The moving hyperbolic unit cylinder has 2 DOFs. At any instant, the first term σ of (18) gives the angular velocity about the axis that is perpendicular to the hyperbolic unit cylinder. The second term $\sigma / (\cosh 2t)^{3/2}$ gives the information about how fast the hyperbolic unit cylinder spins to follow the curve α and, in this way, yields the new tangent direction of the trajectory curve α. This information is used as the inputs of the control system to make hyperbolic unit cylinder follow the trajectory curve α.

3.3.2. Pure-Rolling Motion of the Hyperbolic Unit Sphere on a Hyperbolic Unit Cylinder

Suppose a hyperbolic unit sphere (surface S_2) H_0^2 is rolling without sliding on a hyperbolic unit cylinder (surface S_1) at a contact point M along spacelike curves α and β as in Fig. 3.
Assume that spacelike curves α and β are parameterized by arc lengths s and $\bar{s}$, respectively. Let denote the Darboux frames (the right-handed orthonormal frames) attached to the contact point M of the curves α and β as $(\boldsymbol{T},\boldsymbol{g},\boldsymbol{n})$ and $(\bar{\boldsymbol{T}},\bar{\boldsymbol{g}},\bar{\boldsymbol{n}})$, respectively.

Suppose the parametric equation of the hyperbolic unit cylinder is given by

$$x(u,v)=(-\sinh u, v, 2-\cosh u),$$

and let α be a spacelike helix curve lying on the hyperbolic unit cylinder parameterized as

$\alpha(t)=x\big(u(t),v(t)\big)=x\big(t,2t\big)=(-\sinh t,2t,2-\cosh t)$.

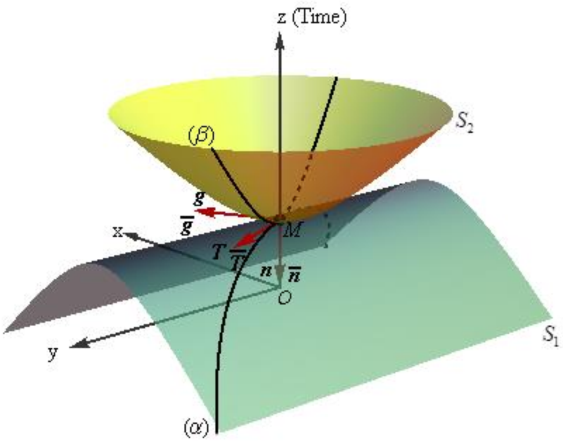

Figure 3. A hyperbolic unit sphere pure-rolling on a hyperbolic unit cylinder along spacelike curves β and α.

The differentiation of arc-length s of the curve α with respect to t is obtained as

$$\frac{ds}{dt}=\sqrt{\left\langle \frac{d\boldsymbol{\alpha}}{dt},\frac{d\boldsymbol{\alpha}}{dt}\right\rangle}=\sqrt{5}.$$

Using Wolfram Mathematica 9, we obtain the invariants as follows (See Appendix B.3. for the complete code for these invariants):

The unit spacelike tangent vector $\boldsymbol{T}$ of the curve α is obtained as

$$\boldsymbol{T}=\boldsymbol{T}(t,2t)=\frac{d\boldsymbol{\alpha}}{ds}=\frac{d\boldsymbol{\alpha}}{dt}\Big/\frac{ds}{dt}=\frac{1}{\sqrt{5}}(-\cosh t,2,-\sinh t). \tag{19}$$

Let the unit timelike normal vector $\boldsymbol{n}$ pointing outwards and it is obtained as

$$\boldsymbol{n}=\boldsymbol{n}(t,2t)=(-\sinh t,0,-\cosh t). \tag{20}$$

The unit vector $\boldsymbol{g}$, which is tangential to S_1, is obtained as

$$\boldsymbol{g}=\boldsymbol{g}(t,2t)=-\boldsymbol{n}\times\boldsymbol{T}=\frac{1}{\sqrt{5}}(2\cosh t,1,2\sinh t). \tag{21}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike helix curve α lying on the hyperbolic unit cylinder are obtained as

$$k_g=\langle d\boldsymbol{T}/dt,\boldsymbol{g}\rangle\Big/\frac{ds}{dt}=0,\ k_n=-\langle d\boldsymbol{T}/dt,\boldsymbol{n}\rangle\Big/\frac{ds}{dt}=\frac{1}{5},\ \tau_g=-\langle d\boldsymbol{g}/dt,\boldsymbol{n}\rangle\Big/\frac{ds}{dt}=-\frac{2}{5}, \tag{22}$$

respectively (See Appendix B.3). It is obvious that the curve α is a geodesic. Now, let us parameterize the hyperbolic unit sphere H_0^2 as

$$y(\bar{u},\bar{v})=(\cos\bar{v}\sinh\bar{u},\sin\bar{v}\sinh\bar{u},\cosh\bar{u}).$$

Let β be a spacelike $\bar{u}$-parametric curve (namely, a hyperbolic unit circle) lying on H_0^2 parameterized as

$$\beta(\bar{u})=y(\bar{u},\arccos(-1/\sqrt{5}))=\left(\frac{-1}{\sqrt{5}}\sinh\bar{u},\frac{2}{\sqrt{5}}\sinh\bar{u},\cosh\bar{u}\right),$$

where $\bar{v}=\arccos(-1/\sqrt{5})$. Since the differentiation of $\bar{s}$ with respect to $\bar{u}$ is $\frac{d\bar{s}}{d\bar{u}}=\left\|\frac{d\boldsymbol{\beta}}{d\bar{u}}\right\|=1$, it is clear that β is a unit-speed curve. Using Wolfram Mathematica 9, we obtain the invariants as follows (See Appendix B.4. to see the complete code for these invariants):

The unit spacelike tangent vector $\bar{\boldsymbol{T}}$ of curve β is given by

$$\bar{\boldsymbol{T}}=\frac{d\boldsymbol{\beta}}{d\bar{u}}=\frac{1}{\sqrt{5}}(-\cosh\bar{u},2\cosh\bar{u},\sqrt{5}\sinh\bar{u}). \tag{23}$$

Let the unit timelike normal vector $\bar{\boldsymbol{n}}$ of H_0^2 be inward (points origin) and it is obtained as

$$\bar{\boldsymbol{n}}=-\frac{\boldsymbol{y}_{\bar{u}}\times\boldsymbol{y}_{\bar{v}}}{\|\boldsymbol{y}_{\bar{u}}\times\boldsymbol{y}_{\bar{v}}\|}=\frac{1}{\sqrt{5}}(\sinh\bar{u},-2\sinh\bar{u},-\sqrt{5}\cosh\bar{u}). \tag{24}$$

The unit vector $\bar{\boldsymbol{g}}$, which is tangential to S_2, is obtained as

$$\bar{\boldsymbol{g}}=-\bar{\boldsymbol{n}}\times\bar{\boldsymbol{T}}=(\frac{2}{\sqrt{5}},\frac{1}{\sqrt{5}},0). \tag{25}$$

The geodesic curvature, the normal curvature, and the geodesic torsion of the spacelike curve β are obtained as

$$\bar{k}_g=\langle d\bar{\boldsymbol{T}}/d\bar{u},\bar{\boldsymbol{g}}\rangle=0,\ \bar{k}_n=-\langle d\bar{\boldsymbol{T}}/d\bar{u},\bar{\boldsymbol{n}}\rangle=-1,\ \bar{\tau}_g=-\langle d\bar{\boldsymbol{g}}/d\bar{u},\bar{\boldsymbol{n}}\rangle=0, \tag{26}$$

respectively (See Appendix B.4). It is clear that the curve β is both principal and geodesic. From (9), the angular velocity of the hyperbolic unit sphere is obtained as

$$\boldsymbol{\omega}=\sigma\left(-\frac{2}{5}\boldsymbol{T}-\frac{6}{5}\boldsymbol{g}\right).$$

Consequently, we can see that this method is expressed by using geometric invariants that can be easily applied to arbitrary spacelike parametric surfaces and spacelike curves.

4. CONCLUSION

In this paper, we adopt the Darboux frame method to investigate the forward kinematics of the instantaneous spin-rolling motion and pure-rolling motion between the spacelike moving surface and the spacelike fixed surface through the contact point in Lorentzian 3-space. We consider that the fixed surface always maintains its causal character during the rolling motion. On the other hand, the moving surface may maintain its causal character locally during the rolling motion. The forward kinematics of the moving surface is determined by the magnitude of rolling velocity σ and induced curvatures k_g^*, k_n^*, and τ_g^*. The result was given with respect to geometric invariants that can be easily generalized to arbitrary spacelike parametric surfaces and spacelike contact curves.

APPENDIX

A. The animation of the Darboux frame

```
r[u_] := {-Sinh[u]/Sqrt[5], 2 Sinh[u]/Sqrt[5], Cosh[u]}; x[u_, v_] := {Sinh[u]*Cos[v], Sinh[u]*Sin[v], Cosh[u]};
y[u_, v_] := {Sinh[u]*Cos[v], Sinh[u]*Sin[v], -Cosh[u]}; T[u_] := {-Cosh[u]/Sqrt[5], 2 Cosh[u]/Sqrt[5], Sinh[u]};
n[u_] := -{-Sinh[u]/Sqrt[5], 2 Sinh[u]/Sqrt[5], Cosh[u]}; g[u_] := {2/Sqrt[5], 1/Sqrt[5], 0};
Animate[Show[ParametricPlot3D[{r[u]}, {u, -Pi, Pi}, PlotStyle -> {Blue, Thick}, PlotLabel -> "DARBOUX FRAME
OF HYPERBOLIC CIRCLE LYING ON HYPERBOLIC UNIT SPHERE \n Red Vector : Unit tangent to the
curve \n \ Magenta Vector : Tangential to the sphere \n Black Vector : Unit normal to the sphere",
LabelStyle -> Directive[Bold, Black], Axes -> False, Boxed -> False],
ParametricPlot3D[{x[u, v]}, {u, 0, 2}, {v, 0, 2 Pi}, PlotStyle -> {Yellow, Opacity[.5]}, PlotPoints -> 20, Axes -> False,
Boxed -> False], ParametricPlot3D[{y[u, v]}, {u, 0, 2}, {v, -Pi, Pi}, PlotStyle -> {Yellow, Opacity[.0]}, Mesh -> None,
PlotPoints -> 55, Axes -> False, Boxed -> False], Graphics3D[{Directive[GrayLevel[.2], Thick, Arrowheads[{{.025, 1}}]],
Arrow[{{0, 0, 0}, {0, 0, 3.2}}]}, Boxed -> False], Graphics3D[{Directive[GrayLevel[.2], Thick, Arrowheads[{{.025, 1}}]],
Arrow[{{0, 0, 0}, {0, 3.2, 0}}]}, Boxed -> False], Graphics3D[{Directive[GrayLevel[.2], Thick, Arrowheads[{{.025, 1}}]],
Arrow[{{0, 0, 0}, {3.2, 0, 0}}]}, Boxed -> False], Graphics3D[{Text[Style["  x", 16], {3.3, 0, 0}], Text[Style["y", 16], {0,
3.3, 0}], Text[Style["          z\ (Time)", 16], {0, 0, 3.3}], Text[Style["O", Italic, 14], {0, 0.1, -0.2}], Text[Style["(\[Beta]) ",
16], {1.6, -3.2, 3.1}], Text[Style[
    "\!\(\*TemplateBox[{\"H\",\"0\",\"2\"},\n\"Subsuperscript\"]\)",
     16], {1, 3.5, 3.4}]}], Graphics3D[{PointSize[Large], Black, Point[Dynamic[{r[s]}]]}],
  Graphics3D[{{Thick, Darker@Red, Arrowheads[0.028], Arrow[{r[s], r[s] + T[s]}]}, {Thick, Darker@Magenta,
   Arrowheads[0.028], Arrow[{r[s], r[s] + g[s]}]}, {Thick, Darker@Black, Arrowheads[0.028], Arrow[{r[s], r[s] +
n[s]}]}}], PlotRange -> {{-1.0 Pi, 1.0 Pi}, {-1.0 Pi, 1.0 Pi}, {-1.0 Pi, 1.0 Pi}}, ViewPoint -> {3.6, -3.5, 1}, ImageSize -> 500],
{s, -Pi, Pi}]
```

B. The code of the invariants with Wolfram Mathematica 9

B.1. The calculation of the invariants (The Formulas (10)-(13))

```
x1 = u; x2 = v; x3 = 1; c1 = Cosh[u] - 1; c2 = Sinh[u]; c3 = 1;dux1 = D[x1, u]; dux2 = D[x2, u]; dux3 = D[x3, u]; dvx1 =
D[x1, v]; dvx2 = D[x2, v]; dvx3 = D[x3, v];     normu = Simplify[Sqrt[Abs[dux1*dux1 + dux2*dux2 - dux3*dux3]], #] /.
Abs -> Identity & /@ {Sqrt[dux1*dux1 + dux2*dux2 - dux3*dux3] >= 0};normv = Simplify[Sqrt[Abs[dvx1*dvx1 +
dvx2*dvx2 - dvx3*dvx3]], #] /. Abs -> Identity & /@ {Sqrt[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3] >= 0}; T11 =
dux1/normu; T12 = dux2/normu; T13 =  dux3/normu; xuxv1 = -dux2*dvx3 + dux3*dvx2; xuxv2 = -dux3*dvx1 +
dux1*dvx3; xuxv3 = dux1*dvx2 - dux2*dvx1; normuxuxv =  Simplify[Sqrt[Abs[(-dux2*dvx3 + dux3*dvx2)^2 + (-
dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]], #] /. Abs -> Identity & /@ {Sqrt[Abs[(-dux2*dvx3 +
dux3*dvx2)^2 + (-dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]] > 0}; n11 = Simplify[xuxv1/normuxuxv];
n12 = Simplify[xuxv2/normuxuxv]; n13 =Simplify[xuxv3/normuxuxv]; g11 = Simplify[T13*n12 - T12*n13]; g12
=Simplify[-T13*n11 + T11*n13]; g13 =Simplify[-T12*n11 + T11*n12]; T21 = dvx1/normv; T22 =dvx2/normv; T23 =
dvx3/normv; g21 =Simplify[T23*n12 - T22*n13]; g22 =Simplify[-T23*n11 + T21*n13]; g23 =Simplify[-T22*n11 +
T21*n12]; dc1 = D[c1, u]; dc2 = D[c2, u]; dc3 =D[c3, u]; normdc =
Simplify[Sqrt[dc1^2 + dc2^2 - dc3^2], #] /.Abs -> Identity & /@ {Sqrt[dc1^2 + dc2^2 - dc3^2] > 0}; T1 =dc1/normdc;
T2 = dc2/normdc; T3 = dc3/normdc; dT1 = D[T1, u]; dT2 =D[T2, u]; dT3 = D[T3, u]; normdT
=Simplify[Sqrt[Abs[dT1*dT1 + dT2*dT2 - dT3*dT3]], #] /.Abs -> Identity & /@ {Sqrt[dT1*dT1 + dT2*dT2 -
dT3*dT3] > 0}; g1 =Simplify[n12*T3 - n13*T2]; g2 = Simplify[n13*T1 - n11*T3]; g3 =Simplify[-n11*T2 + n12*T1] /.
Abs -> Identity; nrmg =Simplify[Sqrt[g1*g1 + g2*g2 - g3*g3], #] /.Abs -> Identity & /@ {Sqrt[g1*g1 + g2*g2 - g3*g3] >
```

```
0}; dg1 =Simplify[D[g1, u]]; dg2 = Simplify[D[g2, u]]; dg3 =Simplify[D[g3, u]];
Subscript[\!\(\*StyleBox["k",StripOnInput->False,FontSlant->Italic]\), n] =Simplify[(-dT1*n11 - dT2*n12 +
dT3*n13)/normdc]; Subscript[\!\(\*StyleBox["k",StripOnInput->False,FontSlant->Italic]\), g] =Simplify[(dT1*g1 +
dT2*g2 - dT3*g3)/normdc]; Subscript[\!\(\*StyleBox["\[Tau]",StripOnInput->False,FontSlant->Italic]\), g] =
Simplify[(-dg1*n11 - dg2*n12 + dg3*n13)/normdc];  u = t;
{"Spacelike Unit Tangent Vector \!\(\*
StyleBox[\"T\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\)\!\(\*
StyleBox[\" \",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\): ", {T1, T2, T3},
"Spacelike Unit Vector \!\(\*
StyleBox[\"g\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\) : ", {g1/nrmg, g2/nrmg,
g3/nrmg}, "Timelike Unit Normal Vector \!\(\*
StyleBox[\"n\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\) : ", {n11, n12, n13},
"Normal curvature  \!\(\*
StyleBox[SubscriptBox[StyleBox[StyleBox[StyleBox[
StyleBox[\"k\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic],\nStripOnInput->False,\nFontSlant->Italic], \
\"n\"],\nStripOnInput->False,\nFontSlant->Italic]\) : ",
Simplify[Subscript[\!\(\*StyleBox["k",
StripOnInput->False,
FontSlant->Italic]\), n]], "Geodesic curvature  \!\(\*SubscriptBox[
StyleBox[StyleBox[StyleBox[
StyleBox[\"k\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic],\nStripOnInput->False,\nFontSlant->Italic], \(g\)]\
\) : ", Simplify[Subscript[\!\(\*
StyleBox["k",StripOnInput->False,
FontSlant->Italic]\), g]], "Geodesic torsion  \!\(\*SubscriptBox[StyleBox[StyleBox[
StyleBox[\"\[Tau]\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic], \(g\)]\) : ", Simplify[Subscript[\!\(\*
StyleBox["\[Tau]",
StripOnInput->False,
FontSlant->Italic]\), g]]} // Column // TraditionalForm
```

B.2. The calculation of the invariants (The Formulas (14)-(17))

```
x1 = \!\(\*OverscriptBox[\(v\), \(_\)]\); x2 = Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; x3 = Cosh[\!\(\*OverscriptBox[\(u\),
\(_\)]\)]; c1 = 0; c2 = Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; c3 = Cosh[\!\(\*OverscriptBox[\(u\), \(_\)]\)];dux1 = D[x1,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dux2 = D[x2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dux3 = D[x3,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dvx1 = D[x1, \!\(\*OverscriptBox[\(v\), \(_\)]\)]; dvx2 = D[x2,
\!\(\*OverscriptBox[\(v\), \(_\)]\)]; dvx3 = D[x3, \!\(\*OverscriptBox[\(v\), \(_\)]\)];  \!\(\*OverscriptBox[\(v\), \(_\)]\) = 0;
normu = Simplify[Sqrt[Abs[dux1*dux1 + dux2*dux2 - dux3*dux3]], #] /. Abs -> Identity & /@ {Sqrt[dux1*dux1 +
dux2*dux2 - dux3*dux3] >= 0}; normv = Simplify[Sqrt[Abs[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3]], #] /.
Abs -> Identity & /@ {Sqrt[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3] >= 0}; T11 = dux1/normu; T12 = dux2/normu; T13 =
dux3/normu; xuxv1 = -dux2*dvx3 + dux3*dvx2; xuxv2 = -dux3*dvx1 + dux1*dvx3; xuxv3 = dux1*dvx2 - dux2*dvx1;
normuxuxv =  Simplify[Sqrt[Abs[(-dux2*dvx3 + dux3*dvx2)^2 + (-dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 -
dux2*dvx1)^2]], #] /. Abs -> Identity & /@ {Sqrt[Abs[(-dux2*dvx3 + dux3*dvx2)^2 + (-dux3*dvx1 +
dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]] > 0}; n11 = -Simplify[xuxv1/normuxuxv]; n12 = -Simplify[
xuxv2/normuxuxv]; n13 = -Simplify[xuxv3/normuxuxv]; g11 = Simplify[T13*n12 - T12*n13]; g12 =
 Simplify[-T13*n11 + T11*n13]; g13 =  Simplify[-T12*n11 + T11*n12]; T21 = dvx1/normv; T22 =  dvx2/normv; T23 =
dvx3/normv; g21 =  Simplify[T23*n12 - T22*n13]; g22 =  Simplify[-T23*n11 + T21*n13]; g23 =  Simplify[-T22*n11 +
T21*n12]; dc1 = D[c1, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dc2 = D[c2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dc3 = D[c3,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; normdc = Simplify[Sqrt[dc1^2 + dc2^2 - dc3^2], #] /. Abs -> Identity & /@
{Sqrt[dc1^2 + dc2^2 - dc3^2] > 0}; T1 =  dc1/normdc; T2 = dc2/normdc; T3 = dc3/normdc; dT1 = D[T1,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dT2 = D[T2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dT3 = D[T3, \!\(\*OverscriptBox[\(u\),
\(_\)]\)]; normdT = Simplify[Sqrt[Abs[dT1*dT1 + dT2*dT2 - dT3*dT3]], #] /.
Abs -> Identity & /@ {Sqrt[dT1*dT1 + dT2*dT2 - dT3*dT3] > 0}; g1 =Simplify[n12*T3 - n13*T2]; g2 =
Simplify[n13*T1 - n11*T3]; g3 =  Simplify[-n11*T2 + n12*T1] /. Abs -> Identity; nrmg =  Simplify[Sqrt[g1*g1 + g2*g2 -
g3*g3], #] /. Abs -> Identity & /@ {Sqrt[g1*g1 + g2*g2 - g3*g3] > 0}; dg1 =  Simplify[D[g1, \!\(\*OverscriptBox[\(u\),
\(_\)]\)]]; dg2 = Simplify[D[g2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]]; dg3 = Simplify[D[g3,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]]; Subscript[\!\(\*StyleBox[OverscriptBox["k", "_"],StripOnInput->False,
FontSlant->Italic]\), n] =  Simplify[(-dT1*n11 - dT2*n12 + dT3*n13)/normdc]; Subscript[\!\(\*StyleBox[
```

```
OverscriptBox["k", "_"],StripOnInput->False, FontSlant->Italic]\), g] =  Simplify[(dT1*g1 + dT2*g2 -
dT3*g3)/normdc]; Subscript[\!\(\*StyleBox[OverscriptBox["\[Tau]", "_"],StripOnInput->False,
FontSlant->Italic]\), g] =
 Simplify[(-dg1*n11 - dg2*n12 + dg3*n13)/normdc];
{"Spacelike Unit Tangent Vector \!\(\*
  StyleBox[OverscriptBox[\"T\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\)\!\(\*
  StyleBox[\" \",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\): ", {T1, T2, T3},
  "Spacelike Unit Vector \!\(\*
  StyleBox[OverscriptBox[\"g\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\) : ", {g1/nrmg,
   g2/nrmg, g3/nrmg}, "Timelike Unit Normal Vector \!\(\*
  StyleBox[OverscriptBox[\"n\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\) : ", {n11, n12,
    n13}, "Normal curvature  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"k\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(n\)]\) \ : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["k", "_"],
StripOnInput->False,
FontSlant->Italic]\), n]],
  "Geodesic curvature  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"k\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(g\)]\) \ : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["k", "_"],
StripOnInput->False,
FontSlant->Italic]\), g]],
  "Geodesic torsion  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"\[Tau]\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(g\)]\) : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["\[Tau]", "_"],
StripOnInput->False,
FontSlant->Italic]\), g]]} // Column // TraditionalForm
```

B.3. The calculation of the invariants (The Formulas (19)-(22))

```
x1 = -Sinh[u]; x2 = v; x3 = 2-Cosh[u]; c1 =  -Sinh[u]; c2 =2u; c3 =2-Cosh[u];dux1 = D[x1, u]; dux2 = D[x2, u]; dux3 =
D[x3, u]; dvx1 =  D[x1, v]; dvx2 = D[x2, v]; dvx3 = D[x3, v];     normu = Simplify[Sqrt[Abs[dux1*dux1 + dux2*dux2 -
dux3*dux3]], #] /. Abs -> Identity & /@ {Sqrt[dux1*dux1 + dux2*dux2 - dux3*dux3] >= 0};normv =
Simplify[Sqrt[Abs[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3]], #] /. Abs -> Identity & /@ {Sqrt[dvx1*dvx1 + dvx2*dvx2 -
dvx3*dvx3] >= 0}; T11 = dux1/normu; T12 = dux2/normu; T13 =  dux3/normu; xuxv1 = -dux2*dvx3 + dux3*dvx2;
xuxv2 = -dux3*dvx1 +  dux1*dvx3; xuxv3 = dux1*dvx2 - dux2*dvx1; normuxuxv =  Simplify[Sqrt[Abs[(-dux2*dvx3 +
dux3*dvx2)^2 + (-dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]], #] /. Abs -> Identity & /@ {Sqrt[Abs[(-
dux2*dvx3 + dux3*dvx2)^2 + (-dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]] > 0}; n11 =
Simplify[xuxv1/normuxuxv]; n12 = Simplify[xuxv2/normuxuxv]; n13 =Simplify[xuxv3/normuxuxv]; g11 =
Simplify[T13*n12 - T12*n13]; g12 =Simplify[-T13*n11 + T11*n13]; g13 =Simplify[-T12*n11 + T11*n12]; T21 =
dvx1/normv; T22 =dvx2/normv; T23 = dvx3/normv; g21 =Simplify[T23*n12 - T22*n13]; g22 =Simplify[-T23*n11 +
T21*n13]; g23 =Simplify[-T22*n11 + T21*n12]; dc1 = D[c1, u]; dc2 = D[c2, u]; dc3 =D[c3, u]; normdc =
Simplify[Sqrt[dc1^2 + dc2^2 - dc3^2], #] /.Abs -> Identity & /@ {Sqrt[dc1^2 + dc2^2 - dc3^2] > 0}; T1 =dc1/normdc;
T2 = dc2/normdc; T3 = dc3/normdc; dT1 = D[T1, u]; dT2 =D[T2, u]; dT3 = D[T3, u]; normdT
=Simplify[Sqrt[Abs[dT1*dT1 + dT2*dT2 - dT3*dT3]], #] /.Abs -> Identity & /@ {Sqrt[dT1*dT1 + dT2*dT2 -
dT3*dT3] > 0}; g1 =Simplify[n12*T3 - n13*T2]; g2 = Simplify[n13*T1 - n11*T3]; g3 =Simplify[-n11*T2 + n12*T1] /.
Abs -> Identity; nrmg =Simplify[Sqrt[g1*g1 + g2*g2 - g3*g3], #] /.Abs -> Identity & /@ {Sqrt[g1*g1 + g2*g2 - g3*g3] >
0}; dg1 =Simplify[D[g1, u]]; dg2 = Simplify[D[g2, u]]; dg3 =Simplify[D[g3, u]];
Subscript[\!\(\*StyleBox["k",StripOnInput->False,FontSlant->Italic]\), n] =Simplify[(-dT1*n11 - dT2*n12 +
dT3*n13)/normdc]; Subscript[\!\(\*StyleBox["k",StripOnInput->False,FontSlant->Italic]\), g] =Simplify[(dT1*g1 +
dT2*g2 - dT3*g3)/normdc]; Subscript[\!\(\*StyleBox["\[Tau]",StripOnInput->False,FontSlant->Italic]\), g] =
Simplify[(-dg1*n11 - dg2*n12 + dg3*n13)/normdc];  u = t;
{"Spacelike Unit Tangent Vector \!\(\*
StyleBox[\"T\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\)\!\(\*
StyleBox[\" \",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\): ", {T1, T2, T3},
"Spacelike Unit Vector \!\(\*
```

```
StyleBox[\"g\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\) : ", {g1/nrmg, g2/nrmg,
g3/nrmg}, "Timelike Unit Normal Vector \!\(\*
StyleBox[\"n\",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\) : ", {n11, n12, n13},
"Normal curvature  \!\(\*
StyleBox[SubscriptBox[StyleBox[StyleBox[StyleBox[
StyleBox[\"k\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic],\nStripOnInput->False,\nFontSlant->Italic], \
\"n\"],\nStripOnInput->False,\nFontSlant->Italic]\) : ",
Simplify[Subscript[\!\(\*StyleBox["k",
StripOnInput->False,
FontSlant->Italic]\), n]], "Geodesic curvature  \!\(\*SubscriptBox[
StyleBox[StyleBox[StyleBox[
StyleBox[\"k\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic],\nStripOnInput->False,\nFontSlant->Italic], \(g\)]\
\) : ", Simplify[Subscript[\!\(\*
StyleBox["k",StripOnInput->False,
FontSlant->Italic]\), g]], "Geodesic torsion  \!\(\*SubscriptBox[StyleBox[StyleBox[
StyleBox[\"\[Tau]\",\nStripOnInput->False,\nFontSlant->Italic],\n\
StripOnInput->False,\nFontSlant->Italic],\nStripOnInput->False,\n\
FontSlant->Italic], \(g\)]\) : ", Simplify[Subscript[\!\(\*
StyleBox["\[Tau]",
StripOnInput->False,
FontSlant->Italic]\), g]]} // Column // TraditionalForm
```

B.4. The calculation of the invariants (The Formulas (23)-(26))

```
x1 = Cos[\!\(\*OverscriptBox[\(v\), \(_\)]\)]*Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; x2 = Sin[\!\(\*OverscriptBox[\(v\),
\(_\)]\)]*Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; x3 = Cosh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; c1 =
Cos[\!\(\*OverscriptBox[\(v\), \(_\)]\)]*Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; c2 = Sin[\!\(\*OverscriptBox[\(v\),
\(_\)]\)]*Sinh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; c3 = Cosh[\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dux1 = D[x1,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dux2 = D[x2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dux3 = D[x3, \!\(\*OverscriptBox[\(u\),
\(_\)]\)]; dvx1 = D[x1, \!\(\*OverscriptBox[\(v\), \(_\)]\)]; dvx2 = D[x2, \!\(\*OverscriptBox[\(v\), \(_\)]\)]; dvx3 = D[x3,
\!\(\*OverscriptBox[\(v\), \(_\)]\)];
\!\(\*OverscriptBox[\(v\), \(_\)]\) = ArcCos[-1/Sqrt[5]];    normu = Simplify[Sqrt[Abs[dux1*dux1 + dux2*dux2 -
dux3*dux3]], #] /. Abs -> Identity & /@ {Sqrt[dux1*dux1 + dux2*dux2 - dux3*dux3] >= 0}; normv =
Simplify[Sqrt[Abs[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3]], #] /.
Abs -> Identity & /@ {Sqrt[dvx1*dvx1 + dvx2*dvx2 - dvx3*dvx3] >= 0}; T11 = dux1/normu; T12 = dux2/normu; T13 =
dux3/normu; xuxv1 = -dux2*dvx3 + dux3*dvx2; xuxv2 = -dux3*dvx1 + dux1*dvx3; xuxv3 = dux1*dvx2 - dux2*dvx1;
normuxuxv =  Simplify[Sqrt[Abs[(-dux2*dvx3 + dux3*dvx2)^2 + (-dux3*dvx1 + dux1*dvx3)^2 - (dux1*dvx2 -
dux2*dvx1)^2]], #] /. Abs -> Identity & /@ {Sqrt[Abs[(-dux2*dvx3 + dux3*dvx2)^2 + (-dux3*dvx1 +
dux1*dvx3)^2 - (dux1*dvx2 - dux2*dvx1)^2]] > 0}; n11 = -Simplify[xuxv1/normuxuxv]; n12 = -Simplify[
xuxv2/normuxuxv]; n13 = -Simplify[xuxv3/normuxuxv]; g11 = Simplify[T13*n12 - T12*n13]; g12 =
 Simplify[-T13*n11 + T11*n13]; g13 =  Simplify[-T12*n11 + T11*n12]; T21 = dvx1/normv; T22 =  dvx2/normv; T23 =
dvx3/normv; g21 =  Simplify[T23*n12 - T22*n13]; g22 =  Simplify[-T23*n11 + T21*n13]; g23 =  Simplify[-T22*n11 +
T21*n12]; dc1 = D[c1, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dc2 = D[c2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dc3 = D[c3,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; normdc = Simplify[Sqrt[dc1^2 + dc2^2 - dc3^2], #] /. Abs -> Identity & /@
{Sqrt[dc1^2 + dc2^2 - dc3^2] > 0}; T1 =  dc1/normdc; T2 = dc2/normdc; T3 = dc3/normdc; dT1 = D[T1,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]; dT2 = D[T2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]; dT3 = D[T3, \!\(\*OverscriptBox[\(u\),
\(_\)]\)]; normdT = Simplify[Sqrt[Abs[dT1*dT1 + dT2*dT2 - dT3*dT3]], #] /.
Abs -> Identity & /@ {Sqrt[dT1*dT1 + dT2*dT2 - dT3*dT3] > 0}; g1 =Simplify[n12*T3 - n13*T2]; g2 =
Simplify[n13*T1 - n11*T3]; g3 =  Simplify[-n11*T2 + n12*T1] /. Abs -> Identity; nrmg =  Simplify[Sqrt[g1*g1 + g2*g2 -
g3*g3], #] /. Abs -> Identity & /@ {Sqrt[g1*g1 + g2*g2 - g3*g3] > 0}; dg1 =  Simplify[D[g1, \!\(\*OverscriptBox[\(u\),
\(_\)]\)]]; dg2 = Simplify[D[g2, \!\(\*OverscriptBox[\(u\), \(_\)]\)]]; dg3 = Simplify[D[g3,
\!\(\*OverscriptBox[\(u\), \(_\)]\)]]; Subscript[\!\(\*StyleBox[OverscriptBox["k", "_"],StripOnInput->False,
FontSlant->Italic]\), n] =  Simplify[(-dT1*n11 - dT2*n12 + dT3*n13)/normdc]; Subscript[\!\(\*StyleBox[
OverscriptBox["k", "_"],StripOnInput->False, FontSlant->Italic]\), g] =  Simplify[(dT1*g1 + dT2*g2 -
dT3*g3)/normdc]; Subscript[\!\(\*StyleBox[OverscriptBox["\[Tau]", "_"],StripOnInput->False,
FontSlant->Italic]\), g] =
 Simplify[(-dg1*n11 - dg2*n12 + dg3*n13)/normdc];
{"Spacelike Unit Tangent Vector \!\(\*
  StyleBox[OverscriptBox[\"T\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\)\!\(\*
```

```
  StyleBox[\" \",\nStripOnInput->False,\nFontSize->13,\n\
FontWeight->Bold,\nFontSlant->Italic]\): ", {T1, T2, T3},
  "Spacelike Unit Vector \!\(\*
  StyleBox[OverscriptBox[\"g\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\) : ", {g1/nrmg,
   g2/nrmg, g3/nrmg}, "Timelike Unit Normal Vector \!\(\*
  StyleBox[OverscriptBox[\"n\", \"_\"],\nStripOnInput->False,\n\
FontSize->13,\nFontWeight->Bold,\nFontSlant->Italic]\) : ", {n11, n12,
    n13}, "Normal curvature  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"k\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(n\)]\) \ : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["k", "_"],
StripOnInput->False,
FontSlant->Italic]\), n]],
  "Geodesic curvature  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"k\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(g\)]\) \ : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["k", "_"],
StripOnInput->False,
FontSlant->Italic]\), g]],
  "Geodesic torsion  \!\(\*SubscriptBox[\(\!\(\*
  StyleBox[OverscriptBox[\"\[Tau]\", \"_\"],\nStripOnInput->False,\n\
FontSlant->Italic]\)\), \(g\)]\) : ", Simplify[Subscript[\!\(\*
StyleBox[
OverscriptBox["\[Tau]", "_"],
StripOnInput->False,
FontSlant->Italic]\), g]]} // Column // TraditionalForm
```

REFERENCES

[1] Cui L. and Dai J. S., A Polynomial Formulation of Inverse Kinematics of Rolling Contact, ASME J. Mech. Rob., 7-4 (2015) 041003_041001-041009.

[2] Cui L. and Dai J. S., A Darboux-Frame-Based Formulation of Spin-Rolling Motion of Rigid Objects With Point Contact, IEEE Trans. Rob., 26-2 (2010) 383–388.

[3] Cui L., Differential Geometry Based Kinematics of Sliding-Rolling Contact and Its Use for Multifingered Hands, Ph.D. Thesis, King's College London, University of London, London, UK, 2010.

[4] Neimark J. I. and Fufaev N. A., Dynamics of Nonholonomic Systems, Providence, RI: Amer. Math. Soc., 1972.

[5] Cai C. and Roth B., On the Spatial Motion of Rigid Bodies with Point Contact, In Proc. IEEE Conf. Robot. Autom., 1987; pp. 686–695.

[6] Cai C. and Roth B., On the Planar Motion of Rigid Bodies with Point Contact, Mech. Mach. Theory, 21 (1986) 453–466.

[7] Montana D. J., The Kinematics of Multi-fingered Manipulation, IEEE Trans. Robot. Autom., 11-4 (1995) 491–503.

[8] Li Z. X. and Canny J., Motion of Two Rigid Bodies with Rolling Constraint, IEEE Trans. Robot. Autom., 6-1 (1990) 62–72.

[9] Sarkar N., Kumar V. and Yun X., Velocity and Acceleration Analysis of Contact Between Three-Dimensional Rigid Bodies, ASME J. Appl. Mech., 63-4 (1996) 974–984.

[10] Marigo A. and Bicchi A., Rolling Bodies with Regular Surface: Controllability Theory and Application, IEEE Trans. Autom. Control, 45-9 (2000) 1586–1599.

[11] Agrachev A. A. and Sachkov Y. L., An Intrinsic Approach to the Control of Rolling Bodies, In Proc. 38th IEEE Conf. Decis. Control, Phoenix, AZ, USA, 1999; 431–435.

[12] Chelouah A. and Chitour Y., On the Motion Planning of Rolling Surfaces, Forum Math., 15-5 (2003) 727–758.

[13] Chitour Y., Marigo A. and Piccoli B., Quantization of the Rolling-Body Problem with Applications to Motion Planning, Syst. Control Lett., 54-10 (2005) 999–1013.

[14] Tchon K., Repeatability of Inverse Kinematics Algorithms for Mobile Manipulators, IEEE Trans. Autom. Control, 47-8 (2002) 1376–1380.

[15] Tchon K. and Jakubiak J., An extended Jacobian Inverse Kinematics Algorithm for Doubly Nonholonomic Mobile Manipulators, In Proc. IEEE Int. Conf. Robot. Autom., Barcelona, Spain, 2005; 1548–1553.

[16] Bottema O. and Roth B., Theoretical Kinematics, North-Holland Publ. Co., Amsterdam, 1979, 556 pp.

[17] Karger A. and Novak J., Space Kinematics and Lie Groups, STNL Publishers of Technical Lit., Prague, Czechoslovakia, 1978.

[18] Müller H. R., Kinematik Dersleri, Ankara Üniversitesi Fen Fakültesi Yayınları, 1963.

[19] Nelson E. W., Best C. L. and McLean W. G., Schaum's Outline of Theory and Problems of Engineering Mechanics, Statics and Dynamics (5^{th} Ed.), McGraw-Hill, New York, 1997.

[20] Birman G. S. and Nomizu K., Trigonometry in Lorentzian Geometry, Ann. Math. Month., 91-9 (1984) 543-549.

[21] O'Neill B., Semi-Riemannian Geometry with Applications to Relativity, Academic Press, London, 1983.

[22] Ratcliffe J. G., Foundations of Hyperbolic Manifolds, Springer, New York, 2006.

[23] Uğurlu H. H. and Çalışkan A., Darboux Ani Dönme Vektörleri ile Spacelike ve Timelike Yüzeyler Geometrisi, Celal Bayar Üniversitesi Yayınları, Manisa, 2012.

[24] Do Carmo M. P., Differential Geometry of Curves and Surfaces, Prentice-Hall, Englewood Cliffs, New Jersey, 1976.