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Partition of the Spectra for the Generalized Difference Operator B(r,s) on the Sequence Space cs

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Abstract: The goal of this study gives the approximate point spectrum, the defect spectrum and the compression spectrum of generalized difference operator $B(r,s)$ over the class of convergent series.

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***Keywords*:** Generalized difference operator, approximate point spectrum, defect spectrum, compression spectrum

Genelleştirilmiş Fark Operatörü B(r,s) nin cs Dizi Uzayı Üzerinde Spektral Ayrışımı

Özet: Bu çalışmanın amacı, yakınsak seriler sınıfında genelleştirilmiş fark operatörünün yaklaşık nokta spektrumunu, eksik spektrumu ve sıkıştırma spektrumu vermektir.

***Anahtar Kelimeler*:** Genelleştirilmiş fark operatörü, yaklaşık nokta spektrum, eksik spektrum, sıkıştırılmış spektrum

1. INTRODUCTION

We know that there is strictly the relationship between matrices and operators. The eigenvalues of matrices are contained in the spectrum of an operator. Spectral theory is a generalization of a set of eigenvalues of a linear operator in a finite dimensional vector space to an infinite dimensional vector space. The spectral theory of finite dimensional linear algebra may be provided as an attempt to expand the known decomposition results in similar situations in the infinite dimension.

Let X and Y be the Banach spaces, and $L: X \to Y$ be a bounded linear operator. By $R(L) = \{y \in Y : y = Lx,\ x \in X\}$, we denote the range of L and by $B(X)$, we show the set of all bounded linear operators on X into itself.

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Let $L:D(L)\to X$ be a linear operator, defined on $D(L)\subset X$, where $D(L)$ denote the domain of L and X is a complex normed space. Let $L_\lambda := \lambda I - L$ for $L\in B(X)$ and $\lambda \in \mathbb{C}$ where I is the identity operator. L_λ^{-1} is known as the resolvent operator of L_λ.

The resolvent set of L is the set of complex numbers λ of L such that L_λ^{-1} exists, is bounded and, is defined on a set which is dense in X, denoted by $\rho(L,X)$. Its complement is given by $\mathbb{C}-\rho(L;X)$ is called the spectrum of L, denoted by $\sigma(L,X)$.

The spectrum $\sigma(L,X)$ is union of three disjoint sets as follows: The point spectrum $\sigma_p(L,X)$ is the set such that L_λ^{-1} does not exist. If the operator L_λ^{-1} is defined on a dense subspace of X and is unbounded then $\lambda \in \mathbb{C}$ belongs to the continuous spectrum $\sigma_c(L,X)$ of L. Furthermore, we say that $\lambda \in \mathbb{C}$ belongs to the residual spectrum $\sigma_r(L,X)$ of L if the operator L_λ^{-1} exists, but its domain of definition (i.e. the range $R(\lambda I - L)$ of $(\lambda I - L)$ is not dense in X than in this case L_λ^{-1} may be bounded or unbounded. From above definitions we have

$$\sigma(L,X)=\sigma_p(L,X)\cup\sigma_c(L,X)\cup\sigma_r(L,X) \tag{1.1}$$

and

$$\sigma_p(L,X)\cap\sigma_c(L,X)=\varnothing,\sigma_p(L,X)\cap\sigma_r(L,X)=\varnothing,\sigma_r(L,X)\cap\sigma_c(L,X)=\varnothing.$$

1.1. Goldberg's Classification of Spectrum

If $T\in B(X)$, then there are three cases for $R(T)$:

(I) $R(T)=X$, (II) $\overline{R(T)}=X$, but $R(T)\neq X$, (III) $\overline{R(T)}\neq X$

and three cases for T^{-1}:

(1) T^{-1} exists and continuous, (2) T^{-1} exists but discontinuous, (3) T^{-1} does not exist.

If these cases are combined in all possible ways, nine different states are created. These are labelled by: I_1, I_2, I_3, II_1, II_2, II_3, III_1, III_2, III_3 (see [10]).

$\sigma(L,X)$ can be divided into subdivisions $I_2\sigma(L,X)=\varnothing$, $I_3\sigma(L,X)$, $II_2\sigma(L,X)$, $II_3\sigma(L,X)$, $III_1\sigma(L,X)$, $III_2\sigma(L,X)$, $III_3\sigma(L,X)$. For example, if $T=\lambda I - L$ is in a given state, III_2 (say), then we write $\lambda\in III_2\sigma(L,X)$.

By w, we will denote the space of all sequences. We will show ℓ_p, c, c_0 and bv for the space of all bounded, convergent, null and bounded variation sequences, respectively. Also by ℓ_p, bv_p we denote the spaces of all $p-$absolutely summable sequences and $p-$bounded variation sequences, respectively.

Many investigators studied the spectrum and fine spectrum of linear operators on some sequence spaces. In 2005, Altay and Başar [1] determined spectra and the fine spectra of generalized difference operator $B(r,s)$ on c_0 and c. In 2008, Bilgiç and Furkan [3] determined spectra and the fine spectra of generalized difference operator $B(r,s)$ on ℓ_p and bv_p, $(1 \leq p < \infty)$. In the last year, the spectral divisions of generalized difference matrices have studied. For example, in [6], Das calculated the spectrum and fine spectrum of the matrix $U(r_1, r_2; s_1, s_2)$ over the sequence space c_0. In [13], Tripathy and Das determined the spectra and fine spectra of $U(r,s)$ on the sequence space

$$cs = \left\{ x = (x_n) \in w : \lim_{n\to\infty} \sum_{i=0}^{n} x_i \text{ exists} \right\},$$

which is a Banach space with respect to the norm $\|x\|_{cs} = \sup_n \left| \sum_{i=0}^{n} x_i \right|$.

Matrices with finite elements or finite difference problems are often banded in numerical analysis. With the help of these matrices, we define relations between problem variables. The bandedness is confirmed with variables which are not conjugate in arbitrarily large distances. We can furthermore divide these matrices. For example, there are banded matrices with every element in the band is nonzero. We generally encounter these matrices while we are separating one-dimensional problems.

The band matrix $B(r,s)$ is represented by the matrix

$$B(r,s) = \begin{pmatrix} r & 0 & 0 & \cdots \\ s & r & 0 & \cdots \\ 0 & s & r & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \ (s \neq 0).$$

2. THE FINE SPECTRA FOR $B(r,s)$

Dutta and Tripathy [9] examined the fine spectra of the matrix $B(r,s)$ on the sequence space cs. Herein we mention the main results.

Lemma 1 ([10], p.59) *T has a dense range if and only if T^* is 1-1.*

Lemma 2 ([10], p.60) *T has a bounded inverse if and only if T^* is onto.*

Lemma 3 ([9], Lemma 2) *$B(r,s): cs \to cs$ is a bounded linear operator with $\|B(r,s)\|_{(cs,cs)} \leq |r| + |s|$.*

Theorem 1 ([9], Theorem 6) $\sigma(B(r,s),cs)=\{\lambda\in\mathbb{C}:|\lambda-r|\leq|s|\}$.

Theorem 2 ([9], Theorem 7) $\sigma_p(B(r,s),cs)=\emptyset$.

Let $T:cs\to cs$ is a bounded linear operator and A is its matrix representation. Then $T^*:cs^*\to cs^*$ is adjoint operator of T and A^t is matrix representation of T^*. Also cs is isomorphic to bv with the norm $\|x\|=\sum_{n=0}^{\infty}|x_n-x_{n+1}|$.

Theorem 3 ([9], Theorem 8) $\sigma_p(B(r,s)^*,cs^*)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

Theorem 4 ([9], Theorem 9) $\sigma_r(B(r,s),cs)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

Theorem 5 ([9], Theorem 10) $\sigma_c(B(r,s),cs)=\{\lambda\in\mathbb{C}:|\lambda-r|=|s|\}$.

Lemma 4

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}a_kb_{nk}\right)=\sum_{k=0}^{\infty}a_k\left(\sum_{n=k+1}^{\infty}b_{nk}\right)$$

where (a_k) and (b_{nk}) are nonnegative real numbers.

Proof.

$$\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}a_kb_{nk}\right)=\sum_{k=0}^{0}a_kb_{1k}+\sum_{k=0}^{1}a_kb_{2k}+\sum_{k=0}^{2}a_kb_{3k}+\sum_{k=0}^{3}a_kb_{4k}+\Lambda$$

$$=a_0b_{10}+(a_0b_{20}+a_1b_{21})+(a_0b_{30}+a_1b_{31}+a_2b_{32})$$

$$+(a_0b_{40}+a_1b_{41}+a_2b_{42}+a_3b_{43})+\Lambda$$

$$=a_0\sum_{n=1}^{\infty}b_{n0}+a_1\sum_{n=2}^{\infty}b_{n1}+a_2\sum_{n=3}^{\infty}b_{n2}+\Lambda$$

$$=\sum_{k=0}^{\infty}a_k\left(\sum_{n=k+1}^{\infty}b_{nk}\right).$$

Theorem 6 $III_1\sigma(B(r,s),cs)=\{\lambda\in\mathbb{C}:|\lambda-r|<|s|\}$.

Proof. We must obtain $x\in cs^*\cong bv$ for all $y\in cs^*\cong bv$ such that $(B(r,s)-\lambda I)^*x=y$. Then we have

$$\begin{aligned}(r-\lambda)x_0+sx_1 &= y_0\\(r-\lambda)x_1+sx_2 &= y_1\\&\vdots\\(r-\lambda)x_k+sx_{k+1} &= y_k\\&\vdots\end{aligned}$$

Assume that $x_0=0$. From the above equations, we get

$$\begin{aligned}x_1 &= \frac{y_0}{s}\\x_2 &= \frac{1}{s}y_1-\frac{r-\lambda}{s^2}y_0\\&\vdots\\x_n &= \frac{1}{s}\left(y_{n-1}-\frac{r-\lambda}{s}y_{n-2}+\left(\frac{r-\lambda}{s}\right)^2y_{n-3}-\left(\frac{r-\lambda}{s}\right)^3y_{n-4}+\cdots+(-1)^{n-1}\left(\frac{r-\lambda}{s}\right)^{n-1}y_0\right)\\&= \frac{1}{s}\sum_{k=0}^{n-1}(-1)^k\left(\frac{r-\lambda}{s}\right)^ky_{n-k-1}\end{aligned}$$

where $n=1,2,3,\cdots$. Now we must show that $x\in bv$.

$$\sum_{n=0}^{\infty}|x_n-x_{n+1}|=|x_0-x_1|+\frac{1}{|s|}\sum_{n=1}^{\infty}\left|\sum_{k=0}^{n-1}(-1)^k\left(\frac{r-\lambda}{s}\right)^ky_{n-k-1}-\sum_{k=0}^{n}(-1)^k\left(\frac{r-\lambda}{s}\right)^ky_{n-k}\right|$$

$$\leq\left|\frac{y_0}{s}\right|+\left|\frac{y_0}{s}\right|\sum_{n=1}^{\infty}\left|\frac{r-\lambda}{s}\right|^n+\frac{1}{|s|}\sum_{n=1}^{\infty}\sum_{k=0}^{n-1}\left|\frac{r-\lambda}{s}\right|^k|y_{n-k-1}-y_{n-k}|.$$

From Lemma 4, we get

$$\sum_{n=0}^{\infty}|x_n-x_{n+1}|\leq\left|\frac{y_0}{s}\right|\sum_{n=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^n+\frac{1}{|s|}\sum_{k=0}^{\infty}\sum_{n=k+1}^{\infty}\left|\frac{r-\lambda}{s}\right|^k|y_{n-k-1}-y_{n-k}|$$

$$=\frac{1}{|s|}\left(|y_0|\sum_{n=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^n+\sum_{k=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^k\sum_{n=k+1}^{\infty}|y_{n-k-1}-y_{n-k}|\right)$$

$$=\frac{1}{|s|}\left(|y_0|\sum_{n=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^n+\sum_{k=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^k\sum_{n=0}^{\infty}|y_n-y_{n+1}|\right)$$

$$=\frac{1}{|s|}\left(|y_0|+\|y\|_{bv}\right)\sum_{n=0}^{\infty}\left|\frac{r-\lambda}{s}\right|^n.$$

That is, for $\lambda\in\sigma_r(B(r,s),cs)$, the operator $(\lambda I-B(r,s))^*$ is surjective if and only if $|r-\lambda|<|s|$. Hence from Lemma 2, $\lambda I-B(r,s)$ has bounded inverse.

3. PARTITION OF THE SPECTRA FOR $B(r,s)$

We recall a sequence (x_k) in X a Weyl sequence for L if $\|x_k\|=1$ and $\|Lx_k\|\to 0$ as $k\to\infty$.

We call the set

$$\sigma_{ap}(L,X):=\{\lambda\in\mathbb{C}: \text{there exists a Weyl sequence for } \lambda I-L\} \tag{3.1}$$

the approximate point spectrum of L. Also,

$$\sigma_{\delta}(L,X):=\{\lambda\in\sigma(L,X): \lambda I-L \text{ is not surjective}\} \tag{3.2}$$

is called the defect spectrum of L. Finally,

$$\sigma_{co}(L,X):=\left\{\lambda\in\mathbb{C}: \overline{R(\lambda I-L)}\neq X\right\} \tag{3.3}$$

is called the compression spectrum. By definitions, we have, $\sigma_p(L,X)\subseteq\sigma_{ap}(L,X)$ and $\sigma_{co}(L,X)\subseteq\sigma_{\delta}(L,X)$. On the other hand, if we consider these subspectra with (1.1) we obtain that

$$\sigma_r(L,X)=\sigma_{co}(L,X)\setminus\sigma_p(L,X) \tag{3.4}$$

and

$$\sigma_c(L,X)=\sigma(L,X)\setminus\sigma_p(L,X)\cup\sigma_{co}(L,X)] \tag{3.5}$$

Proposition 1 ([2], Proposition 1.3) *Let $T\in B(X)$ and its adjoint $T^*\in B(X^*)$ then the following relations are hold:*

(a) $\sigma(T^*,X^*)=\sigma(T,X)$,

(b) $\sigma_c(T^*,X^*)\subseteq\sigma_{ap}(T,X)$,

(c) $\sigma_{ap}(T^*,X^*)=\sigma_{\delta}(T,X)$,

(d) $\sigma_{\delta}(T^*,X^*)=\sigma_{ap}(T,X)$,

(e) $\sigma_p(T^*,X^*)=\sigma_{co}(T,X)$,

(f) $\sigma_{co}(T^*,X^*)\supseteq\sigma_p(T,X)$,

(g) $\sigma(T,X)=\sigma_{ap}(T,X)\cup\sigma_p(T^*,X^*)=\sigma_p(T,X)\cup\sigma_{ap}(T^*,X^*)$.

By the definitions given above, we can write following Table 1.

Table 1. Subdivisions of the spectrum of a linear operator.

		1	2	3
		$R(\lambda;L)$ exits and is unbounded	$R(\lambda;L)$ exits and is unbounded	$R(\lambda;L)$ Does not exits
I	$R(\lambda I-L)=X$	$\lambda\in\rho(L)$ $\lambda\in\rho(L)$	-	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$
II	$\overline{R(\lambda I-L)}\neq X$	$\lambda\in\rho(L)$	$\lambda\in\sigma_c(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$
III	$\overline{R(\lambda I-L)}\neq X$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$	$\lambda\in\sigma_r(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$	$\lambda\in\sigma_p(L)$ $\lambda\in\sigma_{ap}(L)$ $\lambda\in\sigma_\delta(L)$ $\lambda\in\sigma_{co}(L)$

The decomposition of the spectrum which is defined by Goldberg can be obtained in the above-mentioned articles. However, in [7] Durna and Yildirim investigated subdivision of the spectra for factorable matrices on c_0 and in [4] Başar, Durna and Yildirim investigated partition of the spectra for generalized difference operator $B(r,s)$ over certain sequence spaces and in [8] Durna, studied partition of the spectra for Δ^{uv} over the sequence spaces c_0 and c. In [14], Tripathy and Avinoy studied the spectra of the operator $D(r,0,0,s)$ on sequence spaces c_0 and c. In [11], Paul and Tripathy investigated the spectrum of the operator $D(r,0,0,s)$ over the sequence spaces ℓ_p and bv_p. In [12], Paul and Tripathy investigated the spectrum of the operator $D(r,0,0,s)$ on the sequence space bv_0. In [5], Das and Tripathy examined the spectra and fine spectra of the matrix $B(r,s,t)$ on the sequence space cs.

Corollary 1 $III_2\sigma(B(r,s),cs)=\varnothing$.

Proof. It is clear from Theorem 4 and Theorem 6, since

$$III_2\sigma(B(r,s),cs)=\sigma_r((B(r,s),cs))\setminus III_1\sigma((B(r,s),cs)).$$

Corollary 2 $I_3\sigma(B(r,s),cs)=II_3\sigma(B(r,s),cs)=III_3\sigma(B(r,s),cs)=\varnothing$.

Proof. Since $\sigma_p(A,cs)=I_3\sigma(A,cs)\cup II_3\sigma(A,cs)\cup III_3\sigma(A,cs)$ from Table 1, we get the required result from Theorem 2.

Theorem 7 *(a)* $\sigma_{ap}(B(r,s),cs)=\{\lambda\in\mathbb{C}:|\lambda-r|=|s|\}$,

(b) $\sigma_\delta(B(r,s),cs) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$,

(c) $\sigma_{co}(B(r,s),cs) = \{\lambda \in \mathbb{C}: |\lambda - r| < |s|\}$.

Proof. (a) From Table 1,

$$\sigma_{ap}(B(r,s),cs) = \sigma(B(r,s),cs) \setminus III_1\sigma(B(r,s),cs).$$

By Theorem 1 and Corollary 1, we have $\sigma_{ap}(B(r,s),cs) = \{\lambda \in \mathbb{C}: |\lambda - r| = |s|\}$.

(b) From Table 1, we have

$$\sigma_\delta(B(r,s),cs) = \sigma(B(r,s),cs) \setminus I_3\sigma(B(r,s),cs).$$

So using Theorem 1 and Corollary 2, we obtain the result.

(c) By Proposition 1 (e), we get

$$\sigma_p(B(r,s)^*,bv) = \sigma_{co}(B(r,s),cs).$$

Using Theorem 3, we obtain the result.

Corollary 3 *(a)* $\sigma_{ap}(B(r,s)^*,bv) = \{\lambda \in \mathbb{C}: |\lambda - r| \leq |s|\}$,

(b) $\sigma_{ap}(B(r,s)^*,bv) = \{\lambda \in \mathbb{C}: |\lambda - r| = |s|\}$.

Proof. Using Proposition 1 (c) and (d), we have

$$\sigma_{ap}(B(r,s)^*, cs^* \cong bv) = \sigma_\delta(B(r,s),cs)$$

and

$$\sigma_\delta(B(r,s)^*, cs^* \cong bv) = \sigma_{ap}(B(r,s),cs).$$

Using Theorem 7 (a) and (b), we get the required results.

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