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NUMERICAL SOLUTION OF SINGULAR INVERSE NODAL PROBLEM BY USING CHEBYSHEV POLYNOMIALS

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Abstract. In this study, we consider Sturm-Liouville problem in two cases: the first case having no singularity and the second case having a singularity at zero. Then, we calculate the eigenvalues and the nodal points and present the uniqueness theorem for the solution of the inverse problem by using a dense subset of the nodal points in two given cases. Also, we use Chebyshev polynomials of the first kind for calculating the approximate solution of the inverse nodal problem in these cases. Finally, we present the numerical results by providing some examples.

1. Introduction

Inverse spectral problems mean recovering operators by using their spectral characteristics as spectrum, norming constants and nodal points. The use of such problems in mathematical physics has led many researchers to study in this area. Inverse spectral problems are divided into two parts. One of them is inverse eigenvalue problem and the other one is inverse nodal problem. Inverse eigenvalue problem has been studied for along time by many authors [1,8,16,21,22,23,29-30,33]. Inverse nodal problem was first studied by J. R. McLaughlin in 1988 [24]. She obtained some uniqueness results and showed that knowledge of the nodal points could alone determine the potential function of Sturm-Liouville problem up to a constant on an infinite interval. Independently, Shen studied the relation between the nodal points and density function of the string equation [35]. Some numerical results were given by Hald and McLaughlin [14] for reconstruction of the density function of a vibrating string, the elastic modulus of a vibration rod, the potential function in the Sturm-Liouville problem, and the impedance in the impedance equation. Inverse nodal problems have been studied fairly by many authors [4,17,19,20,25-27,36,37,38-40].

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In recent years, inverse Sturm-Liouville problems have been studied by many authors. Some of them solved these type problems by applying numerical methods as Numerov's method, Newton type method, a finite difference method, Rayleigh-Ritz method and other methods [2,5-7,9-10,13,15,31-32,34].

In this study, we consider the equation

$$-y''(x) + (q(x) + V(x))y(x) = \lambda y(x), \tag{1.1}$$

with the boundary conditions

$$y(0, \lambda) = y(1, \lambda) = 0, \tag{1.2}$$

under two following cases for $V(x)$ where $x \in [0, 1]$.

Case A. Suppose that $V(x) = 0$ i.e. consider Sturm-Liouville equation having no singularity.

Case B. Suppose that $V(x) = \frac{1}{x^p}$, $0 < p < 1$, i.e. consider Sturm-Liouville equation having a singularity at zero.

These type singular problems have been studied by many authors [3,11-12,18,28]. In equation (1.1), $\lambda = \rho^2$, ρ is the spectral parameter, $q(x) \in L^1[0, 1]$ is a real-valued function and also, we suppose that $q(1 - x) = q(x)$. In this study, we obtain the numerical values of the potential function $q(x)$ in the given cases under the boundary conditions (1.2) by applying Chebyshev interpolation method and a dense subset of the nodal points. The method of Chebyshev interpolation was used in [31] for calculating the solution of the integro-differential equations. We apply this method to obtain the solution of the inverse nodal problem. The obtained results in this work are original.

In section 2, we present the asymptotic form of the solution, the eigenvalues and the nodal points of the equation (1.1) with the boundary conditions (1.2) in two cases A and B and present the uniqueness theorem for the solution of the inverse nodal problem. In section 3, we use Chebyshev interpolation method for approximating the function $q(x)$ in given cases under the boundary conditions (1.2) by applying a dense subset of the nodal points and present a numerical algorithm for solving the inverse Sturm-Liouville problem and the numerical results are shown by providing some examples in section 4.

2. Preliminaries

In this section, our purpose is to present the asymptotic form of the eigenfunctions, the eigenvalues and the nodal points of the equation (1.1) under the boundary conditions (1.2) in two cases A and B and to express a uniqueness theorem for the inverse Sturm-Liouville problem with the given boundary conditions. For this reason, we study two cases A and B, separately.

Case A.: Let $V(x) = 0$ and $Y(x, \lambda)$ be solution of (1.1) under the initial conditions $Y(0, \lambda) = 0$ and $Y'(0, \lambda) = 1$. Therefore, $Y(x, \lambda)$ is the solution of the integral equation (see [7])

$$Y(x, \lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x - t)}{\rho} q(t) Y(t, \lambda) dt, \tag{2.1}$$

and more precisely, for $\rho \to \infty$

$$Y(x,\lambda) = \frac{\sin \rho x}{\rho} + O(\frac{1}{\rho^2}). \tag{2.2}$$

Substituting (2.2) into the right-hand side of (2.1), one can get

$$\begin{aligned} Y(x,\lambda) &= \frac{\sin \rho x}{\rho} - \frac{1}{\rho^2} q_1(x) \cos \rho x + \frac{1}{2\rho^2} \int_0^x q(t) \cos \rho (x - 2t) dt \\ &+ O(\frac{1}{\rho^3}), \end{aligned} \tag{2.3}$$

where $q_1(x) = \frac{1}{2} \int_0^x q(t)dt$.

Since the eigenvalues $\{\lambda_n\}_{n\geq 1}$ of the boundary value problem (1.1), (1.2) coincide with the zeros of the characteristic function

$$\Delta(\lambda) := Y(1,\lambda), \qquad \lambda = \rho^2,$$

then, we formulate the following theorem by using (2.3).

Theorem 2.1. *The boundary value problem (1.1),(1.2) has a countable set of the eigenvalues $\{\lambda_n\}_{n\geq 1}$ in the form of*

$$\rho_n = \sqrt{\lambda_n} = n\pi + \frac{\omega}{n\pi} + o(\frac{1}{n}), \quad n \to \infty,$$

where $\omega = \frac{1}{2} \int_0^1 q(t)dt$.

Proof. See [8]. □

Let $\lambda_1 < \lambda_2 < ... \to \infty$ be the eigenvalues of the problem (1.1),(1.2) and $0 < x_1^n < ... < x_j^n < 1$, $j = \overline{1, n-1}$, be the nodal points of the n-th eigenfunction. Then, we can express the following theorem for calculating the nodal points.

Theorem 2.2. *Let the equation (1.1) with the initial conditions*

$$Y(0,\lambda) = 0, \quad Y'(0,\lambda) = 1, \tag{2.4}$$

be given. Then, the nodal points of the problem (1.1),(2.4) with $V(x) = 0$ are formulated in the form of

$$x_j^n = \frac{j\pi}{\rho_n} + \frac{1}{\rho_n^2} q_1(x_j^n) - \frac{1}{2\rho_n^2} \int_0^{x_j^n} q(t) \cos(2\rho_n t) dt + O(\frac{1}{\rho_n^3}), \tag{2.5}$$

where $q_1(x_j^n) = \frac{1}{2} \int_0^{x_j^n} q(t)dt$.

Proof. See [40]. □

Now, we present the following uniqueness theorem.

Theorem 2.3. *Suppose that q is integrable. Then, $q - \int_0^1 q$ is uniquely determined by any dense set of the nodal points.*

Proof. See [17,40]. □

Corollary 2.1. *The potential function q of the problem (1.1),(1.2) is uniquely determined by a dense set of the nodal points and the constant*

$$\omega = \frac{1}{2} \int_0^1 q(t)dt.$$

Proof. Let us consider two Sturm-Liouville equations with the potential functions q, $\tilde{q}$ under the boundary conditions (1.2) be given as $x_j^n = \tilde{x}_j^n$, $j = \overline{1, n-1}$, $n > 1$. Also suppose that $\omega = \tilde{\omega}$. Then, $\int_0^1 q = \int_0^1 \tilde{q}$ and consequently by using Theorem 2.3, we get $q = \tilde{q}$ almost everywhere on (0,1) (also see [17]). □

Case B.: Let $V(x) = \frac{1}{x^p}$, $0 < p < 1$, and $Y(x, \lambda)$ be solution of (1.1) under the initial conditions $Y(0, \lambda) = 0$ and $Y'(0, \lambda) = 1$. Therefore, $Y(x, \lambda)$ is the solution of the integral equation

$$Y(x,\lambda) = \frac{\sin \rho x}{\rho} + \int_0^x \frac{\sin \rho(x-t)}{\rho}(q(t) + \frac{1}{t^p})Y(t,\lambda)dt, \tag{2.6}$$

and more precisely, for $\rho \to \infty$

$$Y(x,\lambda) = \frac{\sin \rho x}{\rho} + O(\frac{1}{\rho^2}). \tag{2.7}$$

Substituting (2.7) into the right-hand side of (2.6), we get

$$\begin{aligned} Y(x,\lambda) &= \frac{\sin \rho x}{\rho} - \frac{1}{\rho^2} q_1(x) \cos \rho x + \frac{1}{2\rho^2} \int_0^x (q(t) + \frac{1}{t^p}) \cos \rho(x - 2t) dt \\ &+ O(\frac{1}{\rho^3}), \end{aligned} \tag{2.8}$$

where $q_1(x) = \frac{1}{2} \int_0^x (q(t) + \frac{1}{t^p})dt$. Since

$$\Delta(\lambda) := Y(1,\lambda), \qquad \lambda = \rho^2,$$

then, using (2.8), we can get

$$\rho_n = \sqrt{\lambda_n} = n\pi + \frac{\omega}{n\pi} + o(\frac{1}{n}), \quad n \to \infty,$$

where $\omega = \frac{1}{2} \int_0^1 (q(t) + \frac{1}{t^p})dt$.

Similar to Case A, we can present the following theorem for calculating the nodal points.

Theorem 2.4. *Let the equation (1.1) with the initial conditions*

$$Y(0,\lambda) = 0, \quad Y'(0,\lambda) = 1, \tag{2.9}$$

be given. Then, the nodal points of the problem (1.1),(2.9) with $V(t) = \frac{1}{t^p}$, $0 < p < 1$, are formulated in the form of

$$x_j^n = \frac{j\pi}{\rho_n} + \frac{1}{\rho_n^2} \int_0^{x_j^n} \sin^2(\rho_n t)(q(t) + \frac{1}{t^p})dt + O(\frac{1}{\rho_n^3}). \tag{2.10}$$

Proof. See [40]. □

Now, we present the following uniqueness theorem for this case.

Theorem 2.5. *Suppose that q is integrable. Then, $q - \int_0^1 q$ is uniquely determined by any dense set of the nodal points.*

Proof. See [40]. □

Corollary 2.2. *The potential function q of the problem (1.1),(1.2) is uniquely determined by a dense set of the nodal points and the constant*

$$\omega = \frac{1}{2}\int_0^1 (q(t) + \frac{1}{t^p})dt.$$

3. Main results

In this section, we describe a numerical method based on Chebyshev interpolation method for solving the inverse Sturm-Liouville problem in two cases A and B by using a dense subset of the nodal points. We consider the following inverse nodal problem.

Inverse problem. Given the nodal points $\{x_j^n\}$, $j = 1, 2, ..., n-1$, $n > 1$, construct the potential function $q(x)$.

Since the nodal points $\{x_j^n\}$, $j = 1, 2, ..., n-1$ are the zeroes of the n-th eigenfunction $Y(x, \lambda_n)$, then, we can write

$$Y(x_j^n, \lambda_n) = 0, \quad j = 1, 2, ..., n-1, \quad n > 1.$$

Thus, using (2.1) and (2.6), we get

$$(3.1) \int_0^{x_j^n} \frac{\sin\rho_n(x_j^n - t)}{\rho_n} q(t)Y(t, \lambda_n)dt \;=\; -\frac{\sin\rho_n x_j^n}{\rho_n} - \int_0^{x_j^n} \frac{\sin\rho_n(x_j^n - t)}{\rho_n} V(t)Y(t, \lambda_n)dt,$$

where $V(x) = 0$ in Case A and $V(x) = \frac{1}{x^p}$, $0 < p < 1$, in Case B. In the above integral equation, q is unknown function and Y can be obtained from (2.2).

In order to obtain the solution of inverse nodal problem, it is sufficient that we get the solution of the integral equation (3.1). For this reason, we use Chebyshev polynomials of the first kind as the basic functions for approximating the function q and convert the integral equation (3.1) to the system of linear equations.

The first few Chebyshev polynomials in $[0, 1]$ are given by

$$T_0(x) = 1,$$

$$T_1(x) = 2x - 1,$$

$$T_2(x) = 2(2x-1)^2 - 1,$$

$$T_3(x) = 4(2x-1)^3 - 3(2x-1),$$

$$T_4(x) = 8(2x-1)^4 - 8(2x-1)^2 + 1,$$

$$T_5(x) = 16(2x-1)^5 - 20(2x-1)^3 + 5(2x-1),$$

$$\vdots$$

By using Chebyshev interpolation method for the function $q(t)$, one can show that (see [31])

$$q(t) \cong \sum_{i=0}^{N} q_i l_{i,N}(t), \quad t \in [0,1], \tag{3.2}$$

where

$$l_{i,N}(t) = \frac{2\delta_i}{N} {\sum_{k=0}^{N}}'' T_k(2t-1)\cos(\frac{ki\pi}{N}),$$

$$\delta_i = \begin{cases} 0.5 & i = 0, N, \\ 1 & 0 < i < N, \end{cases}$$

the numbers q_i, $i = 0, 1, ..., N$ are the values of the function $q(t)$ in the points $t_i = (\cos(\frac{i\pi}{N}) + 1)/2$ and the functions $T_k(t)$, $k = 0, 1, ..., N$ are Chebyshev polynomials of the first kind. Also, $\sum''$ is the sum of all terms except the first and last two sentences so that the sum of half of the two sentences is considered.

Substituting (3.2) into (3.1), we obtain

$$\sum_{i=0}^{N} R(x_j^n, t_i) q_i = g(x_j^n), \quad j = 1, 2, ..., n-1, \quad n > 1,$$

where

$$R(x_j^n, t_i) = \frac{2\delta_i}{N} {\sum_{k=0}^{N}}'' I_k(x_j^n)\cos(\frac{ki\pi}{N}),$$

$$I_k(x_j^n) = \int_0^{x_j^n} \frac{\sin\rho_n(x_j^n - t)}{\rho_n} Y(t, \lambda_n) T_k(2t-1)dt,$$

$$g(x_j^n) = -\frac{\sin\rho_n x_j^n}{\rho_n}, \quad \textit{in Case A},$$

and

$$g(x_j^n) = -\frac{\sin\rho_n x_j^n}{\rho_n} + \frac{1}{2(1-p)\rho_n^2}(x_j^n)^{1-p}\cos\rho_n x_j^n$$
$$-\frac{1}{2\rho_n^2}\cos\rho_n x_j^n \int_0^{x_j^n} t^{-p}\cos 2\rho_n t dt$$
$$-\frac{1}{2\rho_n^2}\sin\rho_n x_j^n \int_0^{x_j^n} t^{-p}\sin 2\rho_n t dt, \quad \textit{in Case B}.$$

Therefore, the solution of inverse nodal problem is calculated by using the following algorithm.

Algorithm. Let the numbers $\{x_j^n\}$, $j = 1, 2, ..., n-1$, $n > 1$, be given.
1. Choose N.
2. Find the coefficients q_i, $i = 0, 1, ..., N$ by applying the following linear system:

$$A_N\hat{q} = B_N,$$

where

$$A_N = [R(x_j^n, t_i)], \quad j = 1, 2, ..., n-1, \quad n = N+2, \quad i = 0, 1, ..., N,$$

$$B_N = [g(x_j^n)], \quad j = 1, 2, ..., n-1, \quad n = N+2,$$

$$\hat{q}^T = [q_i], \quad i = 0, 1, ..., N.$$

4. Numerical examples

In this section, we provide some numerical examples for inverse nodal problem in two cases A and B implemented by the given algorithm. We use Matlab software program for drawing the figures. The convergence of the proposed method and the stability of the inverse problem solution are seen in these examples.

Example 4.1. Let $q(x) = \cos(2\pi x)$ be given. Then, the nodal points of the equation (1.1) under the boundary conditions (1.2) in the case A and also in the case B with $p = 1/2$ obtained from the relations (2.5) and (2.10), respectively, are formulated in the following forms

$$x_j^n = \frac{j}{n} + \frac{1}{4\pi^3(n^2-1)}\sin(\frac{2j\pi}{n}), \quad j = \overline{1, n-1}, \quad n > 1, \quad in\ Case\ A,$$

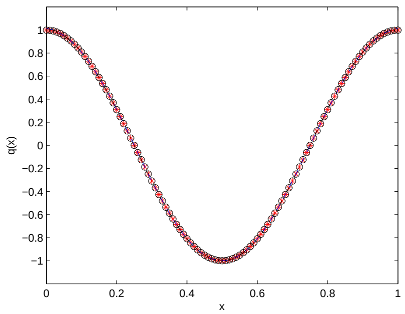
and

$$x_j^n = \frac{j}{n} + \frac{1}{4\pi^3(n^2-1)}\sin(\frac{2j\pi}{n}) + \frac{1}{n^3\pi^2}\sqrt{nj} - \frac{\sqrt{n}}{2n^3\pi^2}FresnelC(2\sqrt{j}),$$

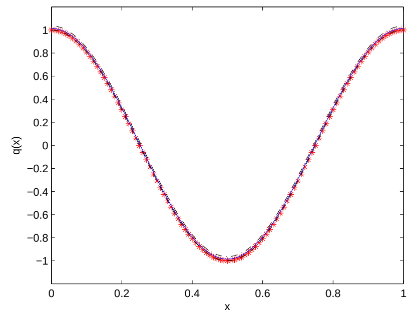
$$j = \overline{1, n-1}, \quad n > 1, \quad in\ Case\ B.$$

Now, we suppose that q is the unknown function and the nodal points given in the above forms are the input data in two cases A and B, separately. We want to get the approximations of the potential q as the solution of inverse problem for two cases A and B by the presented algorithm.

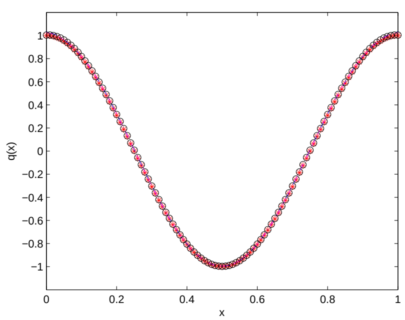
Applying the described algorithm, we obtain the numerical values of the potential function $q(x) = \cos(2\pi x)$ with $N \in \{6, 7, 8\}$ in two Cases A and B and calculate the approximate solutions of inverse problem by substituting the obtained numerical values into (3.2). The exact solution and the numerical approximations obtained with $N \in \{6, 7, 8\}$ in two Cases A and B for no noise in the nodal points are seen in Figure 1 (see (a),(c)). In Figure 1, it can be seen that the best solution can be obtained for the small values of N.



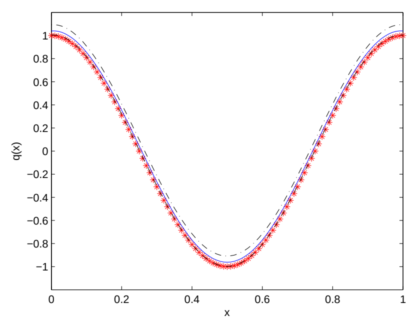
(a) Exact and approximate solutions of the potential function $q(x) = \cos(2\pi x)$ for no noise in the nodal points in Case A: (- . -) for $N = 6$, (- - -) for $N = 7$, (o o o) for $N = 8$ and (***) for exact solution.



(b) Exact and approximate solutions of the potential function $q(x) = \cos(2\pi x)$ with $N = 8$ in Case A: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.



(c) Exact and approximate solutions of the potential function $q(x) = \cos(2\pi x)$ for no noise in the nodal points in Case B: (- . -) for $N = 6$, (- - -) for $N = 7$, (o o o) for $N = 8$ and (***) for exact solution.

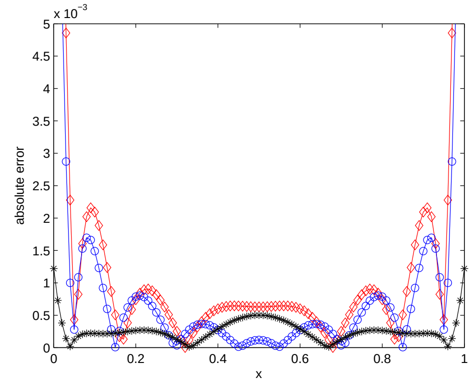


(d) Exact and approximate solutions of the potential function $q(x) = \cos(2\pi x)$ with $N = 8$ in Case B: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.

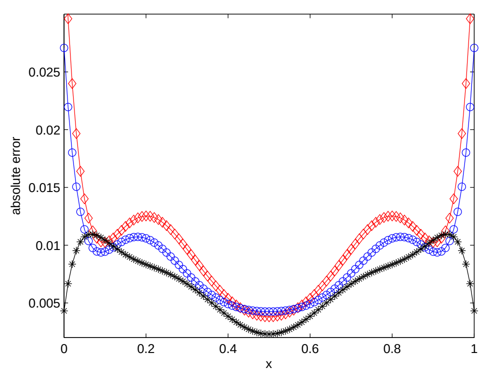
Figure 1. Solution of inverse problem in Example 4.1.

Also, we solve inverse nodal problem by using the noisy data $x_j^n(1+P\epsilon)$, instead of x_j^n, where P and ϵ are the amount of noise and the random real number, respectively, in the interval $[-1, 1]$. The exact solution and the numerical approximations of the function q obtained with $N = 8$ and $P = 0, 2\%$ and 5% in two cases A and B are seen in Figure 1 (see (b),(d)). In Figure 1, it can be shown that the calculated numerical solutions are stable and also become more accurate as the amount of noise P decreases.

Finally, we obtain the absolute errors between the exact and approximate solutions of q for no noise in the nodal points with $N \in \{6, 7, 8\}$ for two cases A and B which are seen in Figure 2. In Figure 2, it can be shown that by increasing the amount of N, the errors are reduced.



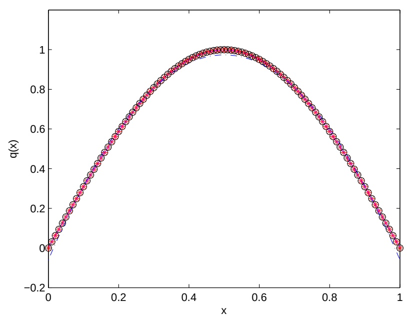
(a) Absolute errors for no noise in the nodal points in Case A: ($\diamond$ $\diamond$ $\diamond$) for $N = 6$, (o o o) for $N = 7$ and (* * *) for $N = 8$.



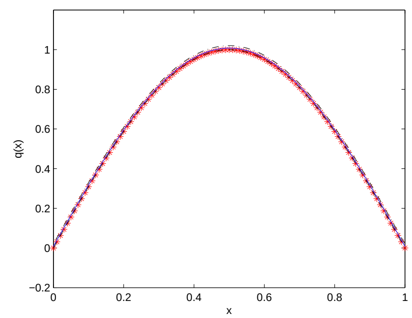
(b) Absolute errors for no noise in the nodal points in Case B: ($\diamond$ $\diamond$ $\diamond$) for $N = 6$, (o o o) for $N = 7$ and (* * *) for $N = 8$.

FIGURE 2. Absolute errors between the exact and approximate solutions in Example 4.1.

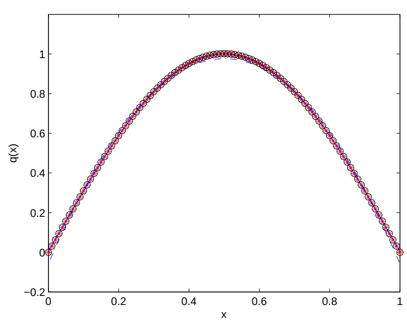
Example 4.2. Let the nodal points $\{x_j^n\}$, $\quad j = \overline{1, n-1}$, $\quad n > 1$ obtained from the formulae (2.5) and (2.10) with $q(x) = \sin(\pi x)$ in the case A and also in the case B with $p = 1/4$ be given. In these cases, we obtain the approximations of the potential function q by the described algorithm. The exact solution and the numerical approximations obtained with $N \in \{3, 4, 5, 6\}$ in two Cases A and B for no noise in the nodal points are seen in Figure 3 (see (a),(c)). Also in Figure 3, the approximate solutions obtained with $N = 6$ and $P = 0, 2\%$ and 5% are shown (see (b),(d)). Similar to Example 4.1, it can be seen that the numerical solutions become more accurate as N increases and P decreases.



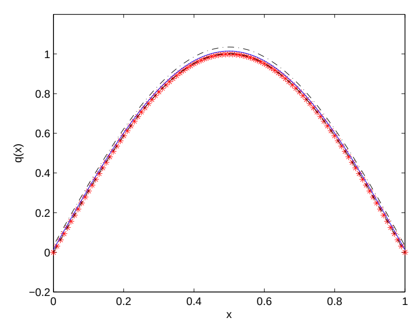
(a) Exact and approximate solutions of the potential function $q(x) = \sin(\pi x)$ for no noise in the nodal points in Case A: (- . -) for $N = 3$, (- - -) for $N = 4$, (—) for $N = 5$, (o o o) for $N = 6$ and (***) for exact solution.



(b) Exact and approximate solutions of the potential function $q(x) = \sin(\pi x)$ with $N = 6$ in Case A: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.

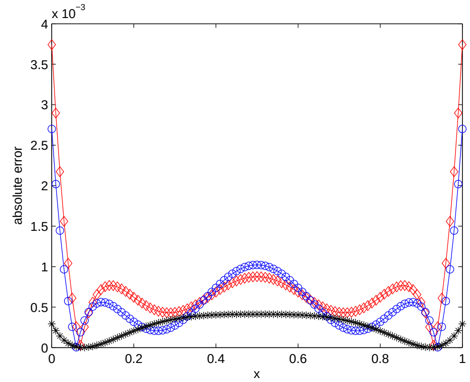


(c) Exact and approximate solutions of the potential function $q(x) = \sin(\pi x)$ for no noise in the nodal points in Case B: (- . -) for $N = 3$, (- - -) for $N = 4$, (—) for $N = 5$, (o o o) for $N = 6$ and (***) for exact solution.

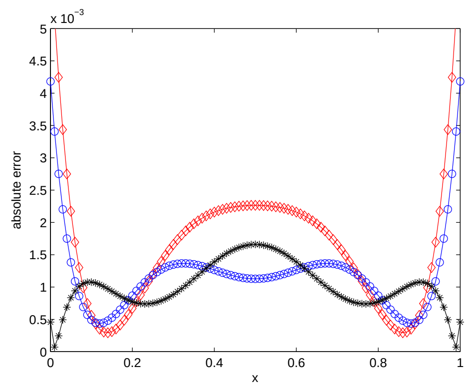


(d) Exact and approximate solutions of the potential function $q(x) = \sin(\pi x)$ with $N = 6$ in Case B: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.

FIGURE 3. Solution of inverse problem in Example 4.2.



(a) Absolute errors for no noise in the nodal points in Case A: ($\diamond$ $\diamond$ $\diamond$) for $N = 4$, (o o o) for $N = 5$ and (* * *) for $N = 6$.

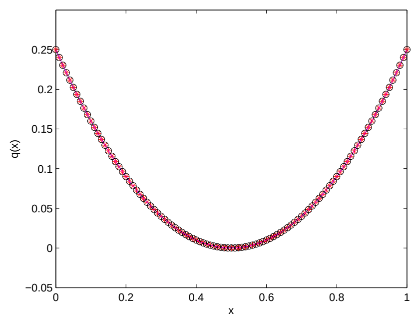


(b) Absolute errors for no noise in the nodal points in Case B: ($\diamond$ $\diamond$ $\diamond$) for $N = 4$, (o o o) for $N = 5$ and (* * *) for $N = 6$.

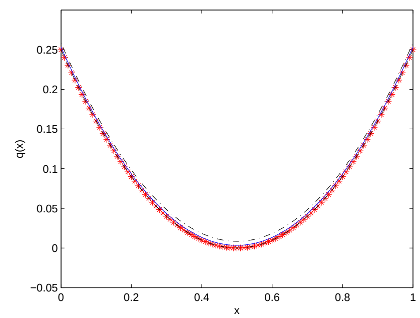
FIGURE 4. Absolute errors between the exact and approximate solutions in Example 4.2.

Finally, we obtain the absolute errors between the exact and approximate solutions of q for no noise in the nodal points with $N \in \{4, 5, 6\}$ in two Cases A and B which are seen in Figure 4.

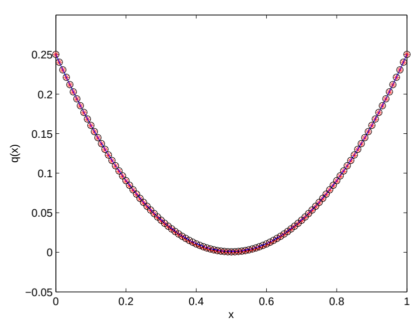
Example 4.3. Let the nodal points $\{x_j^n\}$, $j = \overline{1, n-1}$, $n > 1$ obtained from the formulae (2.5) and (2.10) with $q(x) = (x - \frac{1}{2})^2$ in the case A and also in the case B with $p = 1/8$ be given. The exact solution and the numerical approximations obtained with $N \in \{3, 4, 5, 6\}$ in two Cases A and B for no noise in the nodal points and also the approximate solutions obtained with $N = 6$ and $P = 0, 2\%$ and 5% are shown in Figure 5.



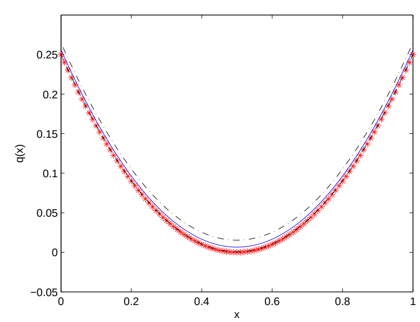
(a) Exact and approximate solutions of the potential function $q(x) = (x - \frac{1}{2})^2$ for no noise in the nodal points in Case A: (- . -) for $N = 3$, (- - -) for $N = 4$, (—) for $N = 5$, (o o o) for $N = 6$ and (***) for exact solution.



(b) Exact and approximate solutions of the potential function $q(x) = (x - \frac{1}{2})^2$ with $N = 6$ in Case A: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.



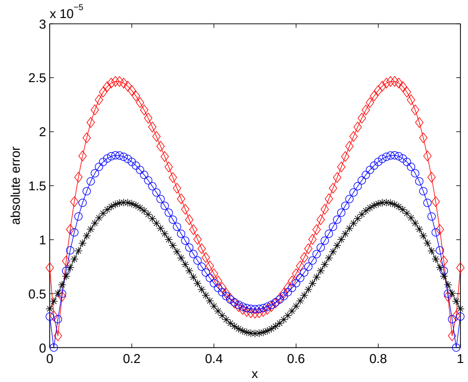
(c) Exact and approximate solutions of the potential function $q(x) = (x - \frac{1}{2})^2$ for no noise in the nodal points in Case B: (- . -) for $N = 3$, (- - -) for $N = 4$, (—) for $N = 5$, (o o o) for $N = 6$ and (***) for exact solution.



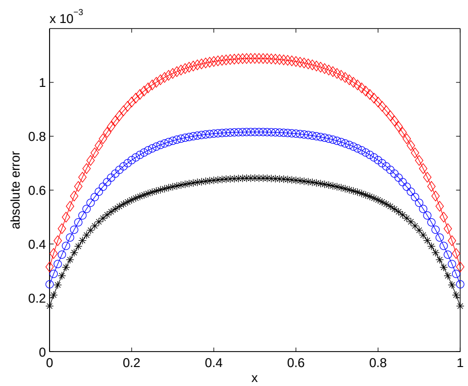
(d) Exact and approximate solutions of the potential function $q(x) = (x - \frac{1}{2})^2$ with $N = 6$ in Case B: (- . -) for $P = 5\%$ noise, (—) for $P = 2\%$ noise, (- - -) for $P = 0$ noise and (***) for exact solution.

FIGURE 5. Solution of inverse problem in Example 4.3.

Also, the absolute errors between the exact and approximate solutions of q for no noise in the nodal points with $N \in \{4, 5, 6\}$ in two Cases A and B are seen in Figure 6.



(a) Absolute errors for no noise in the nodal points in Case A: ($\diamond$ $\diamond$ $\diamond$) for $N = 4$, (o o o) for $N = 5$ and (* * *) for $N = 6$.



(b) Absolute errors for no noise in the nodal points in Case B: ($\diamond$ $\diamond$ $\diamond$) for $N = 4$, (o o o) for $N = 5$ and (* * *) for $N = 6$.

FIGURE 6. Absolute errors between the exact and approximate solutions in Example 4.3.

5. Conclusion

In this study, we obtained the approximate solution of inverse problem by using Chebyshev interpolation method and a dense subset of the nodal points for Sturm-Liouville equation under the separated boundary conditions in two cases A and B that in Case A, the equation did not have any singularity and in Case B, the equation was contained a singularity at zero. For this reason, we used Chebyshev polynomials of the first kind as the basic functions for approximating the function q. Also, we provided some numerical examples and showed stable numerical results in these examples.

References

[1] Ambartsumyan VA. Über eine frage der eigenwerttheorie. Zeitschrift fr Physik. 1929;53:690–695.

[2] Andrew AL. Numerov's method for inverse Sturm-Liouville problem. Inverse Problems. 2005;21:223–238.

[3] Aygar Y. Investigation of spectral analysis of matrix quantum difference equations with spectral singularities. Hacettepe Journal of Mathematics and Statistics. DOI: 10. 15672/HJMS.20164513107.

[4] Browne PJ, Sleeman BD. Inverse nodal problem for Sturm-Liouville equation with eigenparameter dependent boundary conditions. Inverse Problems. 1996;12:377–381.

[5] Drignei MC. A Newton-type method for solving an inverse Sturm-Liouville problem. Inverse Problems in Science and Engineering. http://dx.doi.org/10.1080/17415977.2014.947478.

[6] Efremova L, Freiling G. Numerical solution of inverse spectral problems for Sturm-Liouville operators with discontinuous potentials. Cent. Eur. J. Math. 2013;11:2044–2051.

[7] Fabiano RH, Knobel R, Lowe BD. A finite-difference algorithm for an inverse Sturm-Liouville problem. IMA J. Numer. Anal. 1995;15:75–88. doi:10.1093/imanum/15.1.75.

[8] Freiling G, Yurko V. Inverse Sturm-Liouville problems and their applications. New York: NOVA science publishers; 2001.

[9] Gao Q, Cheng X, Huang Z. Modified Numerov's method for inverse Sturm-Liouville problems. Journal of Computational and Applied Mathematics. 2013;253:181–199.

[10] Gladwell GML. The application of Schur's algorithm to an inverse eigenvalue problem. Inverse Problems. 1991;7:557–565.

[11] Gulsen T, Panakhov ES. Dirac systems that contain discontinuity conditions. AIP Conference Proceeding. 2016;1759:1–6.

[12] Gulsen T, Yilmaz E, Panakhov ES. On a Lipschitz stability problem for p-Laplacian Bessel equation. Communications, Series A1; Mathematics and Statistics. 2017;66:253–262.

[13] Hald OH. The inverse Sturm-Liouville problem and the Rayleigh-Ritz method. Math. Comp. 1978;32:687–705.

[14] Hald OH, McLaughlin JR. Solutions of inverse nodal problems. Inverse Problems. 1989;5:307–347.

[15] Ignatiev M, Yurko V. Numerical methods for solving inverse Sturm-Liouville problems. Result. Math. 2008;52:63–74.

[16] Kerimov NB, Goktas S, Maris EA. Uniform convergence of the spectral expansions in terms of root functions for a spectral problem. Electronic Journal of Differential Equations. 2016;80:1-14.

[17] Koyunbakan H. A new inverse problem for the diffusion operator. Applied Mathematics Letters. 2006;19:995-999.

[18] Koyunbakan H, Panakhov ES. Solution of a discontinuous inverse nodal problem on a finite interval. Mathematical and Computer Modelling. 2006;44:204–209.

[19] Koyunbakan H, Yilmaz E. Reconstruction of the potential function and its derivatives for the diffusion operator. Verlag der Zeitschriftfur Naturforch. 2008;63(a):127–130.

[20] Law CK, Yang CF. Reconstruction of the potential function and its derivatives using nodal data. Inverse Problems. 1999;14:299–312.

[21] Levitan BM, Sargsjan IS. Introduction to spectral theory: Self adjoint ordinary differential operators. American Mathematical Society. Providence, Rhode Island; 1975.

[22] Lowe BD, Pilant M, Rundell W. The recovery of potentials from finite spectral data. SIAM J. Math. Anal. 1992;23:482–504. doi:10.1137/0523023.

[23] Marchenko VA, Maslov KV. Stability of the problem of recovering the Sturm- Liouville operator from the spectral function. Mathematics of the USSR Sbornik. 1970;81:475–502.

[24] McLaughlin JR. Inverse spectral theory using nodal points as data-a uniqueness result. Journal of Differential Equations. 1988;73:342–362.

[25] McLaughlin JR. Stability theorems for two inverse spectral problems. Inverse Problems. 1988;4:529–540.

[26] Neamaty A, Akbarpoor Sh. Numerical solution of inverse nodal problem with an eigenvalue in the boundary condition. Inverse Problems in Science and Engineering. 2017;25:978–994.

[27] Neamaty A, Akbarpoor Sh, Dabbaghian A. Uniqueness theorem for the inverse aftereffect problem and representation the nodal points form. Journal of Mathematical Extension. 2015;9:37–49.

[28] Panakhov ES, Gulsen T. On discontinuous Dirac systems with eigenvalue dependent boundary conditions. AIP Conference Proceeding. 2015;1648:1-4.

[29] Pivovarchik V. Direct and inverse three-point Sturm-Liouville problems with parameter dependent boundary conditions. Asymptotic Analysis. 2001;26:219–238.

[30] Pöschel J, Trubowitz E. Inverse spectral theory. volume 130 of Pure and Applied Mathematics. Academic Press, Inc, Boston, MA; 1987.

[31] Rashed MT. Numerical solution of a special type of integro-differential equations. Applied Mathematics and computation. 2003;143:73–88.

[32] *Röhrl* N. A least-squares functional for solving inverse Sturm-Liouville problems. Inverse Problems. 2005;21:2009–2017. doi:10.1088/0266-5611/21/6/013.

[33] Rundell W, Sacks PE. Reconstruction techniques for classical inverse Sturm-Liouville problems. Math. Comp. 1992;58:161–183.

[34] Sacks PE. An iterative method for the inverse Dirichlet problem. Inverse Problems. 1988;4:1055–1069. doi:10.1088/0266-5611/4/4/009.

[35] Shen CL. On the nodal sets of the eigenfunctions of the string equations. SIAM Journal on Mathematical Analysis. 1988;19:1419–1424.

[36] Shieh CT, Yurko VA. Inverse nodal and inverse spectral problems for discontinuous boundary value problems. Journal of Mathematical Analysis and Applications. 2008;347:266–272.

[37] Yang CF. Reconstruction of the diffusion operator from nodal data. Verlag der Zeitschrift f*ü*r Naturforch. 2010;65a:100–106.

[38] Yang XF. A solution of the inverse nodal problem. Inverse Problems. 1997;13:203–213.

[39] Yang XF. A new inverse nodal problem. J. Differential Equations. 2001;169:633–653.

[40] Yilmaz E, Koyunbakan H, Ic U. Inverse nodal problem for the differential operator with a singularity at zero. Computer Modeling in Engineering and Sciences. 2013;92:303–313.

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