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New Midpoint-type Inequalities of Hermite-Hadamard Inequality with Tempered Fractional Integrals

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ABSTRACT

In this research, we get some midpoint type inequalities of Hermite-Hadamard inequality via tempered fractional integrals. For this, we first obtain an identity. After that, using this identity and with the help of modulus function, Hölder inequality, power mean inequality, ongoing research and the papers mentioned, we have reached our intended midpoint type inequalities. Also, we give the special cases of our results. We see that our special results give earlier works.

Keywords: Hermite-Hadamard-type inequalities, Midpoint-type inequalities, Convex functions, Riemann-Liouville fractional integrals, Tempered fractional integrals.

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Introduction

Convex functions are an important concept that is used in mathematics, economics, engineering, statistics, and many other fields. In the field of statistics, these functions have various applications such as parameter estimation, classification, clustering, and regression. Also, in engineering, convex functions play an important role in many areas such as signal processing, control systems, data compression, optimization, and energy management. Lastly, in the realm of mathematics, convex functions find extensive applications in disciplines like analysis, differential equations, topology, geometry, and mathematical programming. Especially, they have an important place in the issue of inequalities in mathematics. The most known inequality of convex analysis is Hermite-Hadamard inequality, which was investigated by C. Hermite and J. Hadamard. The Hermite-Hadamard is a mathematical inequality states that the mean of the values of a function is an upper bound between the maximum and minimum values of the function. This can be expressed as [1,2]:

Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function $a, b \in \mathbb{R}$ with $a < b$ then the following inequalities hold

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \qquad (1)$$

If f is concave, both inequalities hold in the opposite direction. The left-hand side of the inequality (1) represents midpoint inequality, while the right-hand side of the inequality (1) represents trapezoid inequality.

On the other hand, fractional calculus has a long history, dating back to the correspondence between Leibniz and L'Hopital. Many mathematicians and physicists have contributed to the development of fractional calculus over the past three centuries, and there are now numerous books covering the topic. Recent theories and experiments have shown that fractional calculus is a powerful tool for describing non-classical phenomena in applied sciences and engineering [3-6]. Due to its mathematical properties, fractional calculus is commonly used to study anomalous kinetics in physics, biology, chemistry, and other complex systems [3].

Fractional calculus has two important topics that are fractional derivative and fractional integral. The first fractional integrals were described by Gottfried Wilhelm Leibniz in 1695. However, the modern theory of fractional integration was developed in the 19th century by Augustin Louis Cauchy and Liouville. Cauchy was one of the first to demonstrate that fractional integrals can be calculated analytically. On the other hand, Liouville made important studies on the theory of fractional integrals and obtained many results about fractional derivatives. The modern theory of fractional integration has been developed since the early 20th century. The general theory of fractional integrals was developed in the 1930s and 1940s, especially by Norbert Wiener and Joseph L. Doob. Besides, various types of fractional derivatives, including Riemann-Liouville, Caputo [7,8], Riesz [8], and Hilfer [6,9], have been introduced for practical applications. Fractional integrals have been discovered to be an important tool for modeling stochastic processes. Advances in this area include topics such as fractional derivatives of random walks and fractional Brownian motions. In recent years, fractional integration and derivatives have become a topic with applications in many fields such as mathematical physics, electrotechnics, materials science, biomedical engineering, fluid dynamics, and finance. One can find the papers referenced in [10-14] about fractional calculus.

In this article, we introduce several necessary definitions to present our main results. The definition of Riemann-Liouville integral operators is as follows [15]:

Definition 1. The Riemann-Liouville integrals with $\alpha > 0$ for $f \in L_1[a,b]$ are:

$$J_{a+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a \tag{2}$$

and

$$J_{b-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad x < b. \tag{3}$$

It is evident that the Riemann-Liouville integrals coincide with the classical integrals when $\alpha = 1$.

The trapezoid-type inequalities for convex functions were first established by Dragomir and Agarwal in [16], while Kırmacı was the first to prove midpoint-type inequalities for convex functions in [17]. Fractional midpoint-type inequalities and trapezoid-type inequalities for convex functions were presented by Sarıkaya et al. in [18] and by Iqbal et al. in [19], respectively. To learn more about fractional integral inequalities, see [8,20] and the references mentioned there.

Mohammed et al. defined new function and gave some properties of this function as follows [21]:

Definition 2. For the real numbers $\alpha > 0$ and $x, \lambda \geq 0$ the $\lambda -$ incomplete gamma function is defined as follows:

$$\Upsilon_{\lambda}(\alpha, x) = \int_0^x t^{\alpha-1} e^{-\lambda t} dt.$$

Taking $\lambda = 1$ it turns the incomplete gamma function [22]:

$$\Upsilon(\alpha, x) := \int_0^x t^{\alpha-1} e^{-t} dt.$$

Remark 3. For real numbers $\alpha > 0$; $x, \lambda \geq 0$ and $a < b$, there are the following equalities:

$$1)\ \Upsilon_{\lambda(b-a)}(\alpha, x) = \int_0^1 t^{\alpha-1} e^{-\lambda(b-a)t} dt = \left(\frac{1}{b-a}\right)^{\alpha} \Upsilon_{\lambda}(\alpha, b-a),$$

$$2) \int_0^1 1)\ \Upsilon_{\lambda(b-a)}(\alpha, x)\, dx = \frac{\Upsilon_{\lambda}(\alpha, b-a)}{(b-a)^{\alpha}} - \frac{\Upsilon_{\lambda}(\alpha+1, b-a)}{(b-a)^{\alpha+1}}.$$

Now, we will review the basic definitions and introduce new symbols for the tempered fractional operators [23,24].

Definition 3. Let $f \in L_1[a,b]$ and $\alpha > 0$, $\lambda \geq 0$. The fractional tempered integral operators $J_{a+}^{(\alpha,\lambda)} f(x)$ and $J_{b-}^{(\alpha,\lambda)} f(x)$ are presented by

$$J_{a+}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} e^{-\lambda(x-t)} f(t)dt, \quad x \in [a,b] \tag{4}$$

and

$$J_{b-}^{(\alpha,\lambda)} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} e^{-\lambda(t-x)} f(t)dt, \quad x \in [a,b]. \tag{5}$$

If $\lambda = 0$, then the fractional integrals in (4) and (5) reduce to the Riemann-Liouville fractional integrals in (2) and (3), respectively.

In [21], Mohammed et al. gave Hermite-Hadamard inequality involved λ-incomplete gamma function and tempered fractional operators as follows:

Theorem 4. Let $f: [a,b] \to \mathbb{R}$ be a convex L_1 function on $[a,b]$ with $a < b$. Then, the following inequlaties hold:

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha)}{2(b-a)^{\alpha})\, \Upsilon_{\lambda(b-a)}(\alpha, 1)} \left[J_{a+}^{(\alpha,\lambda)} f(b) + J_{b-}^{(\alpha,\lambda)} f(a)\right] \leq \frac{f(a)+f(b)}{2}, \tag{6}$$

for $\alpha > 0$ and $\lambda \geq 0$.

Tempered fractional calculus is considered as an extension of fractional calculus. Buschman's work [25] introduced the definitions of fractional integration with weak singular and exponential kernels which later led to the development of tempered fractional integration. Meerschaert, Samko, Srivastava [8,26,27], and other researchers have studied various definitions of tempered fractional integration. In [21], Hermite-Hadamard-type inequalities involving tempered fractional integrals were established for convex functions, extending previously published results such as Riemann integrals and Riemann-Liouville fractional integrals. The authors followed the techniques developed by Sarıkaya et al. [18,28] to establish these inequalities.

In this paper, we first obtain an identity for the midpoint side of Hermite-Hadamard inequality [21]. After that using this identity and with the help of modulus function, Hölder inequality, power mean inequality, ongoing research and the papers mentioned above, we get number of midpoint-type inequalities that involve tempered fractional integral operators for the differentiable convex mappings. Also, we give the special cases of these inequalities.

Midpoint-Type Inequalities Involving Tempered Fractional Integrals

Using tempered fractional integrals, we build Hermite-Hadamard type inequalities through differentiable convex functions in this section. To start, we'll define the following identity for obtaining such inequalities.

Lemma 5. Consider that $f:[a,b]\to\mathbb{R}$ is a differentiable function on (a,b) such that $f',f\in L_1[a,b]$. The following equality for Tempered fractional integrals holds with $\alpha>0$, $\lambda\geq 0$:

$$f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left[J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right]=\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}I_k \qquad (7)$$

where

$$I_1=\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(tb+(1-t)a)dt,\qquad I_2=\int_0^{1/2}-\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(ta+(1-t)b)dt,$$

$$I_3=\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right]f'(tb+(1-t)a)dt,$$

$$I_4=\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]f'(ta+(1-t)b)dt.$$

Proof. We can obtain the following result by applying integration by parts

$$I_1=\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(tb+(1-t)a)dt \qquad (8)$$

$$=\frac{\Upsilon_{\lambda(b-a)}(\alpha,t)\,f(tb+(1-t)a)}{b-a}\bigg|_0^{1/2}-\frac{1}{b-a}\int_0^{1/2}t^{\alpha-1}\,e^{-\lambda(b-a)t}f(tb+(1-t)a)dt$$

$$=\frac{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)}{b-a}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_0^{1/2}t^{\alpha-1}\,e^{-\lambda(b-a)t}f(tb+(1-t)a)dt.$$

By computing the remaining integrals in a similar method, we arrive at

$$I_2=\int_0^{1/2}-\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(ta+(1-t)b)dt \qquad (9)$$

$$=\frac{\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)}{b-a}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_0^{1/2}t^{\alpha-1}\,e^{-\lambda(b-a)t}f(ta+(1-t)b)dt,$$

$$I_3=\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right]f'(tb+(1-t)a)dt \qquad (10)$$

$$=\frac{\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]}{b-a}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}t^{\alpha-1}\,e^{-\lambda(b-a)t}f(tb+(1-t)a)dt,$$

and

$$I_4=\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]f'(ta+(1-t)b)dt \tag{11}$$

$$=\frac{\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}\left(\alpha,\frac{1}{2}\right)\right]}{b-a}f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_{\frac{1}{2}}^{1}t^{\alpha-1}e^{-\lambda(b-a)t}f(ta+(1-t)b)dt.$$

By summing up the inequalities (8)-(11), we conclude that

$$\sum_{k=1}^{4}I_k=\frac{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}{b-a}f\left(\frac{a+b}{2}\right) \tag{12}$$

$$-\frac{1}{b-a}\left\{\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}f(tb+(1-t)a)dt+\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}f(ta+(1-t)b)dt\right\}.$$

When the above integrals are rewritten using the variable change $x=tb+(1-t)a$ and $x=ta+(1-t)b$, the following expressions are obtained

$$\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}f(tb+(1-t)a)dt=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J_{a+}^{(\alpha,\lambda)}f(b), \tag{13}$$

$$\int_0^1 t^{\alpha-1}e^{-\lambda(b-a)t}f(ta+(1-t)b)dt=\frac{\Gamma(\alpha)}{(b-a)^{\alpha}}J_{b-}^{(\alpha,\lambda)}f(a). \tag{14}$$

By replacing the equalities (13) and (14) in (12) and applying a multiplication operation to the resulting expression using $\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}$, we arrive at the following equality

$$\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}I_k=f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)(b-a)^{\alpha}}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}.$$

This completes the demonstration of the statement.

Theorem 6. Suppose that $f:[a,b]\to\mathbb{R}$ be a differentiable function on the interval (a,b) with $f,f'\in L_1[a,b]$ and $|f'|$ is a convex function on $[a,b]$ with $a<b$. Then, we have the following inequality for $\alpha>0$, $\lambda\geq 0$ and the Tempered fractional integrals

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right|\leq A_1^{(\alpha,\lambda)}(a,b)(|f'(a)|+|f'(b)|) \tag{15}$$

where

$$A_1^{(\alpha,\lambda)}(a,b)=\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,dt+\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right\}.$$

Proof. By taking the absolute value of both sides of the equation in Lemma 5, we obtain

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right|\leq\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}|I_k|.$$

Using the convexity of the function $|f'|$, we get

$$|I_1|=\left|\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(tb+(1-t)a)dt\right| \tag{16}$$

$$\leq \int_0^{1/2} \left| \Upsilon_{\lambda(b-a)}(\alpha,t)\right| |f'(tb+(1-t)a)|dt$$

$$\leq \int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t) \, [t|f'(b)| + (1-t)|f'(a)|]dt$$

$$= |f'(a)| \int_0^{1/2} (1-t)\Upsilon_{\lambda(b-a)}(\alpha,t)dt + |f'(b)| \int_0^{1/2} t \, \Upsilon_{\lambda(b-a)}(\alpha,t)dt.$$

Utilizing the same approach for the other integrals, the listed expressions are appeared

$$|I_2| \leq |f'(a)| \int_0^{1/2} t \, \Upsilon_{\lambda(b-a)}(\alpha,t)dt + |f'(b)| \int_0^{1/2} (1-t) \, \Upsilon_{\lambda(b-a)}(\alpha,t)dt, \tag{17}$$

$$|I_3| \leq |f'(a)| \int_{1/2}^{1} (1-t)\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt \tag{18}$$

$$+|f'(b)| \int_{1/2}^{1} t\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt$$

and

$$|I_4| \leq |f'(a)| \int_{1/2}^{1} t\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt \tag{19}$$

$$+|f'(b)| \int_{1/2}^{1} (1-t)\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt.$$

By taking the sum of the inequalities from (16) to (19) and after that multiplying the outcome of expression by $\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}$, we conclude that

$$\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)} \sum_{k=1}^{4} I_k = A_1^{(\alpha,\lambda)}(a,b)(|f'(a)| + |f'(b)|).$$

The proof is completed.

Remark 7. Setting $\lambda = 0$ in inequality (15) and applying the property that is $|t_1{}^{\alpha} - t_2{}^{\alpha}| \leq |t_1 - t_2|^{\alpha}$ for $t_1, t_2 \in [0,1]$ and $\alpha \in (0,1]$ to the obtained inequality, then it follows that:

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}} \{J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)\} \right|$$
$$\leq \frac{b-a}{2^{\alpha+1}(\alpha+1)} (|f'(a)| + |f'(b)|),$$

which is proved by Iqbal et al. in [19].

Remark 8. Setting $\alpha = 1$ and $\lambda = 0$ in (15), we obtain

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{b-a} \int_a^b f(x)dx \right| \leq \frac{b-a}{8} (|f'(a)| + |f'(b)|),$$

which is proved by Kırmacı in [17].

Theorem 9. Assuming that $f: [a,b] \to \mathbb{R}$ is a differentiable function on (a,b) with $f, f' \in L_1[a,b]$ and $|f'|^q$, $q > 1$ is a convex function on $[a,b]$, the subsequent inequality is satisfied

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right| \tag{20}$$
$$\leq A_2{}^{(\alpha,\lambda)}(a,b)\left\{\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{1/q}\right\},$$

where

$$A_2^{(\alpha,\lambda)}(a,b)=\frac{b-a}{2\;\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{\left(\int_0^{1/2}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}+\left(\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\right\}$$

and $\frac{1}{p}+\frac{1}{q}=1$.

Proof. By utilizing the properties of the modulus function and Lemma 5, we can conclude that

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right|\leq\frac{b-a}{2\;\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}|I_k|.$$

The Hölder inequality and the convexity of $|f'|^q$ allow us to arrive the following inequality

$$|I_1|=\left|\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(tb+(1-t)a)dt\right| \tag{21}$$
$$\leq\left(\int_0^{1/2}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\left(\int_0^{1/2}|f'(tb+(1-t)a)|^q\,dt\right)^{1/q}$$
$$\leq\left(\int_0^{1/2}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{1/q}.$$

In the same way,

$$|I_2|=\left|\int_0^{1/2}-\Upsilon_{\lambda(b-a)}(\alpha,t)\,f'(ta+(1-t)b)dt\right| \tag{22}$$
$$\leq\left(\int_0^{1/2}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{1/q},$$

$$|I_3|=\left|\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)-\Upsilon_{\lambda(b-a)}(\alpha,1)\right]f'(tb+(1-t)a)dt\right| \tag{23}$$
$$\leq\left(\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{1/q},$$

and

$$|I_4|=\left|\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]f'(ta+(1-t)b)dt\right| \tag{24}$$
$$\leq\left(\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^p dt\right)^{1/p}\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{1/q}.$$

When the inequalities are added up (21)-(24) and subsequently the result is multiplied by $\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}$, the next expression is displayed:

$$\frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}|I_k| \le A_2^{(\alpha,\lambda)}(a,b)\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{8}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{8}\right)^{1/q}\right].$$

Thus, the hypothesis of the theorem is obtained.

Remark 10. The inequality can be obtained by setting $\lambda = 0$ in (20):

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}}\{J_{a+}^{\alpha}f(b)+J_{b-}^{\alpha}f(a)\}\right| \le \frac{b-a}{2^{\alpha+1}(\alpha p+1)^{1/p}}\left[\left(\frac{3|f'(a)|^q+|f'(b)|^q}{4}\right)^{1/q}+\left(\frac{|f'(a)|^q+3|f'(b)|^q}{4}\right)^{1/q}\right],$$

which is proved by Iqbal et al. in [19].

Remark 11. The inequality can be obtained by setting $\alpha = 1$ and $\lambda = 0$ in (20):

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \le \frac{b-a}{16}\left(\frac{4}{p+1}\right)^{1/p}\left\{\left[3|f'(a)|^{p/(p-1)}+|f'(b)|^{p/(p-1)}\right]^{(p-1)/p}\right.$$

$$\left.+\left[|f'(a)|^{p/(p-1)}+3|f'(b)|^{p/(p-1)}\right]^{(p-1)/p}\right\},$$

which is proved by Kırmacı in [17].

Theorem 12. Let $f:[a,b]\to\mathbb{R}$ be a differentiable function on (a,b) such that $f,f'\in L_1[a,b]$. If the function $|f'|^q$ is convex on $[a,b]$ with $q \ge 1$, then the following inequality holds:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right| \tag{25}$$

$$\le \frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}[|f'(a)|+|f'(b)|]\left\{A_3^{(\alpha,\lambda)}(a,b)\left(\int_0^{1/2}\Upsilon_{\lambda(b-a)}(\alpha,t)\,dt\right)^{1-1/q}\right.$$

$$\left.+A_4^{(\alpha,\lambda)}(a,b)\left(\int_{1/2}^{1}[\,\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)]\,dt\right)^{1-1/q}\right\},$$

where

$$A_3^{(\alpha,\lambda)}(a,b)=\left(\int_0^{1/2}t\,\Upsilon_{\lambda(b-a)}(\alpha,t)\,dt\right)^{1/q}+\left(\int_0^{1/2}(1-t)\,\Upsilon_{\lambda(b-a)}(\alpha,t)\,dt\right)^{1/q},$$

$$A_4^{(\alpha,\lambda)}(a,b)=\left(\int_{1/2}^{1}t\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right)^{1/q}+\left(\int_{1/2}^{1}(1-t)\left[\Upsilon_{\lambda(b-a)}(\alpha,1)-\Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right)^{1/q}.$$

Proof. The absolute value of Lemma 5 yields:

$$\left|f\left(\frac{a+b}{2}\right)-\frac{\Gamma(\alpha)}{2\,(b-a)^{\alpha}\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\left\{J_{a+}^{(\alpha,\lambda)}f(b)+J_{b-}^{(\alpha,\lambda)}f(a)\right\}\right| \le \frac{b-a}{2\,\Upsilon_{\lambda(b-a)}(\alpha,1)}\sum_{k=1}^{4}|I_k|.$$

Applying the power-mean integral inequality for $q>1$ and using the fact that for $0\le r<1$, $\sum_{i=1}^{n}(a_i+b_i)^r \le \sum_{i=1}^{n}a_i^r+\sum_{i=1}^{n}b_i^r$ with $a_i,b_i\ge 0$, $i=1,2,\dots,n$, then we obtain:

$$|I_1| = \left|\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, f'(tb+(1-t)a)dt\right| \tag{26}$$

$$\leq \int_0^{1/2} \left|\Upsilon_{\lambda(b-a)}(\alpha,t)\right| |f'(tb+(1-t)a)|dt$$

$$= \int_0^{1/2} \left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^{1/p}\left[\Upsilon_{\lambda(b-a)}(\alpha,t)\right]^{1/q} |f'(tb+(1-t)a)|dt$$

$$\leq \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q} \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, |f'(tb+(1-t)a)|^q\, dt\right)^{1/q}$$

$$\leq \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q} \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, [t|f'(b)|^q + (1-t)|f'(a)|^q]\, dt\right)^{1/q}$$

$$= \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q} \times \left(|f'(b)|^q \int_0^{1/2} t\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt + |f'(a)|^q \int_0^{1/2} (1-t\,)\,\Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1/q}$$

$$\leq \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q}$$

$$\times \left[|f'(b)|\left(\int_0^{1/2} t\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt\right)^{1/q} + |f'(a)|\left(\int_0^{1/2} (1-t\,)\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt\right)^{1/q}\right].$$

In the same way,

$$|I_2| = \left|\int_0^{1/2} -\Upsilon_{\lambda(b-a)}(\alpha,t)\, f'(ta+(1-t)b)dt\right| \tag{27}$$

$$= \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q} \times \left(|f'(a)|^q \int_0^{1/2} t\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt + |f'(b)|^q \int_0^{1/2} (1-t\,)\,\Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1/q}$$

$$\leq \left(\int_0^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt\right)^{1-1/q}$$

$$\times \left[|f'(a)|\left(\int_0^{1/2} t\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt\right)^{1/q} + |f'(b)|\left(\int_0^{1/2} (1-t\,)\,\Upsilon_{\lambda(b-a)}(\alpha,t)dt\right)^{1/q}\right],$$

$$|I_3| = \left|\int_{1/2}^{1}\left[\Upsilon_{\lambda(b-a)}(\alpha,t) - \Upsilon_{\lambda(b-a)}(\alpha,1)\right] f'(tb+(1-t)a)dt\right| \tag{28}$$

$$\leq \left(\int_{\frac{1}{2}}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right)^{1-\frac{1}{q}} \times \left\{ |f'(b)|^q \int_{1/2}^{1} t\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right.$$

$$\left. + |f'(a)|^q \int_{1/2}^{1} (1-t)\left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t)\right]dt\right\}^{1/q}$$

$$\leq \left(\int_{1/2}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1-1/q}$$

$$\times \left\{ |f'(b)| \left(\int_{1/2}^{1} t \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1/q} \right.$$

$$\left. + |f'(a)| \left(\int_{1/2}^{1} (1-t) \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1/q} \right\},$$

and

$$|I_4| = \left| \int_{1/2}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] f'(ta+(1-t)b) dt \right| \tag{29}$$

$$\leq \left(\int_{1/2}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1-1/q}$$

$$\times \left\{ |f'(a)|^q \int_{1/2}^{1} t \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right.$$

$$\left. + |f'(b)|^q \int_{1/2}^{1} (1-t) \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right\}^{1/q}$$

$$\leq \left(\int_{1/2}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1-1/q}$$

$$\times \left\{ |f'(a)| \left(\int_{1/2}^{1} t \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1/q} \right.$$

$$\left. + |f'(b)| \left(\int_{1/2}^{1} (1-t) \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1/q} \right\}.$$

By totaling the inequalities from (26) to (29), it follows that

$$\sum_{k=1}^{4} |I_k| \leq \left(\int_{0}^{1/2} \Upsilon_{\lambda(b-a)}(\alpha,t)\, dt \right)^{1-1/q} \left\{ A_3^{(\alpha,\lambda)}(a,b)[|f'(a)| + |f'(b)|] \right\}$$

$$+ \left(\int_{1/2}^{1} \left[\Upsilon_{\lambda(b-a)}(\alpha,1) - \Upsilon_{\lambda(b-a)}(\alpha,t) \right] dt \right)^{1-1/q} \left\{ A_4^{(\alpha,\lambda)}(a,b)[|f'(a)| + |f'(b)|] \right\}.$$

This completes the proof.

Remark 13. The inequality can be obtained by setting $\lambda = 0$ in (25) and using for $0 \leq r < 1$ $\sum_{i=1}^{n} (a_i + b_i)^r \leq \sum_{i=1}^{n} a_i^r + \sum_{i=1}^{n} b_i^r$ with $a_i, b_i \geq 0$, $i = 1,2,\dots,n$

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)}{2\,(b-a)^{\alpha}} \{ J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a) \} \right| \leq \frac{b-a}{2^{\alpha+1}(\alpha+1)} [|f'(a)| + |f'(b)|] \left\{ \frac{(\alpha+1)^{1/q} + (\alpha+3)^{1/q}}{[2(\alpha+2)]^{1/q}} \right\}$$

which is proved by Iqbal et al. in [19].

Remark 14. The inequality can be obtained by setting $\alpha = 1$ and $\lambda = 0$ in (25):

$$\left|\frac{1}{b-a}\int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)\right| \leq \frac{b-a}{8}\left(\frac{1+2^{1/q}}{3^{1/q}}\right)[|f'(a)| + |f'(b)|],$$

which is proved by Kırmaci et al. in [17].

Conflict interests

The authors declare that they have no conflict interests.

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