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Simulation of Two-Step Block Approach for Solving Oscillatory Differential Equations

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ABSTRACT

This study demonstrates the derivation of a two-step block scheme simulation through a linear block approach. The scheme's fundamental properties were thoroughly analyzed and found to fulfill all necessary conditions. The research focused on examining specific classes of oscillatory differential equations and comparing them to established methods. The findings indicate that the newly proposed methods exhibit superior accuracy and faster convergence compared to the existing methods investigated in this research. Consequently, the results highlight the improved precision and quicker convergence achieved with the new method. All computations were executed using Maple 18 software.

Keywords: Simulation, Linear block approach, Properties of the schemes, Oscillatory differential equation, Accuracy and convergence

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Introduction

The use of mathematics in demonstrating empirical problems in applied science and other areas of study in which noises are introduced into the deterministic models of differential equations [1]. It was found that the deterministic differential equations were ineffective and inadequate to handle differential equations arising from these fields of study with intrinsically complex systems containing millions or billions of interacting particles [2]. The computations of empirical problems will be used for sampling the first order initial value problem given as:

$$\gamma' = f(u,\gamma), u(0) = 0 \tag{1}$$

Most of the empirical problems and studies mentioned above are coded in numbers and mathematical symbols to form an equation in order to have meaning, construction, and application [3]. One of such equations is known as a differential equation of the form (1) [3]. The differential equation (1) may evolve from empirical problems that involve the rate of change of a given variable in the structure (system) with respect to another. These equations came to prominence in the late 17th century with the independent invention of infinitesimal calculus by an English mathematician, Isaac Newton (1642-1727), and a German mathematician, Gottfried Wilhelm Leibniz (1646-1716).

The computational solution of an empirical problem modeled in (1) has great significance to researchers. A lot has been considered in finding analytic solutions to (1). Among others, [4, 5, 6, 7, 8] developed block methods for solving (1). The study of the Obrechkoff method for solving (1) is considered [9, 10]. The collocation method was employed by [11, 12,13, 14, 15, 16] to investigate the resolution of Volterra-Fredholm integro-differential equations and Volterra-Fredholm fractional order integro-differential equations.

Also, [17, 18, 19,20, 21,22,23,24,25] adopt some methods to solve (1). However, it was noticed that many of the problems leading to this type of equation, especially when they are non-linear, could not be easily solved analytically to get the exact solution. As a result, various numerical methods for solving the equations have been developed in order to obtain an approximate solution to (1) [26].

Mathematical Formulation of the Method

This section shows the formulation of the method. The method is derived using the linear block approach [27].

The linear block approach is of the form

$$y_{n+\xi} = \sum_{i=0}^{2} \frac{(\xi h)^i}{i!} y_n^{(i)} + \sum_{i=0}^{4} \left(\phi_{i\xi} f_{n+i}\right), \quad \xi = 0, m, n, 1, a, b, 2 \tag{2}$$

Differentiating Eq. (2) gives

$$y_{n+\xi}^{(a)} = \sum_{i=0}^{1-a} \frac{(\xi h)^i}{i!} y_n^{(i+a)} + \sum_{i=0}^{4} \kappa_{\xi i a} f_{n+i}, a = 1_{(0,m,n,1,a,b,2)} \tag{3}$$

$\phi_{\xi i} = A^{-1}P$ and $\kappa_{\xi i a} = A^{-1}Q$ where

$$A=\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & \frac{(m)^1}{1!} & \frac{(n)^1}{1!} & \frac{(1)^1}{1!} & \frac{(a)^1}{1!} & \frac{(b)^1}{1!} & \frac{(2)^1}{1!} \\ 0 & \frac{(m)^2}{2!} & \frac{(n)^2}{2!} & \frac{(1)^2}{2!} & \frac{(a)^2}{2!} & \frac{(b)^2}{2!} & \frac{(2)^2}{2!} \\ 0 & \frac{(m)^3}{3!} & \frac{(n)^3}{3!} & \frac{(1)^3}{3!} & \frac{(a)^3}{3!} & \frac{(b)^3}{3!} & \frac{(2)^3}{3!} \\ 0 & \frac{(m)^4}{4!} & \frac{(n)^4}{4!} & \frac{(1)^4}{4!} & \frac{(a)^4}{4!} & \frac{(b)^4}{4!} & \frac{(2)^4}{4!} \\ 0 & \frac{(m)^5}{5!} & \frac{(n)^5}{5!} & \frac{(1)^5}{5!} & \frac{(a)^5}{5!} & \frac{(b)^5}{5!} & \frac{(2)^5}{5!} \\ 0 & \frac{(m)^6}{6!} & \frac{(n)^6}{6!} & \frac{(1)^6}{6!} & \frac{(a)^6}{6!} & \frac{(b)^6}{6!} & \frac{(2)^6}{6!} \end{pmatrix}, P=\begin{pmatrix} \frac{(\xi h)^1}{1!} \\ \frac{(\xi h)^2}{2!} \\ \frac{(\xi h)^3}{3!} \\ \frac{(\xi h)^4}{4!} \\ \frac{(\xi h)^5}{5!} \\ \frac{(\xi h)^6}{6!} \\ \frac{(\xi h)^7}{7!} \end{pmatrix}, Q=\begin{pmatrix} \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{6-1}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \\ \frac{(\xi h)^{1-a}}{(1-a)!} \end{pmatrix} \quad (4)$$

Eq. (2) and Eq. (3) are solved step by step using linear block approach through

$$\vartheta_\xi ,\ \xi = 0, m, n, 1, a, b, 2$$

The polynomial $x = x_s + th$, is used in Eq. (3) to yield the block hybrid method of the form:

$$q(x_s th) = \alpha_m y_{s+m}$$
$$h\left(\beta_0 f_s + \beta_m f_{s+m} + \beta_n f_{s+n} + \beta_1 f_{s+1} + \beta_a f_{s+a} + \beta_b f_{s+b} + \beta_2 f_{s+2}\right) \quad (5)$$

Where

$$\alpha_0 = 1$$

$$\beta_0 = \frac{\begin{pmatrix} 168\xi^4 - 210\xi^5 + 60\xi^6 - 210a\xi^3 + 252a\xi^4 - 70a\xi^5 - 210b\xi^3 + 1252b\xi^4 - 70b\xi^5 - 210m\xi^3 \\ + 252m\xi^4 - 70m\xi^5 - 210n\xi^3 + 252n\xi^4 - 70n\xi^5 + 280ab\xi^2 - 315ab\xi^3 + 84ab\xi^4 + \\ 280am\xi^2 - 315\xi^3 + 84am\xi^4 + 280an\xi^2 + 280bm\xi^2 - 315an\xi^2 - 315bm\xi^3 + 84an\xi^4 + \\ 84bm\xi^4 + 280bn\xi^2 - 315bn\xi^3 + 84bn\xi^4 + 280mn\xi^2 - 315mn\xi^3 + 84bn\xi^4 + 280mn\xi^2 \\ - 315mn\xi^3 + 84mn\xi^4 - 420abm\xi - 420abn\xi - 420amn\xi - 420bmn\xi + 420abm\xi^2 - \\ 105abm\xi^3 + 420abn\xi^2 - 105abn\xi^3 + 420amn\xi^2 - 105amn\xi^3 + 420bmn\xi^2 - 105mn\xi^3 \\ + 840abmn - 630abmn\xi + 140abmn\xi^2 \end{pmatrix}}{840abmn}$$

$$\beta_m = -\frac{\xi^2 \begin{pmatrix} -168\xi^3 - 210\xi^4 + 60\xi^5 - 210a\xi^2 + 252a\xi^3 - 70a\xi^4 + 210b\xi^2 - 252b\xi^3 + 70b\xi^4 + 210n\xi^2 \\ - 252n\xi^3 + 70n\xi^4 - 280ab\xi - 280an\xi - 280bn\xi + 315ab\xi^2 - 84ab\xi^3 + 315an\xi^2 - \\ 84an\xi^3 + 315bn\xi^2 + 84bn\xi^3 + 84bn\xi^3 + 420abn - 420abn\xi + 105abn\xi^2 \end{pmatrix}}{420m(m-1)(m-2)(m-n)(b-m)(a-m)}$$

$$\beta_n = \frac{\xi^2 \begin{pmatrix} -168\xi^3 + 210\xi^4 - 60\xi^5 + 210a\xi^2 - 252a\xi^3 + 70a\xi^4 - 210b\xi^2 - 252b\xi^3 + 70b\xi^4 + 210m\xi^2 \\ - 252m\xi^3 + 70m\xi^4 - 280ab\xi - 280an\xi - 280bm\xi + 315ab\xi^2 - 84ab\xi^3 + 315an\xi^2 - \\ 84am\xi^3 + 315bm\xi^2 + 84bm\xi^3 + 84bm\xi^3 + 420abm - 420abm\xi + 105abm\xi^2 \end{pmatrix}}{420n(n-1)(n-2)(m-n)(b-n)(a-n)}$$

$$\beta_1 = -\frac{\xi^2 \begin{pmatrix} -140\xi^4 + 60\xi^5 + 168\xi^3 - 70a\xi^4 + 168b\xi^3 - 70b\xi^4 + 168m\xi^3 - 70m\xi^4 + 168n\xi^3 - 70n\xi^4 \\ - 210ab\xi^2 + 84ab\xi^3 - 210am\xi^2 + 84am\xi^3 - 210an\xi^2 - 210bm\xi^2 + 84an\xi^3 + 84bm\xi^3 - \\ 210bn\xi^2 + 84bn\xi^3 - 210mn\xi^2 + 84mn\xi^3 + 280abm\xi + 280abn\xi + 280amn\xi + 280bmn\xi \\ - 105abm\xi^2 - 105abn\xi^2 - 105amn\xi^2 - 105bmn\xi^2 - 420abmn + 140abmn\xi \end{pmatrix}}{420(n-1)(m-1)(b-1)(a-1)}$$

$$\beta_a = -\frac{\xi^2\begin{pmatrix} -168\xi^3 + 210\xi^4 - 60\xi^5 + 210b\xi^2 - 252b\xi^3 + 70b\xi^4 - 210m\xi^2 - 252m\xi^3 + 70m\xi^4 + 210n\xi^2 \\ -252n\xi^3 + 70n\xi^4 - 280bm\xi - 280bn\xi - 280mn\xi + 315bm\xi^2 - 84bm\xi^3 + 315bn\xi^2 - \\ 84bn\xi^3 + 315bmn\xi^2 + 84mn\xi^3 + 420bmn - 420bm\xi + 105bmn\xi^2 \end{pmatrix}}{420a(a-1)(a-2)(a-n)(a-n)(a-b)}$$

$$\beta_a = -\frac{\xi^2\begin{pmatrix} -168\xi^3 + 210\xi^4 - 60\xi^5 + 210a\xi^2 - 252a\xi^3 + 70a\xi^4 - 210m\xi^2 - 252m\xi^3 + 70m\xi^4 + 210n\xi^2 \\ -252n\xi^3 + 70n\xi^4 - 280bm\xi - 280bn\xi - 280mn\xi + 315am\xi^2 - 84am\xi^3 + 315an\xi^2 - \\ 84an\xi^3 + 315amn\xi^2 - 84mn\xi^3 + 420amn - 420am\xi + 105amn\xi^2 \end{pmatrix}}{420b(b-1)(b-2)(b-n)(b-n)(a-b)}$$

$$\beta_1 = \frac{\xi^2\begin{pmatrix} -70\xi^4 + 60\xi^5 + 84a\xi^3 - 70a\xi^4 + 84b\xi^3 - 70b\xi^4 + 84m\xi^3 - 70m\xi^4 + 84n\xi^3 - 70n\xi^4 \\ -105ab\xi^2 + 84ab\xi^3 - 105am\xi^2 + 84am\xi^3 + 140abn\xi + 140abn\xi + 140amn\xi - 105abn\xi^2 - \\ 105amn\xi^2 - 105bmn\xi^2 - 210abmn + 140bmn\xi \end{pmatrix}}{840(n-2)(m-2)(b-2)(a-2)}$$

To get the unknown values of Eq.(3), we simplify $\kappa_{\xi i a} = A^{-1}Q$ to obtain

$$\left.\begin{aligned} y_{s+m} &= y_s + h(\kappa_{11} f_s + \kappa_{12} f_{s+m} + \kappa_{13} f_{s+n} + \kappa_{14} f_{s+1} + \kappa_{15} f_{s+a} + \kappa_{16} f_{s+b} + \kappa_{17} f_{s+2}) \\ y_{s+n} &= y_s + h(\kappa_{21} f_s + \kappa_{22} f_{s+m} + \kappa_{23} f_{s+n} + \kappa_{24} f_{s+1} + \kappa_{25} f_{s+a} + \kappa_{26} f_{s+b} + \kappa_{27} f_{s+2}) \\ y_{s+1} &= y_s + h(\kappa_{31} f_s + \kappa_{32} f_{s+m} + \kappa_{33} f_{s+n} + \kappa_{34} f_{s+1} + \kappa_{35} f_{s+a} + \kappa_{36} f_{s+b} + \kappa_{37} f_{s+2}) \\ y_{s+a} &= y_s + h(\kappa_{41} f_s + \kappa_{42} f_{s+m} + \kappa_{43} f_{s+n} + \kappa_{44} f_{s+1} + \kappa_{45} f_{s+a} + \kappa_{46} f_{s+b} + \kappa_{47} f_{s+2}) \\ y_{s+b} &= y_s + h(\kappa_{51} f_s + \kappa_{52} f_{s+m} + \kappa_{53} f_{s+n} + \kappa_{54} f_{s+1} + \kappa_{55} f_{s+a} + \kappa_{56} f_{s+b} + \kappa_{57} f_{s+2}) \\ y_{s+2} &= y_s + h(\kappa_{61} f_s + \kappa_{62} f_{s+m} + \kappa_{63} f_{s+n} + \kappa_{64} f_{s+1} + \kappa_{65} f_{s+a} + \kappa_{66} f_{s+b} + \kappa_{67} f_{s+2}) \end{aligned}\right\} \quad (6)$$

(6)

$$y_{s+m}\begin{pmatrix}\kappa_{11}\\ \kappa_{12}\\ \kappa_{13}\\ \kappa_{14}\\ \kappa_{15}\\ \kappa_{16}\\ \kappa_{17}\end{pmatrix} = \begin{pmatrix}\frac{19087}{181440}\\ \frac{2713}{7560}\\ -\frac{15487}{60480}\\ \frac{586}{2835}\\ -\frac{6737}{60480}\\ \frac{263}{7560}\\ -\frac{863}{181440}\end{pmatrix},\ y_{s+n}\begin{pmatrix}\kappa_{21}\\ \kappa_{22}\\ \kappa_{23}\\ \kappa_{24}\\ \kappa_{25}\\ \kappa_{26}\\ \kappa_{27}\end{pmatrix} = \begin{pmatrix}\frac{1139}{11340}\\ \frac{94}{189}\\ \frac{11}{3780}\\ \frac{332}{2835}\\ -\frac{269}{3780}\\ \frac{22}{945}\\ -\frac{37}{11340}\end{pmatrix},\ y_{s+1}\begin{pmatrix}\kappa_{31}\\ \kappa_{32}\\ \kappa_{33}\\ \kappa_{34}\\ \kappa_{35}\\ \kappa_{36}\\ \kappa_{37}\end{pmatrix} = \begin{pmatrix}\frac{137}{1344}\\ \frac{27}{56}\\ \frac{387}{2240}\\ \frac{34}{105}\\ -\frac{243}{2240}\\ \frac{9}{280}\\ -\frac{29}{6720}\end{pmatrix},$$

$$y_{s+a}\begin{pmatrix}\kappa_{41}\\ \kappa_{42}\\ \kappa_{43}\\ \kappa_{44}\\ \kappa_{45}\\ \kappa_{46}\\ \kappa_{47}\end{pmatrix} = \begin{pmatrix}\frac{286}{2835}\\ \frac{464}{945}\\ \frac{128}{945}\\ \frac{1504}{2835}\\ \frac{58}{945}\\ \frac{16}{945}\\ -\frac{8}{2835}\end{pmatrix},\ y_{s+b}\begin{pmatrix}\kappa_{51}\\ \kappa_{52}\\ \kappa_{53}\\ \kappa_{54}\\ \kappa_{55}\\ \kappa_{56}\\ \kappa_{57}\end{pmatrix} = \begin{pmatrix}\frac{3715}{36288}\\ \frac{275}{1512}\\ \frac{2125}{12096}\\ \frac{250}{567}\\ \frac{3875}{12096}\\ \frac{235}{1512}\\ \frac{275}{36288}\end{pmatrix},\ y_{s+2}\begin{pmatrix}\kappa_{61}\\ \kappa_{62}\\ \kappa_{63}\\ \kappa_{64}\\ \kappa_{65}\\ \kappa_{56}\\ \kappa_{67}\end{pmatrix} = \begin{pmatrix}\frac{41}{420}\\ \frac{18}{35}\\ \frac{9}{140}\\ \frac{68}{105}\\ \frac{9}{140}\\ \frac{18}{35}\\ \frac{41}{420}\end{pmatrix}$$

Basic Properties of the Block Method

Order and Error Constant

This subsection establishes the linear operator $\ell[y(x_i);h]$ associated with the newly derived method.

Proposition 1

The local truncation error of the newly derived scheme is $C_{07}h^{07}y^{07}(x_n)+0(h^{08})$.

Proof

The linear difference operators associated with the newly derived method is given as:

$$\left.\begin{aligned}
\ell[y(x_\eta);h] &= y(x_\eta+mh)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2\\
\ell[y(x_\eta);h] &= y(x_\eta+nh)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2\\
\ell[y(x_\eta);h] &= y(x_\eta+h)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2\\
\ell[y(x_\eta);h] &= y(x_\eta+ah)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2\\
\ell[y(x_\eta);h] &= y(x_\eta+bh)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2\\
\ell[y(x_\eta);h] &= y(x_\eta+2h)-\left(\alpha_m(x_\eta+mh)+h\sum_{j=0}^{k}\left(\beta_j(x)f_{n+j}+\beta_k(x)f_{n+k}\right)\right), k=0,m,n,1,a,b,2
\end{aligned}\right\} \quad (7)$$

If $y(x)$ is sufficiently differentiable, we can use the Taylor series to expand Eq. (7) in the power of *h*. It is critical to emphasize that the first non-zero term in each formula in Equation (7) is $C_{07}h^{07}y^{07}(x_n)+0(h^{08})$

Definition 1. [28]

A linear multistep method is of order p if it satisfies the condition

$$c_0=c_1=c_2=c_3=\cdots=c_p=c_{p+1}=0,\ c_{p+2}\neq 0,$$

$$\left.\begin{aligned}
c_0 &= \sum_{j=0}^{k}\alpha_j\\
c_1 &= \sum_{j=0}^{k}\left(j\alpha_j-\beta_j\right)\\
&.\\
&.\\
&.\\
c_p &= \sum_{j=0}^{k}\left[\frac{1}{p!}j^p\alpha_j-\frac{1}{(p-1)!}\left(j^{p-1}\beta_j\right)\right],\ p=2,3,\cdots,q+1
\end{aligned}\right\} \quad (8)$$

$$c_0=c_1=c_2=c_3=\cdots=c_p=c_{p+1}=0,\ c_{p+2}\neq 0,$$ where

(8)

The parameter $c_{p+2}\neq 0$ is referred to as the error constant with the local truncation error defined as $x_{n+k}=c_{p+2}h^{p+2}y^{(p+2)}(x_n)+c_{p+3}h^{p+3}y^{(p+3)}(x_n)+c_{p+4}h^{p+4}y^{(p+4)}(x_n)+0(h^{p+5})$

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{\left(\frac{1}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{2713}{7560}\left(\frac{1}{3}\right)-\frac{15487}{60480}\left(\frac{2}{3}\right)-\frac{586}{2835}(1)-\frac{6737}{60480}\left(\frac{4}{3}\right)+\frac{263}{7560}\left(\frac{5}{3}\right)-\frac{863}{181440}(2)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{2}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{94}{189}\left(\frac{1}{3}\right)-\frac{11}{3780}\left(\frac{2}{3}\right)-\frac{332}{2835}(1)-\frac{269}{3780}\left(\frac{4}{3}\right)+\frac{22}{945}\left(\frac{5}{3}\right)-\frac{37}{11340}(2)\right] \\ \sum_{j=0}^{\infty}\frac{(1)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{27}{56}\left(\frac{1}{3}\right)+\frac{387}{2240}\left(\frac{2}{3}\right)+\frac{34}{105}(1)-\frac{243}{2240}\left(\frac{4}{3}\right)+\frac{9}{280}\left(\frac{5}{3}\right)-\frac{29}{6720}(2)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{4}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{464}{945}\left(\frac{1}{3}\right)+\frac{128}{945}\left(\frac{2}{3}\right)+\frac{1504}{2835}(1)-\frac{58}{945}\left(\frac{4}{3}\right)+\frac{16}{945}\left(\frac{5}{3}\right)-\frac{8}{2835}(2)\right] \\ \sum_{j=0}^{\infty}\frac{\left(\frac{5}{3}\right)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{275}{1512}\left(\frac{1}{3}\right)+\frac{2125}{12096}\left(\frac{2}{3}\right)+\frac{250}{567}(1)-\frac{3875}{12096}\left(\frac{4}{3}\right)+\frac{235}{1512}\left(\frac{5}{3}\right)-\frac{275}{36288}(2)\right] \\ \sum_{j=0}^{\infty}\frac{(2)^j}{j!}-y_n-\sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left[\frac{18}{35}\left(\frac{1}{3}\right)+\frac{9}{140}\left(\frac{2}{3}\right)+\frac{68}{105}(1)-\frac{9}{140}\left(\frac{4}{3}\right)+\frac{18}{35}\left(\frac{5}{3}\right)-\frac{41}{420}(2)\right] \end{bmatrix}$$

Corollary 1 [28].

The newly derived scheme's local truncation error is given by.

$$\left.\begin{matrix} (1.7326\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \\ (1.2903\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \\ (1.5310\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \\ (1.2903\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \\ (1.7326\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \\ (1.2661\times10^{-06})C_{07}h^{07}y^{07}(x_\eta)+0(h^{08}) \end{matrix}\right\} \quad (9)$$

Therefore, the newly derived scheme is of uniform order seven as well as error constant is given by

$$C_9=\begin{pmatrix} 1.7326\times10^{-06} \\ 1.2903\times10^{-06} \\ 1.5310\times10^{-06} \\ 1.2903\times10^{-06} \\ 1.7326\times10^{-06} \\ 1.2661\times10^{-06} \end{pmatrix}$$

Consistent

Traditionally, the method is consistent if the order of the method is greater than or equal to one.

Definition 2. [5]

By definition, the method is said to be zero stable as $h\rightarrow 0$ if the roots of the polynomial $\pi(r)=0$ satisfy $\left|\left[\sum A^0R^{k-1}\right]\right|\leq 1,$ and those roots with *R* = 1 must be simple.

Hence according to [8] it's found as

$$\pi(r)=\left| r\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}-\begin{bmatrix}0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\\0&0&0&0&0&1\end{bmatrix}\right|=\left|\begin{bmatrix}r&0&0&0&0&-1\\0&r&0&0&0&-1\\0&0&r&0&0&-1\\0&0&0&r&0&-1\\0&0&0&0&r&-1\\0&0&0&0&0&r-1\end{bmatrix}\right|=r^6(r-1)$$

Then, solving for r in $r^6(r-1)$,

Gives $r=0,\ 0, 0, 0,\ 0, 1$. Therefore, the method is zero stable.

Dahlquist's theorem states that the scheme is convergent, and consistency and zero-stability are analyzed and fulfilled [29].

Convergence

Theorem 1. [29]

Consistency and zero-stability are both required and sufficient conditions for a linear multistep method to be convergent.

Therefore, the newly derived scheme is convergent since it is consistent and zero-stable.

Linear Stability

Definition 3. [5]

The region of absolute stability of a numerical method is the set of complex values λh for which all solutions of the test problem $y'=-\lambda y$ will remain bounded as $n\to\infty$.

The concept of A-stability according to [8] is discussed by applying the test equation $y^{(k)}=\lambda^{(k)}y$ to yield

$$Y_m=\mu(z)Y_{m-1},\ z=\lambda h \tag{10}$$

Where $\mu(z)$ is the amplification matrix of the form

$$\mu(z)=\left(\xi^0-z\eta^{(0)}-z^1\eta^{(0)}\right)^{-1}\left(\xi^1-z\eta^{(1)}-z^1\eta^{(1)}\right) \tag{11}$$

The matrix $\mu(z)$ has Eigen values $(0,\ 0,\ \cdots,\ \xi_k)$ where ξ_k is called the stability function.

Thus, the stability function of the method is given by

$$\zeta=-\frac{\begin{pmatrix}367275240z^6-10000752628z^5+79785191834z^4+506079675630z^3\\+1827771257925z^2-4328380929600z+4444263936000\end{pmatrix}}{\begin{pmatrix}870912000z^6+-12802406400z^5+106077081600z^4-576108288000z^3\\+2057529600000z^2-4444263936000z+4444263936000\end{pmatrix}}$$

The boundary locus method is used to generate the hybrid method's stability polynomial. The polynomial is defined as

$$\left.\begin{aligned}\bar{h}(w)=&\left(-\frac{1}{5103}w^5+\frac{1}{5103}w^6\right)h^6+\left(-\frac{7}{2430}w^5-\frac{7}{2430}w^6\right)h^5+\left(-\frac{29}{1215}w^5+\frac{29}{1215}w^6\right)h^4\\&+\left(-\frac{7}{54}w^5-\frac{7}{54}w^6\right)h^3+\left(-\frac{25}{54}w^5+\frac{25}{54}w^6\right)h^2+\left(w^5+w^6\right)h+w^5+w^6\end{aligned}\right\} \tag{12}$$

The polynomial is used to plot the region as:

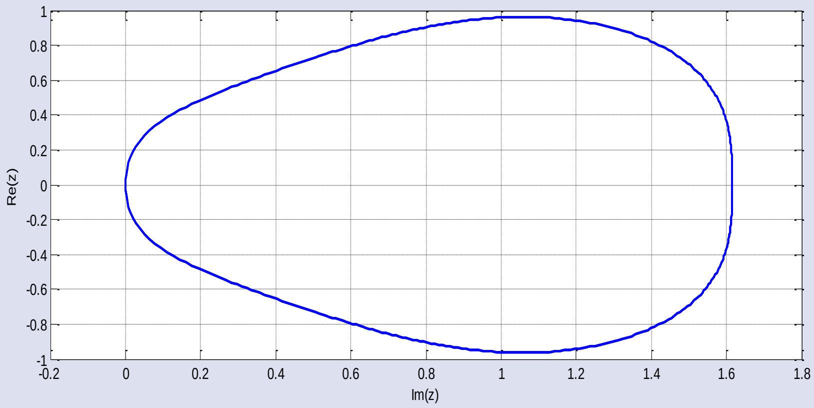

Figure 1. Showing the region of stability of the method.

The region of absolute stability of the method is a region in the complex z plane. The numerical solution of (6) satisfies $y_j \to 0$ *as* $j \to \infty$ for any initial condition [29]. The stability region obtained in Figure 1 is $A - stable$ according [29].

Numerical Problems

To validate the accuracy and convergence of the new method, the following IVPs are considered.
The results obtained from the new method are compared with the existing methods.
The following notations will be used in the tables and figures.
ES means exact solution
CS means computed solution
ENM means error in method
ES11 means error in Sunday 2011 [30]
EJASY means error in James *et al.* 2013 [6]
ESOAJ means error in Sunday *et al.* 2013 [5]
ESJOA means error in Sunday *et al.* 2015 [4]
EOSE means error in Okunuga *et al.* 2013 [31]
E2SEM means error in the Two-Step Implicit Obrechkoff Method of Omar and Adeyeye 2016 [9].
E2SBM means error in New Two-Step Obrechkoff-Type Block Method of Omar and Adeyeye 2016 [9]
EBYP means error in Badmus *et al*., 2015 [20]

Problem 1

A specific radioactive substance is known to decay at a rate proportional to its concentration. A block of this substance with a mass of $100g$ is observed. Its mass is reduced to $90g$ after 40 hours. Find an expression for the mass of the substance at any time and solve this problem for $\forall\ u \in 0, 1$ using the new method. The differential equation for the above problem is

$$\frac{du}{dv} = -\mu u,\ u(0) = 100,\ \forall\ u \in 0, 1. \tag{13}$$

Where u represent the substance's mass at any point in time v and μ are constants that specify the rate at which this particular substance decays. As a result, $100e^{-0.0026v}$ is the theoretical solution to equation (13).
See: [8, 17].

Problem 2: The oscillatory differential equation solved by [4, 5]

$$\frac{du}{dv} = -\sin(v) - 200(u - \cos(v)),\ h = 0.01,\ u(0) = 0, \tag{14}$$

with the exact solution

$$u(v) = \cos(v) - e^{-200v} \tag{15}$$

Problem 3: Consider the oscillatory differential equation [4, 32]

$$\frac{du}{dv} = -10(u-1)^2, h = 0.01, u(0) = 2, \tag{16}$$

with the exact solution

$$u(v) = 1 + \frac{1}{1+10v} \tag{17}$$

Problem 4: Consider the Highly stiff oscillatory differential equation [9, 20]

$$\frac{du}{dv} = -\psi u, h = 0.1, u(0) = \psi = 1, \tag{18}$$

with the exact solution

$$u(v) = \exp(-v) \tag{19}$$

Results and Discussions

Table 1. Showing the result for (13) with that of [6, 30].

V	ES	CS	ENM	ES11	EJASY
0.1	99.97400337970708570600	99.97400337970708570600	0.0000e-00	2.0000e-08	0.0000e-00
0.2	99.94801351765683795200	99.94801351765683795200	0.0000e-00	1.0000e-08	1.4211e-14
0.3	99.92203041209234205300	99.92203041209234205300	0.0000e-00	0.0000e-00	0.0000e-00
0.4	99.89605406125714006400	99.89605406125714006400	0.0000e-00	0.0000e-00	0.0000e-00
0.5	99.87008446339523065700	99.87008446339523065700	0.0000e-00	3.0000e-08	1.4211e-14
0.6	99.87008446339523065700	99.87008446339523065700	0.0000e-00	0.0000e-00	1.4211e-14
0.7	99.84412161675106900700	99.84412161675106900700	0.0000e-00	3.0000e-08	1.4211e-14
0.8	99.81816551956956667200	99.81816551956956667200	0.0000e-00	3.0000e-08	0.0000e-00
0.9	99.79221617009609147100	99.79221617009609147100	0.0000e-00	0.0000e-00	0.0000e-00
0.1	99.76627356657646737200	99.76627356657646737200	0.0000e-00	0.0000e-00	0.0000e-00

Table 2. Showing the result for oscillatory differential equation (13) with that of [4, 5].

V	ES	CS	ENM	ESJOA	ESOAJ
0.001	0.18126874692477177712	0.18126874692205980800	2.7120e-12	3.7249e-10	8.5812e-06
0.002	0.32967795396412439246	0.32967795396502736584	9.0297e-13	5.2169e-10	2.9379e-06
0.003	0.45118386391042716158	0.45118386390934856636	1.0786e-12	6.7870e-10	9.3961e-06
0.004	0.55066303589223450724	0.55066303589344506955	1.2106e-12	7.6010e-10	1.1305e-05
0.005	0.63210805885482676508	0.63210805885459932337	2.2744e-12	7.4126e-10	7.9107e-06
0.006	0.69878778814058064233	0.69878778814179783856	1.2172e-12	7.4495e-10	1.0313e-05
0.007	0.75337853615825529977	0.75337853615843502633	1.7973e-13	7.2211e-10	1.0426e-05
0.008	0.79807148217492301264	0.79807148217601089409	1.0879e-12	6.5649e-10	7.7981e-05
0.009	0.83466061205144457875	0.83466061205178772359	3.4315e-13	6.1326e-10	8.4900e-05
1.000	0.86461471717914105002	0.86461471718005258589	9.1154e-13	5.6367e-10	8.0388e-05

Table 3. Showing the result for oscillatory differential equation (15) with that of [4, 31].

V	ES	CS	ENM	ESJOA	EOSE
0.001	1.90909090884750640830	1.90909090889090909090	4.3403e-11	2.4025e-08	1.0700e-04
0.002	1.83333333337241953740	1.83333333333333333330	3.9086e-11	3.1560e-08	2.3800e-04
0.003	1.76923076920944483900	1.76923076923076923080	2.1324e-11	3.2631e-08	4.5100e-04
0.004	1.71428571432193859870	1.71428571428571428570	3.6224e-11	3.1192e-08	6.2000e-04
0.005	1.66666666668304290430	1.66666666666666666670	1.6376e-11	2.8877e-08	8.8400e-04
0.006	1.62500000002955801560	1.62500000000000000000	2.9558e-11	2.6370e-08	1.0300e-03
0.007	1.58823529413888054590	1.58823529411764705880	2.1234e-11	2.3953e-08	1.2700e-03
0.008	1.55555555557943834040	1.55555555555555555560	2.3883e-11	2.1734e-08	1.5300e-03

0.009	1.52631578949329163390	1.52631578947368421050	1.9607e-11	1.9740e-08	1.7500e-03
1.000	1.50000000001952055900	1.50000000000000000000	1.9521e-11	1.7969e-08	1.8100e-03

Table 4. Showing the result for oscillatory differential equation (15) with that of [9, 20].

V	ES	CS	ENM	E2SEM	E2SBM
0.1	0.90483741803595957316	0.90483741803596084590	1.2727e-15	7.5513e-05	9.0730e-12
0.2	0.81873075307798185867	0.81873075307798400161	2.1429e-15	6.8684e-05	1.1768e-11
0.3	0.74081822068171786607	0.74081822068170938478	8.4813e-15	1.2397e-04	2.3144e-11
0.4	0.67032004603563930074	0.67032004603564280973	3.5090e-15	1.1246e-04	2.8440e-11
0.5	0.60653065971263342360	0.60653065971262806724	5.3564e-15	1.5237e-04	3.1815e-11
0.6	0.54881163609402643263	0.54881163609403074200	4.3094e-15	1.3811e-05	3.4927e-11
0.7	0.49658530379140951470	0.49658530379140642905	3.0857e-15	1.6640e-04	3.6582e-11
0.8	0.44932896411722159143	0.44932896411722629572	4.7043e-15	1.5076e-04	3.8127e-11
0.9	0.40656965974059911188	0.40656965974059764972	1.4622e-15	1.7033e-04	3.8576e-11
1.0	0.36787944117144232160	0.36787944117144713602	4.8144e-15	1.5428e-04	3.9020e-11

Table 5. Showing the result for oscillatory differential equation (15) with that of [9, 20].

V	ES	CS	EBYP
0.1	0.90483741803595957316	0.90483741803596084590	1.5476e-10
0.2	0.81873075307798185867	0.81873075307798400161	1.3823e-10
0.3	0.74081822068171786607	0.74081822068170938478	1.3282e-10
0.4	0.67032004603563930074	0.67032004603564280973	1.1733e-10
0.5	0.60653065971263342360	0.60653065971262806724	1.1342e-10
0.6	0.54881163609402643263	0.54881163609403074200	9.9385e-11
0.7	0.49658530379140951470	0.49658530379140642905	9.6770e-11
0.8	0.44932896411722159143	0.44932896411722629572	8.4003e-11
0.9	0.40656965974059911188	0.40656965974059764972	8.2517e-11
1.0	0.36787944117144232160	0.36787944117144713602	7.0848e-11

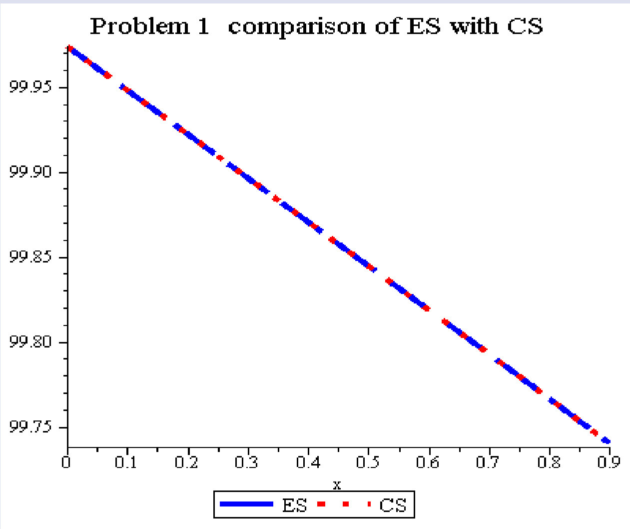

Figure 2. Graphical curves for Problem 1.

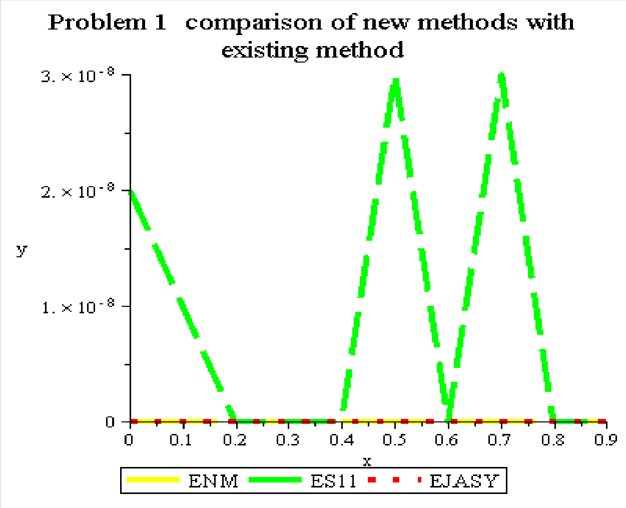

Figure 3. Graphical curves for Problem 1.

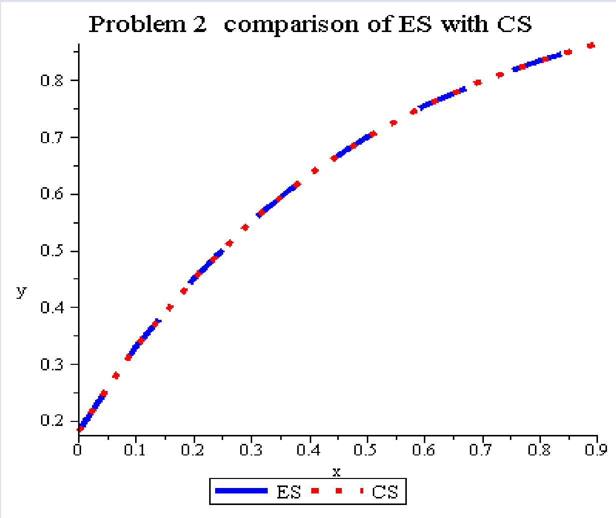

Figure 4. Graphical curves for Problem 2.

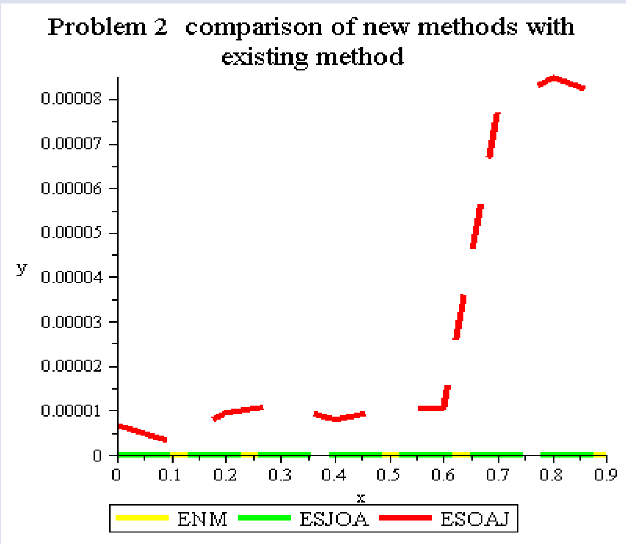

Figure 5. Graphical curves for Problem 2.

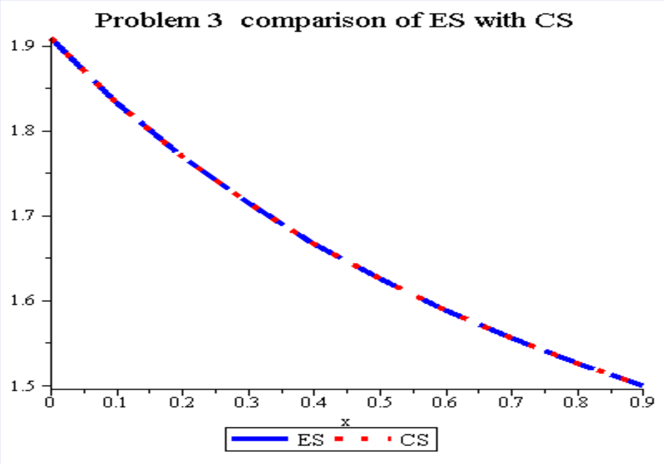

Figure 6. Graphical curves for Problem 3.

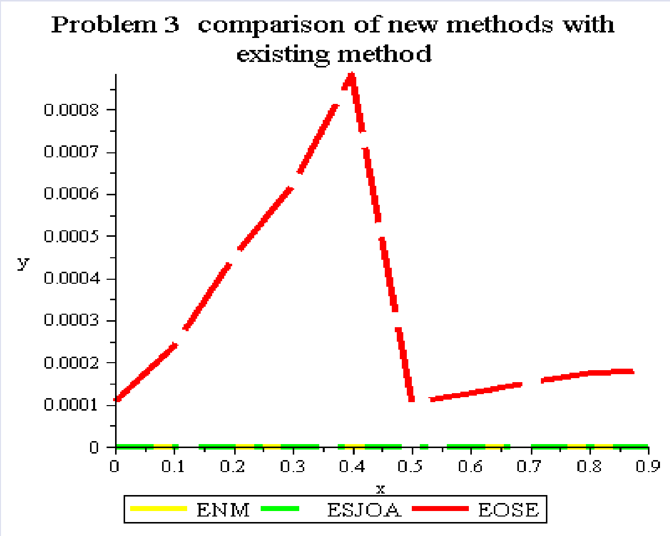

Figure 7. Graphical curves for Problem 3.

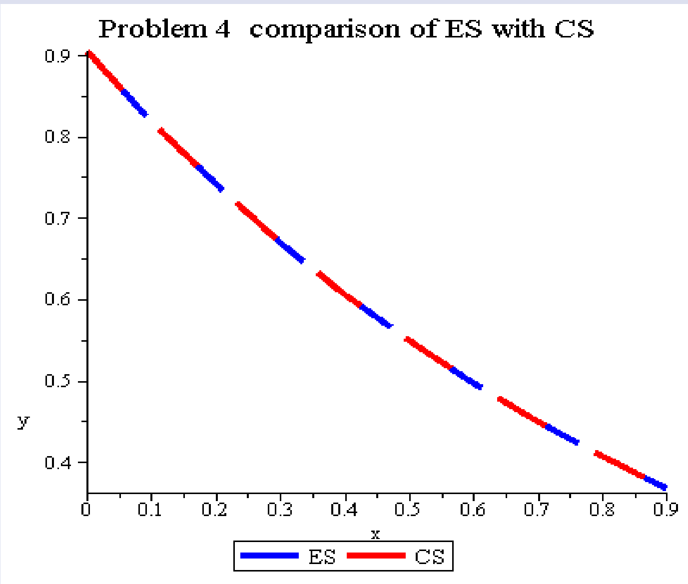

Figure 8. Graphical curves for Problem 4.

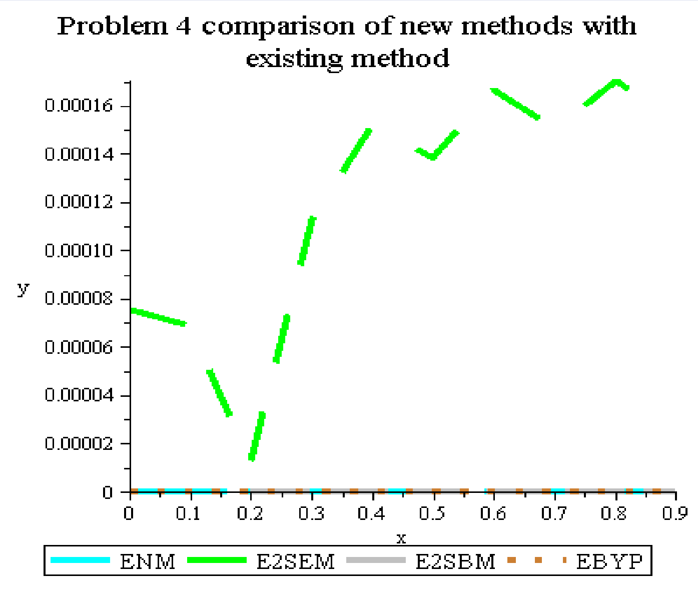

Figure 9. Graphical curves for Problem 4.

Discussion of results

The newly derived method was applied to four sample problems. Problem 1 is the differential equation solved by [6, 30]. From the result obtained in Table 1 above, the new method performs accurately than [6, 30]. Figures 2 and 3 show the graphical curve for problem 1. The oscillatory differential equation in problem 2 was considered by [4, 5]. In Table 2, the proposed method performed accurately than that of [4, 5]. From figures 4 and 5, the textual curve of problem 2 is accurate. The result converges and performs better than that of [4, 5]. The oscillatory differential equation in problem 3 was solved and compared with [4, 31]. The textual results showed the convergence of the new method over the existing one. Finally, the highly stiff oscillatory differential (18) was solved by [20, 30], and the new method was accurate, as seen in tables 4 and 5 and figures 7 and 8 above. Hence, from the results, the new method has shown better accuracy and faster convergence graphically. Figures 3, 5, 7, and 9 have shown the convergence of the new method closer to zero than those in the literature.

Conclusion

In this research, the application of a two-step block scheme is derived using the block approach for solving (1). The basic properties of the scheme were analyzed and satisfied all conditions. Some special classes of differential equations were implemented and compared with existing methods [4, 5, 6, 9, 20, 30, 31]. From tables 1 to 5, the new method proved to have better accuracy and faster convergence than the existing methods considered in this research. The graphical curve for problems 1 to 4 proved the convergence accuracy of the method. Therefore, the new methods have yielded a good result over the existing methods, both graphically and in tabular form.

Conflicts of interest

There are no conflicts of interest in this work.

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