Cumhuriyet Sci. J., 45(1) (2024) 135-146
DOI: https://doi.org/10.17776/csj.1312302

Cumhuriyet Science Journal

| csj.cumhuriyet.edu.tr | Founded: 2002 ISSN: 2587-2680 e-ISSN: 2587-246X Publisher: Sivas Cumhuriyet University

On Almost $C(\alpha)$- Manifold Satisfying Certain Curvature Conditions

Tuğba Mert [1,a,*], Mehmet Atçeken[1,b], Pakize Uygun [1,c]

[1] *Department of Mathematics, Faculty of Sciences, Sivas Cumhuriyet University, Sivas, Türkiye.*
**Corresponding author*

Research Article

History
Received: 09/06/2023
Accepted: 21/12/2023

ABSTRACT

This research article is about the geometry of the almost $C(\alpha)$- manifold. Some important properties of the almost $C(\alpha)$- manifold with respect to the W_3- curvature tensor, such as W_3-flat and W_3- semi-symmetry, are investigated. The relationship of W_3- curvature tensor with Riemann, Ricci, projective, concircular and quasi-conformal curvature tensor is discussed on the almost $C(\alpha)$- manifold and many important results are obtained. In addition, W_3- pseudo symmetry and W_3- Ricci pseudo symmetry are investigated for the almost $C(\alpha)$-manifold. The results obtained are interesting and give an idea about the geometry of the almost $C(\alpha)$-manifold.

Keywords: $C(\alpha)$- Manifold, W_3- Curvature Tensor, Pseudo-symmetric manifold.

[a] *tmert@cumhuriyet.edu.tr* *https://orcid.org/0000-0001-8258-8298*
[b] *mehmet.atceken382@gmail.com* *https://orcid.org/0000-0002-1242-4359,*
[c] *pakizeuygun@hotmail.com* *https://orcid.org/0000-0001-8226-4269*

Introduction

In recent years, many geometers have defined different curvature tensors on many different manifolds such as Sasakian, para-Sasakian, Lorentzian paracontact, Lorentzian para-Sasakian. The curvature tensor is a very important concept for manifolds. These defined curvature tensors have been characterized in accordance with the properties of the manifold used and their relations with different curvature tensors have been established. Many authors have studied the symmetry case of various manifolds. The flatness of the curvature tensor and the semi-symmetrical curvature conditions revealed interesting properties of manifolds.

Motivated by the work of many geometers using different manifolds and different curvature tensors, this research paper deals with some curvature properties of the almost $C(\alpha)$- manifold, which is a subclass of almost contact metric manifolds and the general form of co-Keahler, Kenmotsu, Sasakian manifolds.

The paracontact geometry was studied at first time by Kaneyuki and Williams. And then Zamkovoy gave a characterization of the paracontact metric manifolds and their subclasses [1-2]. Paracontact metric manifolds are studies widely by geometers and they defined a new type of paracontact geometry that is called paracontact metric (k,μ)- spaces [3-8]. And also, semi-symmetric spaces which are generalization of locally symmetric spaces were introduced by some authors [9-10].

A (1,3)-type curvature tensor

$$\begin{aligned} T(\varpi_1,\varpi_2)\varpi_3 = \; & a_0R(\varpi_1,\varpi_2)\varpi_3 + a_1S(\varpi_2,\varpi_3)\varpi_1 + a_2S(\varpi_1,\varpi_3)\varpi_2 \\ & + a_3S(\varpi_1,\varpi_2)\varpi_3 + a_4g(\varpi_2,\varpi_3)Q\varpi_1 + a_5g(\varpi_1,\varpi_3)Q\varpi_2 \\ & + a_6g(\varpi_1,\varpi_2)Q\varpi_3 + a_7r[g(\varpi_2,\varpi_3)\varpi_1 - g(\varpi_1,\varpi_3)\varpi_2] \end{aligned} \tag{1}$$

on an n-dimensional semi-Riemann manifold was defined by M. Tripathi and P. Gupta [11]. Here $a_0,a_1,\ldots,a_7$ are regular functions; R,S,Q and r are respectively Riemann curvature tensor, Ricci tensor, Ricci operator and scalar curvature. If we specifically choose $a_0=1, a_2=-a_4=\frac{-1}{2n}, a_1=a_3=a_5=a_6=a_7=0$ in equation (1), the W_3 −curvature tensor of a $(2n+1)$-dimensional manifold is obtained as

$$W_3(\varpi_1,\varpi_2)\varpi_3 = R(\varpi_1,\varpi_2)\varpi_3 - \frac{1}{2n}[S(\varpi_1,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)Q\varpi_1]. \tag{2}$$

In recent years, many geometers have defined different curvature tensors on many different manifolds such as Sasakian, para-Sasakian, Lorentzian paracontact, Lorentzian para-Sasakian [12-17], [18-19],[23-25]. Based on the many studies mentioned above, in this article, the curvature conditions of almost $C(\alpha)$ −manifold such as $W_3(\varpi_1,\varpi_2)R=0, R(\varpi_1,\varpi_2)W_3=0, W_3(\varpi_1,\varpi_2)P=0, W_3(\varpi_1,\varpi_2)\tilde{Z}=0, W_3(\varpi_1,\varpi_2)S=0$ and $W_3(\varpi_1,\varpi_2)\tilde{C}=0$ are searched. In addition, W_3 −pseudo symmetry and W_3 −Ricci pseudo symmetry are investigated for the almost $C(\alpha)$ −manifold.

Preliminary

Let M be a differentiable manifold with $(2n+1)$ −dimensional. If the condition

$$\phi^2\varpi_1 = -\varpi_1 + \eta(\varpi_1)\xi \text{ and } \eta(\xi) = 1 \tag{3}$$

satisfies on M where ϕ is tesor field with type $(1,1)$, ξ is a vector field and η is a $1-$form, then we say that (ϕ, ξ, η) is an almost contact structure. Also, we say that (M, ϕ, ξ, η) is an almost contact manifold. Let g be a metric with condition

$$g(\phi\varpi_1, \phi\varpi_2) = g(\varpi_1, \varpi_2) - \eta(\varpi_1)\eta(\varpi_2) \text{ and } g(\varpi_1, \xi) = \eta(\varpi_1), \tag{4}$$

for all $\varpi_1, \varpi_2 \in \chi(M)$ and $\xi \in \chi(M)$. In this case, we say that (ϕ, ξ, η, g) is almost contact metric structure and (M, ϕ, ξ, η, g) is almost contact metric manifold. Moreover, we have the property

$$g(\phi\varpi_1, \varpi_2) = -g(\varpi_1, \phi\varpi_2)$$

for all $\varpi_1, \varpi_2 \in \chi(M)$ on M manifold with $(2n+1)-$dimensional. The fundamental $2-$form of (ϕ, ξ, η, g) almost contact metric structure is the Φ transformation such that

$$\Phi(\varpi_1, \varpi_2) = g(\varpi_1, \phi\varpi_2), \eta \wedge \Phi^n \neq 0.$$

for all $\varpi_1, \varpi_2 \in \chi(M)$.

Let M be almost contact metric manifold and R be Riemann curvature tensor of it, which provides the following

$$\begin{aligned} R(\varpi_1, \varpi_2, \varpi_3, \varpi_4) = \quad & R(\varpi_1, \varpi_2, \phi\varpi_3, \phi\varpi_4) + \alpha\{-g(\varpi_1, \varpi_3)g(\varpi_2, \varpi_4) \\ & + g(\varpi_1, \varpi_4)g(\varpi_2, \varpi_3) + g(\varpi_1, \phi\varpi_3)g(\varpi_2, \phi\varpi_4) \\ & - g(\varpi_1, \phi\varpi_4)g(\varpi_2, \phi\varpi_3)\} \end{aligned} \tag{5}$$

for all $\varpi_1, \varpi_2, \varpi_3, \varpi_4 \in \chi(M)$, at least one $\alpha \in \mathbb{R}$. In this case, we say that M is almost $C(\alpha)$-manifold [20]. By the way, we give the Riemann curvature tensor of an almost $C(\alpha)$-manifold which has $c-$constant sectional curvature by

$$\begin{aligned} R(\varpi_1, \varpi_2)\varpi_3 \quad & = \left(\tfrac{c+3\alpha}{4}\right)\{g(\varpi_2, \varpi_3)\varpi_1 - g(\varpi_1, \varpi_3)\varpi_2\} \\ & + \left(\tfrac{c-\alpha}{4}\right)\{g(\varpi_1, \phi\varpi_3)\phi\varpi_2 - g(\varpi_2, \phi\varpi_3)\phi\varpi_1 \\ & + 2g(\varpi_1, \phi\varpi_2)\phi\varpi_3 + \eta(\varpi_1)\eta(\varpi_3)\varpi_2 - \eta(\varpi_2)\eta(\varpi_3)\varpi_1 \\ & + g(\varpi_1, \varpi_3)\eta(\varpi_2)\xi - g(\varpi_2, \varpi_3)\eta(\varpi_1)\xi\}. \end{aligned} \tag{6}$$

So, if we take $\varpi_1 = \xi$ in (6), then we obtain

$$R(\xi, \varpi_2)\varpi_3 = \alpha[g(\varpi_2, \varpi_3)\xi - \eta(\varpi_3)\varpi_2]. \tag{7}$$

If we take $\varpi_3 = \xi$ in (6), then we have

$$R(\varpi_1, \varpi_2)\xi = \alpha[\eta(\varpi_2)\varpi_1 - \eta(\varpi_1)\varpi_2]. \tag{8}$$

Moreover, if we take $\varpi_2 = \xi$ in (8), then we get

$$R(\varpi_1, \xi)\xi = \alpha[\varpi_1 - \eta(\varpi_1)\xi]. \tag{9}$$

Let us take inner product of (6) by $\xi \in \chi(M)$. Then, we get

$$\eta(R(\varpi_1, \varpi_2)\varpi_3) = \alpha[g(\varpi_2, \varpi_3)\eta(\varpi_1) - g(\varpi_1, \varpi_3)\eta(\varpi_2)]. \tag{10}$$

Let M be an almost $C(\alpha)-$manifold with $(2n+1)$ dimensional. Then, we have the following equations.

$$S(\varpi_1, \varpi_2) = \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right] g(\varpi_1, \varpi_2) + \frac{(\alpha - c)(n+1)}{2}\eta(\varpi_1)\eta(\varpi_2) \tag{11}$$

$$S(\varpi_1, \xi) = 2n\alpha\eta(\varpi_1) \tag{12}$$

$$Q\varpi_1 = \left[\frac{\alpha(3n-1) + c(n+1)}{2}\right]\varpi_1 + \frac{(\alpha - c)(n+1)}{2}\eta(\varpi_1)\xi \tag{13}$$

$$Q\xi = 2n\alpha\xi \tag{14}$$

$$Q\phi\varpi_2 = \frac{r-2n\alpha}{2n}Q\varpi_2 \tag{15}$$

for each $\varpi_1, \varpi_2, \in \chi(M)$, where Q, S and r are the Ricci operator, Ricci curvature tensor and scalar curvature of manifold M, respectively.
Let M be a $(2n+1)$ −dimensional Riemannian manifold. Then the $\tilde{Z}$ concircular curvature tensor is defined as

$$\tilde{Z}(\varpi_1,\varpi_2)\varpi_3 = R(\varpi_1,\varpi_2)\varpi_3 - \frac{r}{2n(2n+1)}[g(\varpi_2,\varpi_3)\varpi_1 - g(\varpi_1,\varpi_3)\varpi_2] \tag{16}$$

for each $\varpi_1, \varpi_2, \varpi_3 \in \chi(M)$, where r is the scalar curvature of the manifold [21]. If we choose $\varpi_1 = \xi$ in (16), we get

$$\tilde{Z}(\xi,\varpi_2)\varpi_3 = \left(\alpha - \frac{r}{2n(2n+1)}\right)[g(\varpi_2,\varpi_3)\xi - \eta(\varpi_3)\varpi_2] \tag{17}$$

and if we choose $\varpi_3 = \xi$ in (17), we get

$$\tilde{Z}(\xi,\varpi_2)\xi = \left(\alpha - \frac{r}{2n(2n+1)}\right)[\eta(\varpi_2)\xi - \varpi_2]. \tag{18}$$

The concept of the quasi-conformal curvature tensor was defined by Yano and Sowaki as

$$\begin{aligned}\tilde{C}(\varpi_1,\varpi_2)\varpi_3 &= aR(\varpi_1,\varpi_2)\varpi_3 + b[S(\varpi_2,\varpi_3)\varpi_1 - S(\varpi_1,\varpi_3)\varpi_2 \\ &+g(\varpi_2,\varpi_3)Q\varpi_1 - g(\varpi_1,\varpi_3)Q\varpi_2] \\ &-\frac{r}{2n+1}\left[\frac{a}{2n} + 2b\right][g(\varpi_2,\varpi_3)\varpi_1 - g(\varpi_1,\varpi_3)\varpi_2],\end{aligned} \tag{19}$$

where a and b are constants, Q is the Ricci operator, S is the Ricci tensor and r is the scalar curvature of the manifold [22]. If $\tilde{C} = 0$, then this manifold is called a quasi-conformal flat. If $\varpi_1 = \xi$ is chosen in equation (19),

$$\begin{aligned}\tilde{C}(\xi,\varpi_2)\varpi_3 &= \left[\frac{bc(n+1)+\alpha(2a+7bn-b)}{2} - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right] \\ &\otimes [g(\varpi_2,\varpi_3)\xi - \eta(\varpi_3)\varpi_2].\end{aligned} \tag{20}$$

and if $\varpi_3 = \xi$ is chosen in (20), we reach at

$$\begin{aligned}\tilde{C}(\xi,\varpi_2)\xi &= \left[a\alpha + 2nb\alpha - \frac{r}{2n+1}\left(\frac{a}{2n} + 2b\right)\right][\eta(\varpi_2)\xi - \varpi_2] \\ &+b[2n\alpha\eta(\varpi_2)\xi - Q\varpi_2].\end{aligned} \tag{21}$$

The projective curvature tensor P is defined as

$$P(\varpi_1,\varpi_2)\varpi_3 = R(\varpi_1,\varpi_2)\varpi_3 - \frac{1}{2n}[S(\varpi_2,\varpi_3)\varpi_1 - S(\varpi_1,\varpi_3)\varpi_2] \tag{22}$$

for all $\varpi_1, \varpi_2, \varpi_3 \in \chi(M)$, by K. Yano and S. Sowaki [22]. If $\varpi_1 = \xi, \varpi_2 = \xi$ and $\varpi_3 = \xi$ are chosen respectively in (22), then we get

$$P(\xi,\varpi_2)\varpi_3 = \alpha g(\varpi_2,\varpi_3)\xi - \frac{1}{2n}S(\varpi_2,\varpi_3)\xi, \tag{23}$$

$$P(\varpi_1,\xi)\varpi_3 = -\alpha g(\varpi_1,\varpi_3)\xi + \frac{1}{2n}S(\varpi_1,\varpi_3)\xi \tag{24}$$

and

$$P(\varpi_1,\varpi_2)\xi = 0. \tag{25}$$

Again, if the inner product of both sides of equation (22) is taken by $\xi \in \chi(M)$, we get

$$\begin{aligned}\eta(P(\varpi_1,\varpi_2)\varpi_3) &= \eta(\varpi_1)\left[\alpha g(\varpi_2,\varpi_3) - \frac{1}{2n}S(\varpi_2,\varpi_3)\right] \\ &-\eta(\varpi_2)\left[\alpha g(\varpi_1,\varpi_3) - \frac{1}{2n}S(\varpi_1,\varpi_3)\right].\end{aligned} \tag{26}$$

Finally, if we choose $\varpi_1 = \xi$ in the equation (2), then it reduces the form

$$W_3(\xi,\varpi_2)\varpi_3 = 2\alpha[g(\varpi_2,\varpi_3)\xi - \eta(\varpi_3)\varpi_2] \tag{27}$$

and if we choose $\varpi_3 = \xi$ in the same equation, we get

$$\begin{aligned} W_3(\varpi_1,\varpi_2)\xi &= \left(\tfrac{\alpha(7n-1)+c(n+1)}{4n}\right)\eta(\varpi_2)\varpi_1 - 2\alpha\eta(\varpi_1)\varpi_2 \\ &\quad + \left(\tfrac{(\alpha-c)(n+1)}{4n}\right)\eta(\varpi_1)\eta(\varpi_2)\xi. \end{aligned} \tag{28}$$

Similarly, if we choose $\varpi_2 = \xi$ in equation (2), we get

$$W_3(\varpi_1,\xi)\varpi_3 = \left(\tfrac{(7n-1)\alpha+c(n+1)}{4n}\right)[-g(\varpi_1,\varpi_3)\xi + \eta(\varpi_3)\varpi_1]. \quad 29)$$

Almost $C(\alpha)$-Manifold Satisfying Some Important Conditions on the W_3- Curvature Tensor

In this section, let us first examine the case where the $(2n+1)$ dimensional almost $C(\alpha)$- manifold is W_3- flat, and then consider the special curvature conditions. For this, we give the following theorems.

Theorem 1 *If the $(2n+1)$ dimensional M is an almost $C(\alpha)$- manifold W_3- flat, then the manifold M is an η- Einstein manifold.*

Proof. Let us assume that manifold M is W_3- flat. From (2), we can write

$$W_3(\varpi_1,\varpi_2)\varpi_3 = 0,$$

for each $\varpi_1,\varpi_2,\varpi_3 \in \chi(M)$.Then from (2), we obtain

$$R(\varpi_1,\varpi_2)\varpi_3 = \frac{1}{2n}[S(\varpi_1,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)Q\varpi_1] \tag{30}$$

for each $\varpi_1,\varpi_2,\varpi_3 \in \chi(M)$. In (30), if $\varpi_3 = \xi$ is written and (8), (12) are used, we obtain

$$\frac{1}{2n}\eta(\varpi_2)Q\varpi_1 = 2\alpha\eta(\varpi_1)\varpi_2 - \alpha\eta(\varpi_2)\varpi_1. \tag{31}$$

Taking the inner product on both sides of the last equation by $\varpi_3 \in \chi(M)$ and if we choose $\varpi_2 = \xi$, we get

$$S(\varpi_1,\varpi_3) = -2n\alpha g(\varpi_1,\varpi_3) + 4n\alpha\eta(\varpi_1)\eta(\varpi_3).$$

This proves our assertion.

Theorem 2 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. Then $W_3 \cdot R = 0$ if and only if M is either co-Keahler manifold or M reduces real space form with constant sectinal curvature.*

Proof. Suppose that $W_3(\varpi_1,\varpi_2)\cdot R = 0$. Then, we have

$$\begin{aligned} (W_3(\varpi_1,\varpi_2)R)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\varpi_1,\varpi_2)R(\varpi_4,\varpi_5)\varpi_3 - R(W_3(\varpi_1,\varpi_2)\varpi_4,\varpi_5)\varpi_3 \\ &\quad -R(\varpi_4,W_3(\varpi_1,\varpi_2)\varpi_5)\varpi_3 - R(\varpi_4,\varpi_5)W_3(\varpi_1,\varpi_2)\varpi_3 \\ &= 0. \end{aligned}$$

If we choose $\varpi_1 = \xi$ in here, we get

$$\begin{aligned} (W_3(\xi,\varpi_2)R)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\xi,\varpi_2)R(\varpi_4,\varpi_5)\varpi_3 - R(W_3(\xi,\varpi_2)\varpi_4,\varpi_5)\varpi_3 \\ &\quad -R(\varpi_4,W_3(\xi,\varpi_2)\varpi_5)\varpi_3 - R(\varpi_4,\varpi_5)W_3(\xi,\varpi_2)\varpi_3 \\ &= 0, \end{aligned} \tag{32}$$

for each $\varpi_2,\varpi_3,\varpi_4,\varpi_5 \in \chi(M)$. In (32), using (27), we obtain

$$\begin{aligned} &2\alpha[g(\varpi_2,R(\varpi_4,\varpi_5)\varpi_3)\xi - \eta(R(\varpi_4,\varpi_5)\varpi_3)\varpi_2 \\ &-g(\varpi_2,\varpi_4)R(\xi,\varpi_5)\varpi_3 + \eta(\varpi_4)R(\varpi_2,\varpi_5)\varpi_3 \\ &-g(\varpi_2,\varpi_5)R(\varpi_4,\xi)\varpi_3 + \eta(\varpi_5)R(\varpi_4,\varpi_2)\varpi_3 \\ &-g(\varpi_2,\varpi_3)R(\varpi_4,\varpi_5)\xi + \eta(\varpi_3)R(\varpi_4,\varpi_5)\varpi_2] = 0. \end{aligned} \tag{33}$$

Substituting $\varpi_4 = \xi$ in (33) and using (7), we conclude

$$2\alpha[R(\varpi_2,\varpi_5)\varpi_3 - \alpha(g(\varpi_5,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)\varpi_5)] = 0. \tag{34}$$

From (34), we have

$$\alpha = 0$$

or

$$R(\varpi_2,\varpi_5)\varpi_3 = \alpha[g(\varpi_5,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)\varpi_5].$$

Thus, M is co-Keahler manifold or M is reduced to the real space form with constant sectional curvature. The converse is obvius, and the proof is complete.

Theorem 3 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. Then $W_3 \cdot \tilde{Z} = 0$ if and only if M is either co-Keahler manifold or M reduces real space form with constant sectinal curvature.*

Proof. Suppose that $W_3(\varpi_1,\varpi_2)\cdot\tilde{Z} = 0$. Then we have

$$\begin{aligned}\big(W_3(\varpi_1,\varpi_2)\tilde{Z}\big)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\varpi_1,\varpi_2)\tilde{Z}(\varpi_4,\varpi_5)\varpi_3 - \tilde{Z}(W_3(\varpi_1,\varpi_2)\varpi_4,\varpi_5)\varpi_3\\ &-\tilde{Z}(\varpi_4,W_3(\varpi_1,\varpi_2)\varpi_5)\varpi_3 - \tilde{Z}(\varpi_4,\varpi_5)W_3(\varpi_1,\varpi_2)\varpi_3\\ &= 0.\end{aligned}$$

If we choose $\varpi_1 = \xi$ in here, we get

$$\begin{aligned}(W_3(\xi,\varpi_2)\varpi_3)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\xi,\varpi_2)\varpi_3(\varpi_4,\varpi_5)\varpi_3 - \varpi_3(W_3(\xi,\varpi_2)\varpi_4,\varpi_5)\varpi_3\\ &-\varpi_3(\varpi_4,W_3(\xi,\varpi_2)\varpi_5)\varpi_3 - \varpi_3(\varpi_4,\varpi_5)W_3(\xi,\varpi_2)\varpi_3\\ &= 0,\end{aligned} \tag{35}$$

for each $\varpi_2,\varpi_3,\varpi_4,\varpi_5 \in \chi(M)$. In (35), using (27), we obtain

$$\begin{aligned}&2\alpha\big[g\big(\varpi_2,\tilde{Z}(\varpi_4,\varpi_5)\varpi_3\big)\xi - \eta\big(\tilde{Z}(\varpi_4,\varpi_5)\varpi_3\big)\varpi_2\\ &-g(\varpi_2,\varpi_4)\tilde{Z}(\xi,\varpi_5)\varpi_3 + \eta(\varpi_4)\tilde{Z}(\varpi_2,\varpi_5)\varpi_3 - g(\varpi_2,\varpi_5)\tilde{Z}(\varpi_4,\xi)\varpi_3\\ &+\eta(\varpi_5)\tilde{Z}(\varpi_4,\varpi_2)\varpi_3 - g(\varpi_2,\varpi_3)\tilde{Z}(\varpi_4,\varpi_5)\xi + \eta(\varpi_3)\tilde{Z}(\varpi_4,\varpi_5)\varpi_2] = 0.\end{aligned} \tag{36}$$

Taking $\varpi_4 = \xi$ in (36) and using (17), we obtain

$$2\alpha\left[\tilde{Z}(\varpi_2,\varpi_5)\varpi_3 - \left(\alpha - \frac{r}{2n(2n+1)}\right)(g(\varpi_5,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)\varpi_5)\right] = 0. \tag{37}$$

In (37), using (16), we conclude

$$2\alpha[R(\varpi_2,\varpi_5)\varpi_3 - \alpha(g(\varpi_5,\varpi_3)\varpi_2 - g(\varpi_2,\varpi_3)\varpi_5)] = 0. \tag{38}$$

This proves our assertion. The converse obvious.

Theorem 4 *Let M be $(2n+1)$ dimensional an almost $C(\alpha)$- manifold. Then $W_3 \cdot S = 0$ if and only if M is either co-Keahler manifold or an Einstein manifold.*

Proof. Suppose that $W_3 \cdot S = 0$. Then we can easily see that

$$S(W_3(\varpi_1,\varpi_2)\varpi_3,\varpi_4) + S(\varpi_3,W_3(\varpi_1,\varpi_2)\varpi_4) = 0.$$

If we choose $\varpi_1 = \xi$ in here, we get

$$S(W_3(\xi,\varpi_2)\varpi_3,\varpi_4) + S(\varpi_3,W_3(\xi,\varpi_2)\varpi_4) = 0. \tag{39}$$

In (39), using (27), we obtain

$$2\alpha[2n\alpha\eta(\varpi_4)g(\varpi_2,\varpi_3)-\eta(\varpi_3)S(\varpi_2,\varpi_4)$$
$$+2n\alpha\eta(\varpi_3)g(\varpi_2,\varpi_4)-\eta(\varpi_4)S(\varpi_3,\varpi_2)]=0. \quad (40)$$

Substituting $\varpi_3=\xi$ in (40), we find

$$2\alpha[-S(\varpi_2,\varpi_4)+2n\alpha g(\varpi_2,\varpi_4)]=0. \quad (41)$$

From (41), we get

$$\alpha=0$$

or

$$S(\varpi_2,\varpi_4)=2n\alpha g(\varpi_2,\varpi_4).$$

This proves our assertion. The converse is obvious.

Theorem 5 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. Then $W_3\cdot\tilde{C}=0$ if and only if M is either co-Keahler manifold or M reduces real space form with constant sectinal curvature.*

Proof. Suppose that $W_3(\varpi_1,\varpi_2)\cdot\tilde{C}=0$. Then, we have

$$\left(W_3(\varpi_1,\varpi_2)\tilde{C}\right)(\varpi_3,\varpi_4,\varpi_5)=W_3(\varpi_1,\varpi_2)\tilde{C}(\varpi_3,\varpi_4)\varpi_5-\tilde{C}(W_3(\varpi_1,\varpi_2)\varpi_3,\varpi_4)\varpi_5$$
$$-\tilde{C}(\varpi_3,W_3(\varpi_1,\varpi_2)\varpi_4)\varpi_5-\tilde{C}(\varpi_3,\varpi_4)W_3(\varpi_1,\varpi_2)\varpi_5$$
$$=0.$$

If we choose $\varpi_1=\xi$ in here, we get

$$\left(W_3(\xi,\varpi_2)\tilde{C}\right)(\varpi_3,\varpi_4,\varpi_5)=W_3(\xi,\varpi_2)\tilde{C}(\varpi_3,\varpi_4)\varpi_5-\tilde{C}(W_3(\xi,\varpi_2)\varpi_3,\varpi_4)\varpi_5$$
$$-\tilde{C}(\varpi_3,W_3(\xi,\varpi_2)\varpi_4)\varpi_5-\tilde{C}(\varpi_3,\varpi_4)W_3(\xi,\varpi_2)\varpi_5 \quad (42)$$
$$=0,$$

for each $\varpi_2,\varpi_3,\varpi_4,\varpi_5\in\chi(M)$. Using (27) in (42), we get

$$2\alpha\big[g\big(\varpi_2,\tilde{C}(\varpi_3,\varpi_4)\varpi_5\big)\xi-\eta\big(\tilde{C}(\varpi_3,\varpi_4)\varpi_5\big)\varpi_2$$
$$-g(\varpi_2,\varpi_3)\tilde{C}(\xi,\varpi_4)\varpi_5+\eta(\varpi_3)\tilde{C}(\varpi_2,\varpi_4)\varpi_5-g(\varpi_2,\varpi_4)\tilde{C}(\varpi_3,\xi)\varpi_5 \quad (43)$$
$$+\eta(\varpi_4)\tilde{C}(\varpi_3,\varpi_2)\varpi_5-g(\varpi_2,\varpi_5)\tilde{C}(\varpi_3,\varpi_4)\xi+\eta(\varpi_5)\tilde{C}(\varpi_3,\varpi_4)\varpi_2\big]=0.$$

Taking $\varpi_3=\xi$ in (43) and using (20), (21), we obtain

$$2\alpha\left\{\left[\frac{bc(n+1)+\alpha(2a+7bn-b)}{2}-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\otimes\right.$$
$$\left.[g(\varpi_2,\varpi_5)\varpi_4-g(\varpi_4,\varpi_5)\varpi_2]+\tilde{C}(\varpi_2,\varpi_4)\varpi_5\right\}=0. \quad (44)$$

Substituting $\varpi_2\to\phi\varpi_2$ and $\varpi_4\to\phi\varpi_4$ in (44), we conclude

$$2\alpha\left\{\left[\frac{bc(n+1)+\alpha(2a+7bn-b)}{2}-\frac{r}{2n+1}\left(\frac{a}{2n}+2b\right)\right]\otimes\right.$$
$$\left.[g(\phi\varpi_2,\varpi_5)\phi\varpi_4-g(\phi\varpi_4,\varpi_5)\phi\varpi_2]+\tilde{C}(\phi\varpi_2,\phi\varpi_4)\varpi_5\right\}=0. \quad (45)$$

In the last equation, if (19) is written in its place and necessary adjustments are made, we get

$$\alpha=0 \quad (46)$$

or

$$R(\phi\varpi_2,\phi\varpi_4)\varpi_5=A[g(\phi\varpi_4,\varpi_5)\phi\varpi_2-g(\phi\varpi_2,\varpi_5)\phi\varpi_4], \quad (47)$$

where

$$A=\frac{\alpha[(2a+7nb-5b)+b(4r-3n+1)]}{2a}-\frac{r[2(n+1)a+(6n+1)b]}{2n(2n+1)a}$$

It is clear from (46) and (47) that M is either co-Keahler manifold or M reduces real space form with constant sectinal curvature. The converse is obvious.

Theorem 6 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. Then $W_3 \cdot P = 0$ if and only if M is either co-Keahler manifold or η- Einstein manifold.*

Proof. Suppose that $W_3 \cdot P = 0$. Then we have

$$\begin{aligned}(W_3(\varpi_1,\varpi_2)P)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\varpi_1,\varpi_2)P(\varpi_4,\varpi_5)\varpi_3 - P(W_3(\varpi_1,\varpi_2)\varpi_4,\varpi_5)\varpi_3 \\ &-P(\varpi_4,W_3(\varpi_1,\varpi_2)\varpi_5)\varpi_3 - P(\varpi_4,\varpi_5)W_3(\varpi_1,\varpi_2)\varpi_3 \\ &= 0.\end{aligned}$$

If we choose $\varpi_1 = \xi$ in here, we get

$$\begin{aligned}(W_3(\xi,\varpi_2)P)(\varpi_4,\varpi_5,\varpi_3) &= W_3(\xi,\varpi_2)P(\varpi_4,\varpi_5)\varpi_3 - P(W_3(\xi,\varpi_2)\varpi_4,\varpi_5)\varpi_3 \\ &-P(\varpi_4,W_3(\xi,\varpi_2)\varpi_5)\varpi_3 - P(\varpi_4,\varpi_5)W_3(\xi,\varpi_2)\varpi_3 \\ &= 0,\end{aligned} \tag{48}$$

for each $\varpi_2,\varpi_3,\varpi_4,\varpi_5 \in \chi(M)$. In (48), using (27), we obtain

$$\begin{aligned}&2\alpha[g(\varpi_2,P(\varpi_4,\varpi_5)\varpi_3)\xi - \eta(P(\varpi_4,\varpi_5)\varpi_3)\varpi_2 \\ &-g(\varpi_2,\varpi_4)P(\xi,\varpi_5)\varpi_3 + \eta(\varpi_4)P(\varpi_2,\varpi_5)\varpi_3 - g(\varpi_2,\varpi_5)P(\varpi_4,\xi)\varpi_3 \\ &+\eta(\varpi_5)P(\varpi_4,\varpi_2)\varpi_3 - g(\varpi_2,\varpi_3)P(\varpi_4,\varpi_5)\xi + \eta(\varpi_3)P(\varpi_4,\varpi_5)\varpi_2] = 0.\end{aligned} \tag{49}$$

If (22) and (23) are used in (49), then we get

$$\begin{aligned}&2\alpha\Big\{g(\varpi_2,R(\varpi_4,\varpi_5)\varpi_3)\xi - \tfrac{1}{2n}S(\varpi_5,\varpi_3)g(\varpi_2,\varpi_4)\xi \\ &+\tfrac{1}{2n}S(\varpi_4,\varpi_3)g(\varpi_2,\varpi_5)\xi - \eta(R(\varpi_4,\varpi_5)\varpi_3)\varpi_2 \\ &+\tfrac{1}{2n}S(\varpi_5,\varpi_3)\eta(\varpi_4)\varpi_2 - \tfrac{1}{2n}S(\varpi_4,\varpi_3)\eta(\varpi_5)\varpi_2 \\ &-\alpha g(\varpi_2,\varpi_4)g(\varpi_5,\varpi_3)\xi + \tfrac{1}{2n}g(\varpi_2,\varpi_4)S(\varpi_5,\varpi_3)\xi \\ &+\eta(\varpi_4)R(\varpi_2,\varpi_5)\varpi_3 - \tfrac{1}{2n}S(\varpi_5,\varpi_3)\eta(\varpi_4)\varpi_2 \\ &+\tfrac{1}{2n}S(\varpi_2,\varpi_3)\eta(\varpi_4)\varpi_5 + \alpha g(\varpi_2,\varpi_5)g(\varpi_4,\varpi_3)\xi \\ &-\tfrac{1}{2n}g(\varpi_2,\varpi_5)S(\varpi_4,\varpi_3)\xi + \eta(\varpi_5)R(\varpi_4,\varpi_2)\varpi_3 \\ &-\tfrac{1}{2n}S(\varpi_2,\varpi_3)\eta(\varpi_5)\varpi_4 + \tfrac{1}{2n}S(\varpi_4,\varpi_3)\eta(\varpi_5)\varpi_2 \\ &+\eta(\varpi_3)R(\varpi_4,\varpi_5)\varpi_2 - \tfrac{1}{2n}S(\varpi_5,\varpi_2)\eta(\varpi_3)\varpi_4 \\ &+\tfrac{1}{2n}S(\varpi_4,\varpi_2)\eta(\varpi_3)\varpi_5\Big\} = 0\end{aligned} \tag{50}$$

Taking $\varpi_4 = \xi$ in (50) and using (7), we obtain

$$\begin{aligned}&2\alpha\{\alpha g(\varpi_5,\varpi_3)\eta(\varpi_2)\xi - \alpha g(\varpi_5,\varpi_3)\varpi_2 \\ &+\tfrac{1}{2n}S(\varpi_5,\varpi_3)\varpi_2 - g(\varpi_5,\varpi_3)\eta(\varpi_2)\xi \\ &+R(\varpi_2,\varpi_5)\varpi_3 - \tfrac{1}{2n}S(\varpi_5,\varpi_3)\varpi_2 \\ &+\tfrac{1}{2n}S(\varpi_2,\varpi_3)\varpi_5 + \alpha g(\varpi_2,\varpi_5)\eta(\varpi_3)\xi \\ &+\alpha\eta(\varpi_5)g(\varpi_2,\varpi_3)\xi - \tfrac{1}{2n}S(\varpi_2,\varpi_3)\eta(\varpi_5)\xi \\ &-\tfrac{1}{2n}S(\varpi_5,\varpi_2)\eta(\varpi_3)\xi\Big\} = 0.\end{aligned} \tag{51}$$

If we take the inner product of both sides of (51) by $\xi \in \chi(M)$, we have

$$2\alpha\{-g(\varpi_5,\varpi_3)\eta(\varpi_2)+\alpha g(\varpi_5,\varpi_3)\eta(\varpi_2)$$
$$+\alpha g(\varpi_2,\varpi_5)\eta(\varpi_3)-\frac{1}{2n}S(\varpi_5,\varpi_2)\eta(\varpi_3)\Big\}=0. \quad (52)$$

If we choose $\varpi_3=\xi$ in (52), we get

$$2\alpha\left[(\alpha-1)\eta(\varpi_5)\eta(\varpi_2)+\alpha g(\varpi_2,\varpi_5)-\frac{1}{2n}S(\varpi_5,\varpi_2)\right]=0. \quad (53)$$

It is clear from (53) that

$$S(\varpi_5,\varpi_2)=2n\alpha g(\varpi_5,\varpi_2)+2n(\alpha-1)\eta(\varpi_5)\eta(\varpi_2)$$

or

$$\alpha=0.$$

This proves our assertion. The converse obvious.

Definition 1 Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold, R be the Riemann curvature tensor of M and S be the Ricci curvature tensor of M. If the pair $R\cdot W_3$and $Q(g,W_3)$ are linearly dependent, that is, if a λ_1 function can be found on the set $M_1=\{\varpi_1\in M|g(\varpi_1)\neq W_3(\varpi_1)\}$ such that

$$R\cdot W_3=\lambda_1 Q(g,W_3) \quad (54)$$

the M manifold is called a W_3- pseudo symmetric manifold. Particularly, if $\lambda_1=0$, then this manifold is said to be semi-symmetric.

Let us now investigate the case of W_3- pseudo symmetry.

Theorem 7 Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. M is W_3- pseudo symmetric if and only if M is either Einstein manifold or $\lambda_1=\alpha$.

Proof. Let us assume that the manifold M is a W_3- pseudo symmetric manifold. Then, we can write

$$(R(\varpi_1,\varpi_2)\cdot W_3)(\varpi_3,\varpi_5,\varpi_4,)=\lambda_1 Q(g,W_3)(\varpi_3,\varpi_4,\varpi_5;\varpi_1,\varpi_2), \quad (55)$$

for each $\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5\in\chi(M)$. In this case, we get

$$R(\varpi_1,\varpi_2)W_3(\varpi_3,\varpi_5)\varpi_4-W_3(R(\varpi_1,\varpi_2)\varpi_3,\varpi_5)\varpi_4$$
$$-W_3(\varpi_3,R(\varpi_1,\varpi_2)\varpi_5)\varpi_4-W_3(\varpi_3,\varpi_5)R(\varpi_1,\varpi_2)\varpi_4$$
$$=-\lambda_1\left\{W_3\left((\varpi_1\Lambda_g\varpi_2)\varpi_3,\varpi_5,\varpi_4\right)+W_3\left(\varpi_3,(\varpi_1\Lambda_g\varpi_2)\varpi_5,\varpi_4\right)\right. \quad (56)$$
$$\left.+W_3\left(\varpi_3,\varpi_5,(\varpi_1\Lambda_g\varpi_2)\varpi_4\right)\right\}.$$

If necessary arrangements are made here, we obtain and we choose $\varpi_1=\xi$ in (56), we get

$$R(\xi,\varpi_2)W_3(\varpi_3,\varpi_5)\varpi_4-W_3(R(\xi,\varpi_2)\varpi_3,\varpi_5)\varpi_4$$
$$-W_3(\varpi_3,R(\xi,\varpi_2)\varpi_5)\varpi_4-W_3(\varpi_3,\varpi_5)R(\xi,\varpi_2)\varpi_4$$
$$=-\lambda_1\{g(\varpi_2,\varpi_3)W_3(\xi,\varpi_5)\varpi_4-g(\xi,\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4 \quad (57)$$
$$+g(\varpi_2,\varpi_5)W_3(\varpi_3,\xi)\varpi_4-g(\xi,\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4$$
$$+g(\varpi_2,\varpi_4)W_3(\varpi_3,\varpi_5)\xi-g(\xi,\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2\}.$$

Using (7), (27), (28) and (29) in the last equation, we obtain

$$\alpha g(\varpi_2, W_3(\varpi_3,\varpi_5)\varpi_4)\xi - \alpha\eta(W_3(\varpi_3,\varpi_5)\varpi_4)\varpi_2 - 2\alpha^2 g(\varpi_2,\varpi_3)g(\varpi_5,\varpi_4)\xi$$
$$+2\alpha^2\eta(\varpi_4)g(\varpi_2,\varpi_3)\varpi_5 + \alpha\eta(\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4 +$$
$$\frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}g(\varpi_2,\varpi_5)g(\varpi_3,\varpi_4)\xi - \frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_4)g(\varpi_2,\varpi_5)\varpi_3$$
$$+\alpha\eta(\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4 - \frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_5)g(\varpi_2,\varpi_4)\varpi_3$$
$$+2\alpha^2\eta(\varpi_3)g(\varpi_2,\varpi_4)\varpi_5 - \frac{\alpha(\alpha-c)(n+1)}{4n}g(\varpi_2,\varpi_4)\eta(\varpi_3)\eta(\varpi_5)\xi$$
$$+\alpha\eta(\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2 = -\lambda_1\{2\alpha g(\varpi_2,\varpi_3)g(\varpi_5,\varpi_4)\xi \tag{58}$$
$$-2\alpha\eta(\varpi_4)g(\varpi_2,\varpi_3)\varpi_5 - \eta(\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4$$
$$-\frac{[(7n-1)\alpha+c(n+1)]}{4n}g(\varpi_2,\varpi_5)g(\varpi_3,\varpi_4)\xi + \frac{[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_4)g(\varpi_2,\varpi_5)\varpi_3$$
$$-\eta(\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4 - \eta(\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2$$
$$+\frac{[(7n-1)\alpha+c(n+1)]}{4n}g(\varpi_2,\varpi_4)\eta(\varpi_5)\varpi_3 - 2\alpha\eta(\varpi_3)g(\varpi_2,\varpi_4)\varpi_5$$
$$\left.+\frac{(\alpha-c)(n+1)}{4n}g(\varpi_2,\varpi_4)\eta(\varpi_3)\eta(\varpi_5)\xi\right\}.$$

If we choose $\varpi_3 = \xi$ in (58), we get

$$-2\alpha^2 g(\varpi_5,\varpi_4)\varpi_2 + \alpha W_3(\varpi_2,\varpi_5)\varpi_4 + 2\alpha^2 g(\varpi_2,\varpi_4)\varpi_5$$
$$= -\lambda_1\{2\alpha g(\varpi_5,\varpi_4)\eta(\varpi_2)\xi - W_3(\varpi_2,\varpi_5)\varpi_4 + 2\alpha\eta(\varpi_5)\eta(\varpi_4)\varpi_2 \tag{59}$$
$$-2\alpha g(\varpi_5,\varpi_2)\eta(\varpi_4)\xi - 2\alpha g(\varpi_2,\varpi_4)\varpi_5\}.$$

If we use (2) in (59), we have

$$-2\alpha^2 g(\varpi_5,\varpi_4)\varpi_2 + \alpha R(\varpi_2,\varpi_5)\varpi_4 - \frac{\alpha}{2n}S(\varpi_2,\varpi_4)\varpi_5$$
$$+\frac{\alpha}{2n}g(\varpi_5,\varpi_4)Q\varpi_2 + 2\alpha^2 g(\varpi_2,\varpi_4)\varpi_5 = -\lambda_1\{2\alpha g(\varpi_5,\varpi_4)\eta(\varpi_2)\xi$$
$$-R(\varpi_2,\varpi_5)\varpi_4 + \frac{1}{2n}S(\varpi_2,\varpi_4)\varpi_5 - \frac{1}{2n}g(\varpi_5,\varpi_4)Q\varpi_2 \tag{60}$$
$$+2\alpha\eta(\varpi_5)\eta(\varpi_4)\varpi_2 - 2\alpha g(\varpi_5,\varpi_2)\eta(\varpi_4)\xi - 2\alpha g(\varpi_2,\varpi_4)\varpi_5\}.$$

In (60), if we choose $\varpi_4 = \xi$ and we use (8) and (12), we get

$$-\alpha^2\eta(\varpi_5)\varpi_2 + \frac{\alpha}{2n}\eta(\varpi_5)Q\varpi_2$$
$$= -\lambda_1\{2\alpha\eta(\varpi_2)\eta(\varpi_5)\xi + \alpha\eta(\varpi_5)\varpi_2 \tag{61}$$
$$\left.-\frac{1}{2n}\eta(\varpi_5)Q\varpi_2 - 2\alpha g(\varpi_5,\varpi_2)\xi\right\}.$$

If $\varpi_5 = \xi$ is chosen in the last equation and we take the inner product of both sides of (61) by $\varpi_4 \in \chi(M)$, then we have

$$(\lambda_1 - \alpha)\left[\alpha g(\varpi_2,\varpi_4) - \frac{1}{2n}S(\varpi_2,\varpi_4)\right] = 0 \tag{62}$$

It is clear from (62) that either

$$\lambda_1 = \alpha$$

or the manifold M is an Einstein manifold. This ends our proof.

Corollary 1 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. M is a semi-symmetric manifold if and only if M is either a co-Keahler manifold or an Einstein manifold.*

Definition 2 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold, R be the Riemann curvature tensor of M and S be the Ricci curvature tensor of M. If the pair $\mathrm{R}\cdot\mathrm{W}_3$ and $Q(S,W_3)$ are linearly dependent, that is, if a λ_2 function can be found on the set $M_2=\{\varpi_1\in M|S(\varpi_1)\neq W_3(\varpi_1)\}$ such that*

$$R\cdot W_3=\lambda_2 Q(S,W_3)$$

the M manifold is called a W_3 −Ricci-pseudo symmetric manifold.

Theorem 8 *Let M be a $(2n+1)$ dimensional almost $C(\alpha)$- manifold. M is W_3- Ricci pseudo symmetric if and only if M is an η- Einstein manifold provided $\lambda_2\neq\frac{1}{2n}$.*

Proof. Let us assume that the manifold M is a W_3- Ricci-pseudo symmetric manifold. Then, we can write

$$(R(\varpi_1,\varpi_2)\cdot W_3)(\varpi_3,\varpi_5,\varpi_4)=\lambda_2 Q(S,W_3)(\varpi_3,\varpi_4,\varpi_5;\varpi_1,\varpi_2), \tag{63}$$

for each $\varpi_1,\varpi_2,\varpi_3,\varpi_4,\varpi_5\in\chi(M)$. In this case, we get

$$\begin{aligned}&R(\varpi_1,\varpi_2)W_3(\varpi_3,\varpi_5)\varpi_4-W_3(R(\varpi_1,\varpi_2)\varpi_3,\varpi_5)\varpi_4\\&-W_3(\varpi_3,R(\varpi_1,\varpi_2)\varpi_5)\varpi_4-W_3(\varpi_3,\varpi_5)R(\varpi_1,\varpi_2)\varpi_4\\&=-\lambda_2\{W_3\big((\varpi_1\Lambda_S\varpi_2)\varpi_3,\varpi_5,\varpi_4\big)+W_3(\varpi_3,(\varpi_1\Lambda_S\varpi_2)\varpi_5,\varpi_4)\\&+W_3(\varpi_3,\varpi_5,(\varpi_1\Lambda_S\varpi_2)\varpi_4)\}.\end{aligned} \tag{64}$$

If necessary arrangements are made here, we obtain and we choose $\varpi_1=\xi$ in (64), we get

$$\begin{aligned}&R(\xi,\varpi_2)W_3(\varpi_3,\varpi_5)\varpi_4-W_3(R(\xi,\varpi_2)\varpi_3,\varpi_5)\varpi_4\\&-W_3(\varpi_3,R(\xi,\varpi_2)\varpi_5)\varpi_4-W_3(\varpi_3,\varpi_5)R(\xi,\varpi_2)\varpi_4\\&=-\lambda_2\{S(\varpi_2,\varpi_3)W_3(\xi,\varpi_5)\varpi_4-S(\xi,\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4\\&+S(\varpi_2,\varpi_5)W_3(\varpi_3,\xi)\varpi_4-S(\xi,\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4\\&+S(\varpi_2,\varpi_4)W_3(\varpi_3,\varpi_5)\xi-S(\xi,\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2\}.\end{aligned} \tag{65}$$

Using (7), (27), (28) and (29) in the last equation, we obtain

$$\begin{aligned}&\alpha g(\varpi_2,W_3(\varpi_3,\varpi_5)\varpi_4)\xi-\alpha\eta(W_3(\varpi_3,\varpi_5)\varpi_4)\varpi_2\\&-2\alpha^2 g(\varpi_2,\varpi_3)g(\varpi_5,\varpi_4)\xi+2\alpha^2\eta(\varpi_4)g(\varpi_2,\varpi_3)\varpi_5\\&+\alpha\eta(\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4+\frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}g(\varpi_2,\varpi_5)g(\varpi_3,\varpi_4)\xi\\&-\frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_4)g(\varpi_2,\varpi_5)\varpi_3+\alpha\eta(\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4\\&-\frac{\alpha[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_5)g(\varpi_2,\varpi_4)\varpi_3+2\alpha^2\eta(\varpi_3)g(\varpi_2,\varpi_4)\varpi_5\\&-\frac{\alpha(\alpha-c)(n+1)}{4n}g(\varpi_2,\varpi_4)\eta(\varpi_3)\eta(\varpi_5)\xi+\alpha\eta(\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2\\&=-\lambda_2\{2\alpha S(\varpi_2,\varpi_3)g(\varpi_5,\varpi_4)\xi-2\alpha\eta(\varpi_4)S(\varpi_2,\varpi_3)\varpi_5\\&-2n\alpha\eta(\varpi_3)W_3(\varpi_2,\varpi_5)\varpi_4-\frac{[(7n-1)\alpha+c(n+1)]}{4n}S(\varpi_2,\varpi_5)g(\varpi_3,\varpi_4)\xi\\&+\frac{[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_4)S(\varpi_2,\varpi_5)\varpi_3-2n\alpha\eta(\varpi_5)W_3(\varpi_3,\varpi_2)\varpi_4\\&-\frac{[(7n-1)\alpha+c(n+1)]}{4n}\eta(\varpi_5)S(\varpi_2,\varpi_4)\varpi_3-2n\alpha\eta(\varpi_4)W_3(\varpi_3,\varpi_5)\varpi_2\\&-2\alpha\eta(\varpi_3)S(\varpi_2,\varpi_4)\varpi_5+\frac{(\alpha-c)(n+1)}{4n}S(\varpi_2,\varpi_4)\eta(\varpi_3)\eta(\varpi_5)\xi\Big\}.\end{aligned} \tag{66}$$

In (66), if we choose $\varpi_3=\xi$, we get

$$-2\alpha^2 g(\varpi_5,\varpi_4)\varpi_2 + \alpha W_3(\varpi_2,\varpi_5)\varpi_4 + 2\alpha^2 g(\varpi_2,\varpi_4)\varpi_5$$
$$= -\lambda_2\{4n\alpha^2 g(\varpi_5,\varpi_4)\eta(\varpi_2)\xi - 2n\alpha W_3(\varpi_2,\varpi_5)\varpi_4 - 4n\alpha^2 g(\varpi_2,\varpi_4)\eta(\varpi_5)\xi$$
$$+4n\alpha^2\eta(\varpi_4)\eta(\varpi_5)\varpi_2 + 2\alpha\eta(\varpi_5)S(\varpi_2,\varpi_4)\xi$$
$$-2\alpha S(\varpi_2,\varpi_4)\varpi_5 - 4n\alpha^2 g(\varpi_5,\varpi_2)\eta(\varpi_4)\xi\}. \quad (67)$$

If we use (2) in (67), we have

$$-2\alpha^2 g(\varpi_5,\varpi_4)\varpi_2 + \alpha R(\varpi_2,\varpi_5)\varpi_4 - \frac{\alpha}{2n}S(\varpi_2,\varpi_4)\varpi_5$$
$$+\frac{\alpha}{2n}g(\varpi_5,\varpi_4)Q\varpi_2 + 2\alpha^2 g(\varpi_2,\varpi_4)\varpi_5 = -\lambda_2\{4n\alpha^2 g(\varpi_5,\varpi_4)\eta(\varpi_2)\xi$$
$$-2n\alpha R(\varpi_2,\varpi_5)\varpi_4 + \alpha S(\varpi_2,\varpi_4)\varpi_5 - \alpha g(\varpi_5,\varpi_4)Q\varpi_2 \quad (68)$$
$$-4n\alpha^2 g(\varpi_2,\varpi_4)\eta(\varpi_5)\xi + 4n\alpha^2\eta(\varpi_5)\eta(\varpi_4)\varpi_2 + 2\alpha\eta(\varpi_5)S(\varpi_2,\varpi_4)\xi$$
$$-2\alpha S(\varpi_2,\varpi_4)\varpi_5 - 4n\alpha^2 g(\varpi_5,\varpi_2)\eta(\varpi_4)\xi\}.$$

If we choose $\varpi_4 = \xi$ in (68) and we use (8) and (12), we get

$$-\alpha^2\eta(\varpi_5)\varpi_2 + \frac{\alpha}{2n}\eta(\varpi_5)Q\varpi_2$$
$$= -\lambda_2\{2n\alpha^2\eta(\varpi_5)\varpi_2 + 4n\alpha^2\eta(\varpi_2)\varpi_5 \quad (69)$$
$$-\alpha\eta(\varpi_5)Q\varpi_2\}.$$

If $\varpi_5 = \xi$ is chosen in the last equation and we take the inner product of both sides of (69) by $\varpi_4 \in \chi(M)$, then we have

$$S(\varpi_2,\varpi_4) = \frac{2n\alpha(1-2n)}{(1-2n\lambda_2)}g(\varpi_2,\varpi_4) - \frac{8n^2}{(1-2n\lambda_2)}\eta(\varpi_2)\eta(\varpi_4). \quad (70)$$

It is clear from (70) that the manifold M is an η −Einstein manifold provided $\lambda_2 \neq \frac{1}{2n}$. This ends our proof.

Acknowledgments

The authors declare that there is no conflict of interest.

Conflicts of interest

There are no conflicts of interest in this work.

References

[1] Kaneyuki, S., Williams, F.L., Almost paracontact and parahodge structures on manifolds, *Nagoya Math. J.*, 99 (1985) 173-187.

[2] Zamkovoy, S., Canonical connections on paracontact manifolds, *Ann. Global Anal. Geom*., 36 (2009) 37-60.

[3] Atçeken, M., Uygun, P., Characterizations for totally geodesic submanifolds of (k,µ)-paracontact metric manifolds, *Korean J. Math.,* 28 (2020) 555-571.

[4] Atçeken, M., Dirik, S., On the geometry of pseudo-slant submanifolds of a Kenmotsu manifold, *Gulf Journal of Mathematics*, 2 (2014) 51-66.

[5] Laha, B., Das, B., Bhattacharyya, A., Submanifolds of some indefinite contact and paracontact metric manifolds, *Gulf Journal of Mathematics*, 1 (2013) 180-188.

[6] Yıldırım, Ü., Atçeken, M., Dirik,S., Generalized B-curvature tensor of normal paracntact metric manifold, *Hagia Sophia Journal of Geometry*, 1 (2009) 1-7.

[7] Yıldırım, Ü., Atçeken, M., Dirik, S., Pseudo projective curvature tensor satisfying some properties on a normal paracontact metric manifold, *Commun. Fac. Sci. Univ. Ank. Ser. A1, Math. Stat*., 68 (2019) 997-1006.

[8] Capelletti-Montano, B., Küpeli Erken, I., Murathan, C., Nullity conditions in paracontact geometry, *Differential Geom. Appl*., 30 (2012) 665-693.

[9] Kowalczyk, D., On some subclass of semisymmetric manifolds, *Soochow J. Math*., 27 (2001) 445-461.

[10] Szabo, Z.I., Structure theorems on riemannian spaces satisfying R(ϖ_1,ϖ_2)R=0, I: The local Version, *J. Differential Geom*., 17 (1982) 531-582.

[11] Tripathi, M., Gupta, P., T-Curvature tensor on a semi-Riemannian manifold, *J. Adv. Math. Studies*, 4 (1) (2011) 117-129.

[12] Yıldız, A., De, U.C., Murathan, C., Arslan K., On the weyl projective curvature tensor of an N(k)-contact metric manifold, *Mathematica Pannonica*, 21 (2010) 1-14.

[13] De, U.C., Sarkar, A., On the Projective curvature tensor of generalized Sasakian space forms, *Questiones Mathematicae*, 33 (2010) 245-252.

[14] Atçeken, M., On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor, *Bulletin of Mathematical Analysis and Applications*, 6 (2014) 1-8.

[15] Özgür, C., De, U.C., On the quasi-conformal curvature tensor of a Kenmotsu manifold, *Mathematica Pannonica*, 17 (2006) 221-228.

[16] Arslan, K., Murathan, C., Özgür, C., On contact manifolds satisfying certain curvature conditions, *An. Univ. Bucuresti Math.,* 49 (2000) 17-26.

[17] Pokhariyal, G.P., Mishra, R.S., Curvature tensors and their relativistic significance II, *Yokohama Math. J.*, 19 (1971) 97-103.

[18] Ojha, R., M-projectively flat Sasakian manifolds, *Indian J. Pure Appl. Math*., 17 (1986) 481-484.

[19] Uygun, P., Atçeken, M., On (κ,μ)-paracontact metric spaces satisfying some conditions on the W_0-curvature tensor, *New Trends in Mathematical Sciences*, 9 (2) 2021 26-37.

[20] Janssens, D., Vanhecke, L., Almost contact structure and curvature tensors, *Kodai Math. J.,* 4 (1981) 1-27.

[21] Yano, K., Concircular geometry I, Concircular transformations, *Proc. Imp. Acad., Tokyo*, 16 (1940) 195-200.

[22] Yano, K., Sawaki, S., Riemannian manifolds admitting a conformal transformation group, *J. Diff. Geom*., 2 (1968) 161-184.

[23] Mert, T., On a Classification of Almost C(α)-Manifolds, *Journal of Mathematics*, (2022).

[24] Mert, T., Characterization of Some Special Curvature Tensor on Almost C(α)-Manifold, *Asian Journal of Mathematics and Computer Research*, 29 (1) (2022) 27-41.

[25] Mert, T., Atçeken, M., Almost C(α)-Manifold on W_0^*-Curvature Tensor, *Applied Mathematical Sciences*, 15 (15) (2021) 693-703.