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A NEW GENERALIZED FRACTIONAL DERIVATIVE AND INTEGRAL

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Abstract. In this article, we introduce a new general definition of fractional derivative and fractional integral, which depends on an unknown kernel. By using these definitions, we obtain the basic properties of fractional integral and fractional derivative such as Product Rule, Quotient Rule, Chain Rule, Roll's Theorem and Mean Value Theorem. We give some examples.

1. Introduction

The main aim of this paper is to introduced limit definition of the derivative of a function which obeys classical properties including: linearity, Product Rule, Quotient Rule, Chain Rule, Rolle's Theorem and Mean Value Theorem.

Today, there are many fractional integral and fractional derivative definitions such as Riemann-Liouville, Caputo, Grünwald-Letnikov, Hadamard, Riesz. For these, please see [3], [5], [11]. For more information on the Fractional Calculus, please see ([1], [6], [8], [9], [10], [12]-[14]).

Here, all fractional derivatives do not provide some properties such as Product Rule, Quotient Rule, Chain Rule, Roll's Theorem and Mean Value Theorem.

To overcome some of these and other difficulties, Khalil et al. [7], came up with an interesting idea that extends the familiar limit definition of the derivative of a function given by the following T_α

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon t^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{1.1}$$

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In [2], Almeida et al. introduced limit definition of the derivative of a function as follows,

$$f^{(\alpha)}(t) = \lim_{\varepsilon \to 0} \frac{f\left(t + \varepsilon k(t)^{1-\alpha}\right) - f(t)}{\varepsilon}. \tag{1.2}$$

Recently, in [4] Katugampola introduced the idea of fractional derivative

$$D_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f\left(te^{\varepsilon t^{-\alpha}}\right) - f(t)}{\varepsilon}. \tag{1.3}$$

2. Generalized new fractional derivative

In this paper, we introduce a new fractional derivative which is generalized the results obtained in [2], [4], [7].

In this section we present the definition of the Generalized new fractional derivative. We provided representations for the Product Rule, Quotient Rule, Chain Rule, Roll's Theorem and Mean Value Theorem. Also, we give some applications.

Definition 2.1. Let $k : [a,b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t)$, $k'(t) \neq 0$, whenever $t > a$. Given a function $f : [a,b] \to \mathbb{R}$ and $\alpha \in (0,1)$ a real, we say that the generalized fractional derivative of f of order α is defined by,

$$D^\alpha(f)(t) := \lim_{\epsilon \to 0} \frac{f\left(t - k(t) + k(t)\, e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) - f(t)}{\epsilon} \tag{2.1}$$

exist. If f is $\alpha-$differentiable in some $(0,a)$, $\alpha > 0$, $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exist, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t). \tag{2.2}$$

We can write $f^{(\alpha)}(t)$ for $D^\alpha(f)(t)$ to denote the generalized fractional derivatives of f of order α.

Remark 2.1. When $k(t) = t$ in (2.1), it turns out to be the definition for derivatives of a function, in [4].

Remark 2.2. When $\alpha \to 1$ and $k(t) = t$ in (2.1), it turns out to be the classical definition for derivatives of a function, $f^{(\alpha)}(t) = f'(t)$.

Theorem 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function and $t > a$. Then, f is a $\alpha-$differentiable at t and*

$$f^{(\alpha)}(t) = \frac{(k(t))^{1-\alpha}}{k'(t)} \frac{df}{dt}(t).$$

Also, if f' is continuous at $t = a$, then

$$f^{(\alpha)}(a) = \frac{(k(a))^{1-\alpha}}{k'(a)} \frac{df}{dt}(a).$$

Proof. From definition 2.1, we have

$$\begin{aligned} D^{\alpha}(f)(t) &= \lim_{\epsilon\to 0}\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)-f(t)}{\epsilon} \\ &= \lim_{\in\to 0}\frac{\left(t-k(t)+k(t)\left[1+\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}+\frac{\left(\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}\right)^{2}}{2!}+\ldots\right]\right)-f(t)}{\varepsilon} \\ &= \lim_{\epsilon\to 0}\frac{f\left(t+\epsilon\frac{(k(t))^{1-\alpha}}{k'(t)}[1+O(\epsilon)]\right)-f(t)}{\epsilon}. \end{aligned}$$

Taking

$$h=\epsilon\frac{(k(t))^{1-\alpha}}{k'(t)}\left[1+O(\epsilon)\right]$$

we have,

$$\begin{aligned} D^{\alpha}(f)(t) &= \frac{(k(t))^{1-\alpha}}{k'(t)}\lim_{\epsilon\to 0}\left[1+O(\epsilon)\right]\lim_{h\to 0}\frac{f(t+h)-f(t)}{h} \\ &= \frac{(k(t))^{1-\alpha}}{k'(t)}\frac{df}{dt}(t). \end{aligned}$$

$\square$

Theorem 2.2. *If a function $f:[a,b]\to\mathbb{R}$ is $\alpha-$differentiable at $a>0,\ \alpha\in(0,1]$, then f is continuous at a.*

Proof. Since

$$f\left(a-k(a)+k(a)e^{\varepsilon\frac{(k(a))^{-\alpha}}{k'(a)}}\right)-f(a)=\frac{f\left(a-k(a)+k(a)e^{\varepsilon\frac{(k(a))^{-\alpha}}{k'(a)}}\right)-f(a)}{\epsilon}\epsilon,$$

we have

$$\lim_{\epsilon\to 0}\left[f\left(a-k(a)+k(a)e^{\varepsilon\frac{(k(a))^{-\alpha}}{k'(a)}}\right)-f(a)\right]=\lim_{\epsilon\to 0}\frac{\left[f\left(a-k(a)+k(a)e^{\varepsilon\frac{(k(a))^{-\alpha}}{k'(a)}}\right)-f(a)\right]}{\epsilon}\lim_{\epsilon\to 0}\epsilon.$$

Let $h=\epsilon\frac{(k(t))^{1-\alpha}}{k'(t)}\left[1+O(\epsilon)\right]$. Then,

$$\lim_{h\to 0}\left[f(a+h)-f(a)\right]=D^{\alpha}(f)(a).0$$

and

$$\lim_{h\to 0}f(a+h)=f(a).$$

This completes the proof. $\square$

Theorem 2.3. *Let $\alpha\in(0,1]$ and f,g be $\alpha-$differentiable at a point $t>0$. Then,*

1. $D^{\alpha}(af+bg)(t)=aD^{\alpha}(f)(t)+bD^{\alpha}(g)(t)$, for all $a,b\in\mathbb{R}$ (linearity).

2.$D^{\alpha}(t^{n})=\frac{(k(t))^{1-\alpha}}{k'(t)}nt^{n-1}$ for all $n\in\mathbb{R}$.

3. $D^{\alpha}(c) = 0$, for all constant functions $f(t) = c$.

4. $D^{\alpha}(fg)(t) = f(t) D^{\alpha}(g)(t) + g(t) D^{\alpha}(f)(t)$ (Product Rule).

5. $D^{\alpha}\left(\frac{f}{g}\right)(t) = \dfrac{f(t) D_{\alpha}(g)(t) - g(t) D_{\alpha}(f)(t)}{[g(t)]^2}$ (Quotient Rule).

6. $D^{\alpha}(f \circ g)(t) = \frac{(k(t))^{1-\alpha}}{k'(t)} f'(g(t)) D'(g)(t)$ (Chain rule).

Proof. Part (1) and (3) follow directly from the definition. Let us prove (2), (4), (5) and (6) respectively. Now, for fixed $\alpha \in (0,1]$, $n \in \mathbb{R}$ and $t > 0$, we have

$$\begin{aligned} D^{\alpha}(t^n) &= \lim_{\epsilon \to 0} \frac{\left(t - k(t) + k(t) e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right)^n - t^n}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{\left(t + \epsilon \frac{(k(t))^{1-\alpha}}{k'(t)} [1 + O(\epsilon)]\right)^n - t^n}{\epsilon} \\ &= \frac{(k(t))^{1-\alpha}}{k'(t)} n t^{n-1}. \end{aligned}$$

This completes proof of (2). Then, we shall prove (4). To this end, since f, g are $\alpha-$differentiable at $t > 0$, note that,

$$\begin{aligned} & D^{\alpha}(fg)(t) \\ = & \lim_{\epsilon \to 0} \frac{f\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) g\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) - f(t)g(t)}{\epsilon} \\ = & \lim_{\epsilon \to 0} \left[\frac{f\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) g\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) - f(t) g\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right)}{\epsilon} \right. \\ & \left. + \frac{f(t) g\left(t-k(t)+k(t)e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) - f(t)g(t)}{\epsilon} \right] \end{aligned}$$

$$
\begin{aligned}
&= \lim_{\epsilon\to 0}\left[\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)-f(t)}{\epsilon}g\left(t-k\left(t\right)+k\left(t\right)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)\right] \\
&\quad +f\left(t\right)\lim_{\epsilon\to 0}\frac{g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)-g(t)}{\epsilon} \\
&= D^{\alpha}\left(f\right)\left(t\right)\lim_{\epsilon\to 0}\left[g\left(t-k\left(t\right)+k\left(t\right)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)\right]+f\left(t\right)D^{\alpha}\left(g\right)\left(t\right) \\
&= g\left(t\right)D^{\alpha}\left(f\right)\left(t\right)+f\left(t\right)D^{\alpha}\left(g\right)\left(t\right).
\end{aligned}
$$

Since g is continuous at t, $\lim_{\epsilon\to 0}\left[g\left(t-k\left(t\right)+k\left(t\right)e^{\varepsilon k(t)^{-\alpha}}\right)\right]=g\left(t\right)$. This completes the proof of (4). Next, we prove (5). Similarly,

$$
\begin{aligned}
&D^{\alpha}\left(\frac{f}{g}\right)\left(t\right) \\
&= \lim_{\epsilon\to 0}\frac{\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)}{g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)}-\frac{f(t)}{g(t)}}{\epsilon} \\
&= \lim_{\epsilon\to 0}\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)g(t)-f(t)g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)}{\epsilon g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)g(t)} \\
&= \lim_{\epsilon\to 0}\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)g(t)-f(t)g(t)+f(t)g(t)-f(t)g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)}{\epsilon g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)g(t)} \\
&= \lim_{\epsilon\to 0}\frac{1}{g\left(t-k\left(t\right)+k\left(t\right)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)g\left(t\right)} \\
&\quad\times\left[\frac{f\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)-f(t)}{\epsilon}g\left(t\right)-f\left(t\right)\frac{g(t)-g\left(t-k(t)+k(t)e^{\varepsilon\frac{(k(t))^{-\alpha}}{k'(t)}}\right)}{\epsilon}\right] \\
&= \frac{f\left(t\right)D^{\alpha}\left(g\right)\left(t\right)-g\left(t\right)D^{\alpha}\left(f\right)\left(t\right)}{\left(g\left(t\right)\right)^{2}}.
\end{aligned}
$$

We have implicitly assumed here that $f^{(\alpha)}$ and $g^{(\alpha)}$ exist and that $g(t) \neq 0$. Finally, we prove (6). We have from the definition that

$$\begin{aligned} D^{\alpha}(f \circ g)(t) &= \lim_{\epsilon \to 0} \frac{(f \circ g)\left(t - k(t) + k(t)\, e^{\varepsilon \frac{(k(t))^{-\alpha}}{k'(t)}}\right) - (f \circ g)(t)}{\epsilon} \\ &= \lim_{\epsilon \to 0} \frac{(f \circ g)\left(t + \epsilon \frac{(k(t))^{1-\alpha}}{k'(t)}\left[1 + O(\epsilon)\right]\right) - (f \circ g)(t)}{\epsilon}. \end{aligned}$$

Let $h = \epsilon \frac{(k(t))^{1-\alpha}}{k'(t)}\left[1 + O(\epsilon)\right]$ such that

$$\begin{aligned} D^{\alpha}(f \circ g)(t) &= \lim_{\epsilon \to 0} \frac{(f \circ g)\left(t + \epsilon \frac{(k(t))^{1-\alpha}}{k'(t)}\left[1 + O(\epsilon)\right]\right) - (f \circ g)(t)}{\epsilon} \\ &= \lim_{h \to 0} \frac{(f \circ g)(t+h) - (f \circ g)(t)}{\frac{k'(t)(k(t))^{\alpha-1} h}{1+O(\epsilon)}}. \end{aligned}$$

Therefore, we have

$$D^{\alpha}(f \circ g)(t) = \frac{(k(t))^{1-\alpha}}{k'(t)} f'(g(t))\, D'(g)(t).$$

This completes the proof of the theorem. □

Now, we will give the derivatives of some special functions.

Theorem 2.4. *Let $a, n \in \mathbb{R}$ and $\alpha \in (0,1]$. Then we have the following results.*

1. $D^{\alpha}(1) = 0$,
2. $D^{\alpha}(e^{ax}) = a\frac{(k(x))^{1-\alpha}}{k'(x)} e^{ax}$,
3. $D^{\alpha}(\sin(ax)) = a\frac{(k(x))^{1-\alpha}}{k'(x)} \cos(ax)$,
4. $D^{\alpha}(\cos(ax)) = -a\frac{(k(x))^{1-\alpha}}{k'(x)} \sin(ax)$,
5. $D^{\alpha}(\log_a bx) = \dfrac{1}{x}\frac{(k(x))^{1-\alpha}}{k'(x)}\frac{1}{\ln a}$,
6. $D^{\alpha}\left(a^{bx}\right) = b\frac{(k(x))^{1-\alpha}}{k'(x)} a^{bx} \ln a$.

When $\alpha = 1$ and $k(t) = t$ in Theorem 2.4, it turns out to be the classical derivatives of a function.

Theorem 2.5 (Rolle's theorem for $\alpha-$generalized Fractional Differentiable functions)**.** *Let $a > 0$ and $f : [a,b] \to \mathbb{R}$ be a function with the properties that,*

i. f is continuous on $[a,b]$,
ii. f is a α-differentiableon (a,b) for some $\alpha \in (0,1)$,
iii. $f(a) = f(b)$.
Then, there exist $c \in (a,b)$, such that $D^{\alpha}(f)(c) = 0$.

Proof. We will prove this theorem by using contradiction. Since f is continuous on $[a,b]$ and $f(a)=f(b)$, there is $c\in(a,b)$ at which the function has a local extrema. Then,

$$D^{\alpha}(f)(c)=\lim_{\epsilon\to 0^{-}}\frac{\left[f\left(c-k(c)+k(c)e^{\varepsilon\frac{(k(c))^{-\alpha}}{k'(c)}}\right)-f(c)\right]}{\epsilon}=\lim_{\epsilon\to 0^{+}}\frac{\left[f\left(c-k(c)+k(c)e^{\varepsilon\frac{(k(c))^{-\alpha}}{k'(c)}}\right)-f(c)\right]}{\epsilon}.$$

But, the two limits have opposite signs. Hence, $D^{\alpha}(f)(c)=0$. □

When $\alpha=1$ and $k(t)=t$ in Theorem 2.5, it turns out to be the classical Rolles's Theorem.

Theorem 2.6 (Mean value theorem for Generalized fractional differentiable functions)**.** *Let $\alpha\in(0,1]$ and $f:[a,b]\to\mathbb{R}$ be a continuous on $[a,b]$ and an α-generalized fractional differentiable mapping on (a,b) with $0\le a<b$. Let $k:[a,b]\to\mathbb{R}$ be a continuous nonnegative map such that $k(t)$, $k'(t)\neq 0$. Then, there exists $c\in(a,b)$, such that*

$$D^{\alpha}(f)(c)=\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}. \tag{2.3}$$

Proof. Let h be a constant. Consider the function,

$$G(x)=f(x)+h\frac{k^{\alpha}(x)}{\alpha}. \tag{2.4}$$

G is continuous functions on $[a,b]$ and integrable $\forall x\in(a,b)$. Here, if we choose $G(a)=G(b)$, then we have

$$f(a)+h\frac{k^{\alpha}(a)}{\alpha}=f(b)+h\frac{k^{\alpha}(b)}{\alpha}.$$

Thus,

$$h=-\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}. \tag{2.5}$$

Using (2.5) in (2.4), it follows that

$$G(x)=f(x)-\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}\frac{k^{\alpha}(x)}{\alpha}. \tag{2.6}$$

$$\begin{aligned}
D^{\alpha}(G)(x) &= D^{\alpha}(f)(x)-\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}D^{\alpha}\left(\frac{k^{\alpha}(x)}{\alpha}\right)\\
&= D^{\alpha}(f)(x)-\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}\frac{(k(t))^{1-\alpha}}{k'(t)}\frac{d}{dt}\left(\frac{k^{\alpha}(x)}{\alpha}\right)\\
&= D^{\alpha}(f)(x)-\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}.
\end{aligned}$$

Then, the function g satisfies the condition of the generalized fractional Rolle's theorem. Hence, there exist $c\in(a,b)$, such that $D^{\alpha}(G)(c)=0$. Using the fact that $D^{\alpha}\left(\frac{k^{\alpha}(x)}{\alpha}\right)=1$, we have

$$f^{(\alpha)}(x)=\frac{f(b)-f(a)}{\frac{k^{\alpha}(b)}{\alpha}-\frac{k^{\alpha}(a)}{\alpha}}.$$

Therefore, we get desired result. $\square$

When $\alpha = 1$ and $k(t) = t$ in Theorem 2.5, it turns out to be the classical Mean Value Theorem.

3. Generalized new fractional integral

Now we introduce the generalized fractional integral as follows:

Definition 3.1 (Generalized Fractional Integral)**.** Let $a \geq 0$ and $t \geq a$. Also, let f be a function defined on $(a, t]$ and $\alpha \in \mathbb{R}$. Let $k : [a, b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t),\ k'(t) \neq 0$. Then, the $\alpha-$generalized fractional integral of f is defined by,

$$I^{\alpha}(f)(t) = \int_a^b \frac{k'(x) f(x)}{(k(x))^{1-\alpha}} dx$$

if the Riemann improper integral exist.

Theorem 3.1 (Inverse property)**.** *Let $a \geq 0$ and $\alpha \in (0, 1)$. Also, let f be a continuous function such that $I^a f$ exist. Let $k : [a, b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t),\ k'(t) \neq 0$. Then, for all $t > a$, we have*

$$D^a [I^a f(t)] = f(t).$$

Proof. Since f is continuous, then $I^a f(t)$ is clearly differentiable. Hence,

$$\begin{aligned} D^a [I^a (f)(t)] &= \frac{(k(t))^{1-\alpha}}{k'(t)} \frac{d}{dt} I^a(t) \\ &= \frac{(k(t))^{1-\alpha}}{k'(t)} \frac{d}{dt} \int_a^t \frac{f(x) k'(x)}{(k(x))^{1-\alpha}} dx \\ &= \frac{(k(t))^{1-\alpha}}{k'(t)} \frac{f(t) k'(t)}{(k(t))^{1-\alpha}} \\ &= f(t). \end{aligned}$$

$\square$

Theorem 3.2. *Let $f : (a, b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Let $k : [a, b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t),\ k'(t) \neq 0$. Then, for all $t > a$ we have*

$$I^a [D^a (f)(t)] = f(t) - f(a). \tag{3.1}$$

Proof.

$$\begin{aligned} I^a\left[D^a(f)(t)\right] &= \int_a^t \frac{k'(x)}{(k(x))^{1-\alpha}} D^a(f)(x)dx \\ &= \int_a^t \frac{k'(x)}{(k(x))^{1-\alpha}} \frac{(k(x))^{1-\alpha}}{k'(x)} \frac{df}{dx}(x)dx \\ &= \int_a^t \frac{df}{dx}(x)dx \\ &= f(t) - f(a). \end{aligned}$$

□

Theorem 3.3. ***(Integration by parts)*** *Let $f, g : [a,b] \to \mathbb{R}$ be two functions such that fg is $\alpha-$differentiable. Then*

$$\int_a^b \frac{f(x) D^\alpha(g)(x) k'(x)}{(k(x))^{1-\alpha}} dx = f(x) g(x)|_a^b - \int_a^b \frac{g(x) D^\alpha(f)(x) k'(x)}{(k(x))^{1-\alpha}} dx.$$

Proof. If f and g are two $\alpha-$differentiable functions, then the product rule gives us:

$$D^\alpha\left[f(t) g(t)\right] = f(t) D^\alpha(g)(t) + g(t) D^\alpha(f)(t). \tag{3.2}$$

Multiplying (3.2) by $\frac{k'(x)}{(k(x))^{1-\alpha}}$ and integrating with respect to x over (a,b), we obtain:

$$\begin{aligned} \int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}} \left[D^\alpha\left[f(x) g(x)\right]\right] dx &= \int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}} \left[f(x) D^\alpha(g)(x)\right] dx \\ &\quad + \int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}} \left[g(x) D^\alpha(f)(x)\right] dx. \end{aligned}$$

By Theorem 3.2, we have:

$$f(b) g(b) - f(a) g(a) = \int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}} \left[f(x) D^\alpha(g)(x)\right] dx + \int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}} \left[g(x) D^\alpha(f)(x)\right] dx.$$

Thus,

$$\int_a^b \frac{f(x) D^\alpha(g)(x) k'(x)}{(k(x))^{1-\alpha}} dx = f(x) g(x)|_a^b - \int_a^b \frac{g(x) D^\alpha(f)(x) k'(x)}{(k(x))^{1-\alpha}} dx.$$

This is the integration by parts formula. □

For $\alpha = 1$ and $k(t) = t$ this reduces to the classical integration by parts formula.

Theorem 3.4. *Let f and g be functions satisfying the following* *i*. continuous on $[a,b]$,

ii. bounded and integrable functions on $[a,b]$.

In addition, let $g(x)$ be nonnegative (or nonpositive) on $[a,b]$. Let $k:[a,b]\to\mathbb{R}$ be a continuous nonnegative map such that $k(t)\,,\ k'(t)\neq 0$. Let us set $m=\inf\{f(x):x\in[a,b]\}$ and $M=\sup\{f(x):x\in[a,b]\}$. Then there exists a number ξ in (a,b) such that

$$\int_a^b \frac{f(x)\,g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx=\xi\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx. \tag{3.3}$$

If f continuous on $[a,b]$, then for $\exists x_0\in[a,b]$

$$\int_a^b \frac{f(x)\,g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx=f(x_0)\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx. \tag{3.4}$$

Proof. If $m=\inf f$, $M=\sup f$ and $g(x)\geq 0$ in $[a,b]$, then, we get

$$mg(x)<f(x)g(x)<Mg(x). \tag{3.5}$$

Multiplying (3.5) by $\frac{k'(x)}{(k(x))^{1-\alpha}}$ and integrating (3.5) with respect to x over (a,b), we obtain:

$$m\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx<\int_a^b \frac{f(x)g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx<M\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx. \tag{3.6}$$

Then there exists a number ξ in $[m,M]$ such that

$$\int_a^b \frac{f(x)g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx=\xi\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx.$$

When $g(x)<0$, the proof is done in a similar way.

By the intermediate value theorem, f attains every value of the interval $[m,M]$, so for some x_0 in $[a,b]$, $f(x_0)=\xi$. Then

$$\int_a^b \frac{f(x)g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx=f(x_0)\int_a^b \frac{g(x)\,k'(x)}{(k(x))^{1-\alpha}}dx.$$

If $g(x)=0$, equality (3.3) becomes obvious; if $g(x)>0$, then (3.6) implies

$$\xi=\frac{\int_a^b \frac{f(x)g(x)k'(x)}{(k(x))^{1-\alpha}}dx}{\int_a^b \frac{g(x)k'(x)}{(k(x))^{1-\alpha}}dx}.$$

which yields the desired result (3.3). In particular, when $g(x) = 1$, we get from Theorem 3.4 the following result

$$\begin{aligned}\int_a^b \frac{f(x)\,k'(x)}{(k(x))^{1-\alpha}}dx &= f(x_0)\int_a^b \frac{k'(x)}{(k(x))^{1-\alpha}}dx \\ &= f(x_0)\left(\frac{k^\alpha(b)}{\alpha} - \frac{k^\alpha(a)}{\alpha}\right).\end{aligned}$$

Thus, we have

$$f(x_0) = \frac{1}{\frac{k^\alpha(b)}{\alpha} - \frac{k^\alpha(a)}{\alpha}}\int_a^b \frac{f(x)\,k'(x)}{(k(x))^{1-\alpha}}dx. \tag{3.7}$$

This (3.7) is called the mean value or variance of the f function. □

For $\alpha = 1$ and $k(t) = t$ this reduces to the classical mean value theorem of integral calculus,

$$f(x_0) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Theorem 3.5. *Let $a \geq 0$ and $\alpha \in (0,1]$. Also, let $f, g : [a,b] \to \mathbb{R}$ be a continuous function. Let $k : [a,b] \to \mathbb{R}$ be a continuous nonnegative map such that $k(t)$, $k'(t) \neq 0$. Then,*

i. $\int_a^b (f(x) + g(x))\frac{k'(x)}{(k(x))^{1-\alpha}}dx = \int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx + \int_a^b \frac{g(x)k'(x)}{(k(x))^{1-\alpha}}dx,$

ii. $\int_a^b \lambda\frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx = \lambda\int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx,\ \lambda \in \mathbb{R},$

iii. $\int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx = -\int_b^a \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx,$

iv. $\int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx = \int_a^c \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx + \int_c^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx,$

v. $\int_a^a \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx = 0,$

vi. if $f(x) \geq 0$ for all $x \in [a,b]$, then $\int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx \geq 0,$

vii. $\left|\int_a^b \frac{f(x)k'(x)}{(k(x))^{1-\alpha}}dx\right| \leq \int_a^b \frac{|f(x)|k'(x)}{(k(x))^{1-\alpha}}dx.$

Proof. The relations follow from Definition 3.1 and Theorem 3.2, analogous properties of generalized fractional integral, and the properties of section 2 for the generalized fractional derivative. □

4. CONCLUSION

In this note, we have defined a new derivative and integral, which includes some existing definitions in the literature with appropriate selection of $k(x)$. With the help of these definitions, we have achieved some important results. Also, we gave some examples.

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