Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.39-3(2018) 565-572

A Strongly Ill-Posed Problem for the Equation

$$Lu \equiv x\Delta u + k(x,'y)u_x = xf(x,'y)$$

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Received: 01.08.2018; Accepted: 11.09.2018 *http://dx.doi.org/10.17776/csj.450207*

Abstract. In this work, we consider an inverse problem for an elliptic equation which is strongly ill-posed in Hadamard sense. We prove the uniqueness of the solution of the problem by using Carleman estimates.

***Keywords*:** Elliptic equation, Inverse problem, Carleman estimate

$$Lu \equiv x\Delta u + k(x,'y)u_x = xf(x,'y)$$

Denklemi için Bir Kuvvetli Kötü Konulmuş Problem

Özet. Bu çalışmada eliptik denklem için Hadamard anlamında kuvvetli kötü konulmuş olan bir ters problem ele alınmıştır. Bu problemin çözümünün tekliği Carleman değerlendirmeleri yardımıyla ispatlanmıştır.

***Anahtar Kelimeler*:** Eliptik denklem, Ters problem, Carleman değerlendirmesi

1. INTRODUCTION

In this study, a strongly ill-posed inverse problem in Hadamard sense is investigated. In the domain

$$D = \{(x,y)|\, x \in (0,1)\,, y_i \in (0,1)\,, i = 1,2,\dots,n\},$$

we consider the equation

$$Lu \equiv x\Delta u + ku_x = xf(x,'y) \tag{1}$$

and the conditions

$$u|_{y_n=0} = u_0(x,'y);\; u_{y_n}\big|_{y_n=0} = u_1(x,'y), \tag{2}$$

where $'y = (y_2, y_3, \dots, y_n)$.

Equation 1 is an elliptic equation which has important applications in potential theory, [6]. Moreover, recently, inverse and ill-posed problems for differential equations are attracting the interest of many scientists because of various applications in science and engineering such as tomography and seismology, [1-5].

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Additional condition for the solution of Cauchy problem (1)-(2) is

$$u|_{y_1=0} = u_2(x,'y). \tag{3}$$

Problem 1: We now aim to get the pair $\big(u(x,y), xf(x,'y)\big)$ from equation (1) which satisfies conditions (2)-(3).

Theorem 1. Problem 1 can have only one solution in the space $C^3(\bar{D}) \times C^1(\bar{D})$ with $u\epsilon C^3(\bar{D})$, $f\epsilon C^1(\bar{D})$.

Theorem 1 is related with the uniqueness of the solution of problem 1. In general, one can not prove the existence of solution for this problem. Because, problem (1)-(3) is strongly ill-posed in Hadamard sense. In order to prove the theorem, the following lemmas are needed.

2. PROOF OF THE MAIN RESULT

Let us suppose that $\big(u^{(1)}, f^{(1)}\big), \big(u^{(2)}, f^{(2)}\big)$ are two solutions for problem (1)-(3) in $C^3(\bar{D}) \times C^1(\bar{D})$. Then we have

$$x\Delta u^{(i)} + k u_x^{(i)} = x f^{(i)}(x,'y), \tag{4}$$

$$u^{(i)}, u_{y_n}^{(i)}\Big|_{y_n=0} = u_0, u_1, \tag{5}$$

$$u^{(i)}\Big|_{y_1=0} = u_2, \; i = 1,2. \tag{6}$$

For $\bar{u} = u^{(2)} - u^{(1)}, \bar{f} = f^{(2)} - f^{(1)}$ we can write

$$x\Delta\bar{u} + k\bar{u}_x = x\bar{f}(x,'y), \tag{7}$$

$$\bar{u}, \bar{u}_{y_n}\Big|_{y_n=0} = 0, \tag{8}$$

$$\bar{u}|_{y_1=0} = 0. \tag{9}$$

Taking derivative of both sides of equation (7) with respect to y_1 , we get

$$x\Delta\bar{u}_{y_1} + k\bar{u}_{y_1 x} = 0. \tag{10}$$

In equation (10), if we take $w = \bar{u}_{y_1}$ we have

$$x\Delta w + k w_x = 0 \tag{11}$$

and conditions (8) become

$$w, w_{y_n}\Big|_{y_n=0} = 0. \tag{12}$$

Let us first consider the following general problem,

$$\left|\Delta_{x,y} z\right| \le C(|z| + |\nabla z|), \tag{13}$$

$$z(x,' y, 0) = 0, z_{y_n}(x,' y, 0) = 0. \tag{14}$$

It is easy to prove that there is only zero solution of problem (11)-(12) in $C^2(\bar{D})$ if (13)-(14) has only zero solution in $C^2(\bar{D})$.

We now attempt to prove that problem (13)-(14) can have zero solution in $C^2(\bar{D})$ using the Carleman method.

For the sake of simplicity we have

$$y_n = u_1, x = u_2, y_i = u_{2+i},\ i = 1,2,3, \dots, n-1.$$

Let

$$\Omega(u^0) = \left\{u: u_1 > 0, 0 < \delta u_1 < \gamma - \tfrac{1}{2}\sum_{i=2}^{n+1}\left(u_i - u_i^0\right)^2, \delta > 1, \gamma < 1\right\}$$

and

$$\psi(u) = \delta u_1 + \tfrac{1}{2}\sum_{i=2}^{n+1}\left(u_i - u_i^0\right)^2 + \alpha_0, \alpha_0 > 0, \gamma + \alpha_0 = \eta < 1.$$

If we choose

$$u^0 = (u_2^0, u_3^0, \dots, u_{n+1}^0)\epsilon D^1, D^1 = \{(x,y): (x, y_1, \dots, y_{n-1})\epsilon(0,1)^n\},$$

then it is clear that

$$\alpha_0 < \psi(u) < \eta \text{ in the domain } \left(0, u^0\epsilon\Omega(u^0)\right) \subset D \text{ and } \Omega(u^0).$$

Let $H = exp(\lambda\psi^{-\nu})$, where λ, ν are positive parameters.

Lemma 1. For every $\varphi\epsilon C^2(\bar{D})$

$$-\varphi\Delta\varphi H^2 = \left[|\nabla\varphi|^2 - \sum_{i=1}^{n+1}\left(2\lambda^2\nu^2\psi^{-2\nu-2}\psi_{u_i}^2 + \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{u_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i}\right)\varphi^2\right]H^2 + d_1(\varphi) \tag{15}$$

where

$$d_1(\varphi) = -\sum_{i=1}^{n+1}\left(\varphi\varphi_{u_i}H^2 + \psi^{-\nu-1}\lambda\nu\psi_{u_i}\varphi^2H^2\right)_{u_i}.$$

Proof. From the definition, we have

$$-\varphi\Delta\varphi H^2 = -\varphi\left(\sum_{i=1}^{n+1}\varphi_{u_iu_i}\right)e^{2\lambda\psi^{-\nu}}.$$

On the other hand,

$$-\varphi\varphi_{u_iu_i}H^2 = -\left(\varphi\varphi_{u_i}e^{2\lambda\psi^{-\nu}}\right)_{u_i} + \varphi_{u_i}^2e^{2\lambda\psi^{-\nu}} - 2\lambda\nu\varphi\varphi_{u_i}e^{2\lambda\psi^{-\nu}}\psi_{u_i}\psi^{-\nu-1}$$

and

$$-2\lambda\nu\psi^{-\nu-1}\psi_{u_i}H^2\varphi\varphi_{u_i} = -\left(\lambda\nu\psi_{u_i}\psi^{-\nu-1}H^2\varphi^2\right)_{u_i} - 2\lambda^2\nu^2\psi_{u_i}^2\psi^{-2\nu-2}H^2\varphi^2$$

$$-\lambda\nu(\nu+1)\psi_{u_i}^2H^2\psi^{-\nu-2}\varphi^2 + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i}H^2\varphi^2,\ i = 1,2,3, \dots, n+1.$$

By the last two equations, (15) is obtained.

Lemma 2.

For

$$\nu_0 = \frac{(n+2)\eta}{\delta(1-\eta)},\ \lambda_0 = 2(\nu+2)^2(\delta^2+\gamma^2)^2 + 2\,,$$

if $\nu \geq 2+\nu_0$, $\lambda \geq \lambda_0$, then for any $\varphi \epsilon C^2(\overline{D})$ the inequality

$$\psi^{\nu+1}(\Delta\varphi)^2H^2 \geq 3\lambda^3\nu^4\delta^4\psi^{-2\nu-2}H^2\varphi^2 - 2\lambda\nu|\nabla\varphi|^2H^2 + d_2(\varphi) \tag{16}$$

is valid, where

$$d_2(\varphi H) = \sum_{i=1}^{4} d_{2i}(\varphi H)\,,$$

$$d_{21}(\varphi H) = 4\lambda\nu\sum_{i,j=1}^{n+1}\left[\psi_{u_i}\left(\varphi_{u_i} - \lambda\nu\psi_{u_i}\psi^{-\nu-1}\varphi\right)\left(\varphi_{u_i} - \lambda\nu\psi_{u_i}\psi^{-\nu-1}\psi\right)\right]_{u_j}$$

$$-2\lambda\nu\sum_{i,j=1}^{n+1}\left[\psi_{u_i}\left(\varphi_{u_i} - \lambda\nu\psi_{u_i}\psi^{-\nu-1}\varphi\right)\right]_{u_i} + 2\nu^2\lambda^2\sum_{i=1}^{n+1}\left(\psi^{-\nu-1}\psi_{u_i}\varphi^2\right)_{u_i},$$

$$d_{22}(\varphi H) = 2\lambda^3\nu^3\left(\psi^{-2\nu-2}\varphi^2|\nabla\psi|^2\sum_{i=1}^{n+1}\psi_{u_i}\right)_{u_i},$$

$$d_{23}(\varphi H) = -2\lambda^2\nu^2(\nu+1)\sum_{i=1}^{n+1}\left(|\nabla\psi|^2\psi^{-\nu-2}\psi_{u_i}\varphi^2\right)_{u_i},$$

$$d_{24}(\varphi H) = 2\lambda^2\nu^2\sum_{i=1}^{n+1}\left(\psi_{u_i}\psi^{-\nu-1}\varphi^2\right)_{u_i}.$$

Proof. For the auxiliary function $w = H\varphi$, we have

$$\varphi_{u_iu_i} = H^{-1}(w_{u_iu_i} + 2\lambda\nu\psi_{u_i}\psi^{-\nu-1}w_{u_i} + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i}w - \lambda\nu(\nu+1)\psi_{u_i}^2\psi^{-\nu-2}w$$

$$+\lambda^2\nu^2\psi^{-2\nu-2}w\psi_{u_i}^2).$$

Multiplying both sides of the last expression with H and adding from 1 to (n+1) with respect to i, we obtain

$$H^2(\Delta\varphi)^2 = (\sum_{i=1}^{n+1} w_{u_iu_i} + 2\lambda\nu\psi^{-\nu-1}\sum_{i=1}^{n+1}\psi_{u_i}w_{u_i} + \lambda\nu\psi^{-\nu-1}w\sum_{i=1}^{n+1}\psi_{u_iu_i}$$

$$-\lambda\nu(\nu+1)w\psi^{-\nu-2}\sum_{i=1}^{n+1}\psi_{u_i}^2 + \lambda^2\nu^2\psi^{-2\nu-2}w\sum_{i=1}^{n+1}\psi_{u_i}^2)^2.$$

The both hand sides of the last equality are multiplied with $\psi^{\nu+1}$, we get

$$\psi^{\nu+1}H^2(\Delta\varphi)^2 = \psi^{\nu+1}(\Delta w + \sum_{i=1}^{n+1}(\lambda^2\nu^2\psi^{-2\nu-2}\psi_{u_i}^2$$

$$-\lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{u_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i})w + 2\lambda\nu\psi^{-\nu-1}\sum_{i=1}^{n+1}\psi_{u_i}w_{u_i})^2.$$

Having

$$a = \Delta w + \sum_{i=1}^{n+1}(\lambda^2\nu^2\psi^{-2\nu-2}\psi_{u_i}^2 - \lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{u_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i})w,$$

$$b = 2\lambda\nu\psi^{-\nu-1}\sum_{i=1}^{n+1}\psi_{u_i}w_{u_i}$$

and applying the inequality $(a+b)^2 \geq 2ab$ to the last expression, it is obvious that

$$\psi^{\nu+1}H^2(\Delta\varphi)^2 = \psi^{\nu+1}(\Delta w + \sum_{i=1}^{n+1}(\lambda^2\nu^2\psi^{-2\nu-2}\psi_{u_i}^2$$

$$-\lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{u_i}^2 + \lambda\nu\psi^{-\nu-1}\psi_{u_iu_i})w + 2\lambda\nu\psi^{-\nu-1}\sum_{i=1}^{n+1}\psi_{u_i}w_{u_i})^2$$

$$\geq 4\lambda\nu(\sum_{i=1}^{n+1}\psi_{u_i}w_{u_i})\,[\Delta w + \sum_{j=1}^{n+1}\lambda\nu\psi^{-\nu-1}(\lambda\nu\psi^{-\nu-1}\psi_{u_j}^2$$

$$-(\nu+1)\psi^{-1}\psi_{u_j}^2 + \psi_{u_ju_j})w]\,. \tag{17}$$

Now let us consider each term in (17) respectively:

1. $4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_i}w_{u_ju_j} = 4\lambda\nu\sum_{i,j=1}^{n+1}\left(\psi_{u_i}w_{u_j}w_{u_i}\right)_{u_j} - 4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_j}w_{u_iu_j}$

$$= 4\lambda\nu\sum_{i,j=1}^{n+1}\left(\psi_{u_i}w_{u_j}w_{u_i}\right)_{u_j} - 2\lambda\nu\sum_{i,j=1}^{n+1}\left(\psi_{u_i}w_{u_j}^2\right)_{u_i} + 2\lambda\nu\sum_{i=1}^{n+1}w_{u_i}^2\,.$$

Additionally, since

$$w = \varphi H = e^{\lambda\psi^{-\nu}}\varphi,$$

$$w_{u_i} = H\left(\varphi_{u_i} - \lambda\nu\psi^{-\nu-1}\psi_{u_i}\varphi\right), \nu \geq 2, \delta > 1$$

then,

$$4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_i}w_{u_ju_j} = 4\lambda\nu\sum_{i,j=1}^{n+1}\left[\psi_{u_i}H(\varphi_{u_i} - \lambda\nu\psi^{-\nu-1}\psi_{u_i}\varphi)H\left(\varphi_{u_j} - \lambda\nu\psi^{-\nu-1}\psi_{u_j}\varphi\right)\right]_{u_j}$$

$$-2\lambda\nu\sum_{i,j=1}^{n+1}\left[\psi_{u_i}H^2\left(\varphi_{u_j} - \lambda\nu\psi^{-\nu-1}\psi_{u_j}\varphi\right)^2\right]_{u_j} + 2\lambda\nu\sum_{i=1}^{n+1}H^2\left(\varphi_{u_j} - \lambda\nu\psi^{-\nu-1}\psi_{u_j}\varphi\right)^2$$

$$= d_{21}(H\varphi) + 2\lambda\nu|\nabla\varphi|^2H^2|\nabla\psi|^2 + 2\lambda^3\nu^3\psi^{-2\nu-2}H^2\varphi^2|\nabla\psi|^2$$

$$-2\lambda^2\nu^2\sum_{i=1}^{n+1}\left(H^2\psi^{-\nu-1}\varphi^2\psi_{u_i}\right)_{u_i} - 2\lambda^2\nu^2(\nu+1)\varphi^2\psi^{-\nu-2}H^2|\nabla\psi|^2 + 2\lambda^2\nu^2\varphi^2\psi^{-\nu-1}H^2$$

$$\geq d_{21}(H\varphi) + 2\lambda\nu|\nabla\varphi|^2H^2 + 2\lambda^3\nu^3\psi^{-2\nu-2}|\nabla\psi|^2\varphi^2H^2. \tag{18}$$

2. From the conditions $|\nabla\psi| \geq \delta > 1, \eta^{-1} < \psi^{-1}, \nu > \frac{(n+2)\eta}{\delta(1-\eta)}$, we have the inequality is

$$\psi^{-1} - \frac{n+2}{2\nu}|\nabla\psi|^{-2} > 1 \text{ and}$$

$$4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_i}\lambda^2\nu^2\psi^{-2\nu-2}\psi_{u_j}^2w = 2\lambda^3\nu^3\sum_{i,j=1}^{n+1}\left(\psi_{u_i}\psi_{u_j}^2\psi^{-2\nu-2}w^2\right)_{u_i}$$

$$+4\lambda^3\nu^3(\nu+1)\psi^{-2\nu-3}|\nabla\psi|^4w^2 - 2\lambda^3\nu^3\psi^{-2\nu-2}|\nabla\psi|^2w^2 - 4\lambda^3\nu^3\psi^{-2\nu-2}(\sum_{i=2}^{n+1}\psi_{u_i}^2)w^2$$

$$\geq d_{22}(w) + 4\lambda^3\nu^4\psi^{-2\nu-2}|\nabla\psi|^4w^2\left(\psi^{-1} - \frac{n+1}{2\nu}|\nabla\psi|^{-2}\right)$$

$$\geq d_{22}(w) + 4\lambda^3\nu^4\delta^4\psi^{-2\nu-2}w^2$$

$$= d_{22}(H\varphi) + 4\lambda^3\nu^4\delta^4\psi^{-2\nu-2}\varphi^2H^2. \quad (19)$$

3.

$$-4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_i}\lambda\nu(\nu+1)\psi^{-\nu-2}\psi_{u_j}^2 w = -2\lambda^2\nu^2(\nu+1)\sum_{i,j=1}^{n+1}\left(\psi_{u_i}\psi_{u_j}^2\psi^{-\nu-2}w^2\right)_{u_i}$$

$$+2\lambda^2\nu^2(\nu+1)\psi^{-\nu-2}|\nabla\psi|^2w^2 + 4\lambda^2\nu^2(\nu+1)\left(\sum_{i=2}^{n+1}\psi_{u_i}^2\right)\psi^{-\nu-2}w^2$$

$$-2\lambda^2\nu^2(\nu+2)(\nu+1)\psi^{-\nu-3}|\nabla\psi|^4w^2$$

$$\geq d_{23}(w) - 2\lambda^2\nu^2(\nu+2)(\nu+1)\psi^{-\nu-3}|\nabla\psi|^4w^2$$

$$= d_{23}(H\varphi) - 2\lambda^2\nu^2(\nu+1)(\nu+2)\psi^{-\nu-3}|\nabla\psi|^4\varphi^2H^2. \quad (20)$$

4.

$$4\lambda\nu\sum_{i,j=1}^{n+1}\psi_{u_i}w_{u_i}\lambda\nu\psi^{-\nu-1}\psi_{u_iu_j}w = 2\lambda^2\nu^2\sum_{i=1}^{n+1}\left(\psi_{u_i}\psi^{-\nu-1}w^2\right)_{u_i} - 2\lambda^2\nu^2\psi^{-\nu-1}w^2$$

$$+2\lambda^2\nu^2(\nu+1)\psi^{-\nu-2}|\nabla\psi|^2w^2$$

$$\geq d_{24}(w) - 2\lambda^2\nu^2\psi^{-\nu-1}w^2$$

$$= d_{24}(H\varphi) - 2\lambda^2\nu^2\psi^{-\nu-1}\varphi^2H^2. \quad (21)$$

By inequalities (17)-(21), we obtain

$$\psi^{\nu+1}H^2(\Delta\varphi)^2 \geq d_2(\varphi H) + 2\lambda\nu|\nabla\varphi|^2H^2$$

$$+\lambda^2\nu^2\varphi^2H^2(4\lambda\nu^2\delta^4\psi^{-\nu-1} - 2 - 2(\nu+1)(\nu+2)\psi^{-2}|\nabla\psi|^4)\psi^{-\nu-1}. \quad (22)$$

Since

$$\lambda\nu^2\delta^4\psi^{-\nu-1} - 2(\nu+1)(\nu+2)\psi^{-2}|\nabla\psi|^4 - 2 \geq 0$$

for $\lambda \geq \lambda_0$, then (16) results from inequality (22). Thus Lemma 2 is proven.

Lemma 3.

Let the conditions of Lemma 2 be satisfy and let

$$\nu \geq \nu_1 = max\{\nu_0 + n + 1, 4(n+1) + 8(n+1)(\delta^2 + \eta^2) + 1\}.$$

Then

$$-2(n+1)\lambda\nu\varphi(\Delta\varphi)H^2 + \psi^{\nu+1}(\Delta\varphi)^2H^2$$

$$\geq 2\lambda^3\nu^3\psi^{-2\nu-2}\varphi^2H^2 + 2\lambda\nu n|\nabla\varphi|^2H^2 + 2(n+1)\lambda\nu d_1(\varphi) + d_2(\varphi). \quad (23)$$

Proof.

Multiplying equation (15) with $2(n+1)\lambda\nu$ and adding the resulting expression with (16), one gets

$$-2(n+1)\lambda\nu\varphi(\Delta\varphi)H^2 + \psi^{\nu+1}(\Delta\varphi)^2H^2$$

$\geq 2\lambda\nu n|\nabla\varphi|^2 H^2 + (3\lambda^3\nu^4\delta^4\psi^{-2\nu-2} - 4(n+1)\lambda^3\nu^3\delta^3\psi^{-2\nu-2}|\nabla\psi|^2$

$-2(n+1)\lambda^2\nu^2(\nu+1)\psi^{-\nu-2}\sum_{i=1}^{n+1}\psi_{u_i}^2 + 2(n+1)\lambda^2\nu^2\psi^{-\nu-1})\varphi^2 H^2$

$+2(n+1)\lambda\nu d_1(\varphi) + d_2(\varphi).$

Since $\nu \geq \nu_1$, (23) is obtained from the last inequality. This proves Lemma 3.

Proof of Theorem 1.

Considering Lemma 3 and inequality (13), we have

$$[(n+1)^2\lambda^2\nu^2 z^2 + 2c^2(z^2 + |\nabla z|^2)]H^2 + 2\psi^{\nu+1}H^2(|z|^2 + |\nabla z|^2)C^2$$
$$\geq ((n+1)^2\lambda^2\nu^2 z^2 + (\Delta z)^2)H^2 + \psi^{\nu+1}(\Delta z)^2 H^2$$
$$\geq 2(n+1)\lambda\nu|z||\Delta z|H^2 + (\Delta z)^2 H^2\psi^{\nu+1}$$
$$\geq -2(n+1)\lambda\nu|z||\Delta z|H^2 + \psi^{\nu+1}H^2(\Delta z)^2$$
$$\geq -2\lambda^3\nu^3\psi^{-2\nu-2}z^2 H^2 + 2\lambda\nu n|\nabla z|^2 H^2 + 2(n+1)\lambda\nu d_1(\varphi) + d_2(\varphi)$$

or

$$0 \geq H^2 z^2(2\lambda^3\nu^3\psi^{-2\nu-2} - [(n+1)^2\lambda^2\nu^2 + 2C^2 + 2\psi^{\nu+1}C^2]$$
$$+H^2|\nabla z|^2(2\lambda\nu n - 2C^2\psi^{\nu+1} - 2C^2) + 2(n+1)\lambda\nu d_1(\varphi) + d_2(\varphi). \quad (24)$$

If we take $\nu \geq \nu_1$, then it is clear that there exists $\overline{\lambda_0}$ such that

$$2\lambda^3\nu^3\psi^{-2\nu-2} - [(n+1)^2\lambda^2\nu^2 + 2C^2 + 2\psi^{\nu+1}C^2] \geq \lambda,$$
$$2\lambda\nu - 2C^2\psi^{\nu+1} - 2C^2 \geq \lambda$$

for $\lambda > \overline{\lambda_0}$.

In addition, in the domain $\Omega(u^0)$, $H \geq 1$. Then in $\Omega(u^0)$ for $\lambda \geq \overline{\lambda_0}$, it is possible to write the following inequality from (24):

$$0 \geq \lambda(z^2 + |\nabla z|^2) + 4\lambda\nu d_1(zH) + d_2(zH). \quad (25)$$

Integrating (25) over $\Omega(u^0)$ and taking the limit for $\lambda \to \infty$ of the resulting inequality, then from the definition of the function H and conditions (14) we have

$$\int_{\Omega(u^0)}(z^2 + |\nabla z|^2)d\Omega(u^0) \leq 0.$$

That is, in the domain $\Omega(u^0)$, we get $z \equiv 0$.

Varying the point u^0 at the boundary of the domain D where $\{y_n = 0\}$, one can prove that $z = 0$ in the set of points which satisfy the condition $0 \leq y_n \leq \delta$.

Similarly, we can prove that $z = 0$ for the remaining part of D.

To complete the proof of theorem 1, it is necessary to show that every solution of problem (11)-(12) satisfies the conditions of problem (13)-(14) at D for points $x > 0$.

Obviously each solution of (11) satisfies inequality (13) in the domain $\Omega(u^0)$ and for C dependent on the constant k. A similar case is valid for conditions (12)-(14). That is, the solution w of problem (11)-(12) satisfies conditions (13)-(14). Then, $w \equiv 0$ in the domain D. Moreover, $\bar{u} \equiv 0$ in D from the equality $w = \bar{u}_{y_1}$ and condition (9). On the other hand we see that $f \equiv 0$ from (7). Thus theorem 1 is proven.

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