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Relations Among Minimal Elements of a Family of Sets with Respect to Various Set Order Relations

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ABSTRACT

In this article, some set order relations given for set-optimization criterion which is one of the solution concept of set-valued optimization problems are considered. Minimal elements of a family of sets with respect to these set order relations are compared in detail. For this comparison relations between set order relations mentioned in this article are used. Also, in cases where a family of minimal sets is not a subset of the other one, examples are given.

Keywords: Set-valued optimization, Set order relations, Minimal solution.

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Introduction

A set-valued optimization problem is a problem which has a set-valued objective map. The set optimization approach, which is based on the idea of comparing values of objective map, is one of the solution concepts to solve set-valued optimization problems. For set comparison Nishnianidze [1], Young [2], Kuroiwa [3], Jahn and Ha [4] and Karaman et. al. [5] defined set order relations. One can see [6-20] for further works based on these order relations, including existence theorems for minimal elements, scalarizations, derivatives and optimality conditions etc.

In this article, the set of minimal elements of a family of sets with respect to set order relations $\preccurlyeq_c, \preccurlyeq_{mc}, \preccurlyeq_{mn}, \preccurlyeq_m, \preccurlyeq_s, \preccurlyeq_l, \preccurlyeq_u, \preccurlyeq_{m_1}, \preccurlyeq_{m_2}$ are compared each other in detail. The cases when a set of minimal elements includes the other one is proved and counter examples are given for other cases. Some of the counter examples in this article also can be given by using relations given in Example 3.4 in [4]. Also, in [Example 2.2, 11], it is shown that an m_2 −minimal element may not be an l −minimal element and vice versa. But in this article, it is also aimed to present different examples to contribute to the literature. Finally, all the relations presented in this article are summarized in figures.

Known Set Order Relations

In this section, we recall the known set order relations defined by Nischnianidze [1], Young [2], Kuroiwa [3], Jahn and Ha [4], Karaman et.al. [5].

For a normed space $(Y,\|\cdot\|)$ the algebraic sum and Minkowski (Pontryagin) difference of A and B is defined by

$$A+B:=\{a+b \mid a\in A, b\in B\},$$
$$A\dot{-}B:=\{x\in Y \mid x+B\subset A\},$$

respectively.

A set $K\subset Y$ is called a cone if $\lambda y\in K$ for all $\lambda\geq 0$ and $y\in K$. In this work, $K\subset Y$ is a nonempty, convex, pointed $(K\cap(-K)=\{0_Y\})$, cone with nonempty interior. $\operatorname{conv} A$ and $\operatorname{int} A$ denotes the convex hull of A and the topological interior of A, respectively. Also, $B(x,\varepsilon)$ denotes the closed ball with center x and radius ε, and

$$\mathcal{P}(Y)=\{A\subset Y \mid A\neq\emptyset\}.$$

A partial order relation $\leq_K$ is defined on Y via cone K as the following way:

$$x\leq_K y :\Leftrightarrow y-x\in K.$$

By using this vector order relation, the set of minimal and maximal elements of a nonempty subset A of Y are given as follows:

$$\min A:=\{x\in A \mid A\cap(x-K)=\{x\}\}.$$
$$\max A:=\{x\in A \mid A\cap(x+K)=\{x\}\}.$$

The set less order relation $\preccurlyeq_s$ was defined by Nishnianidze [1] and Young [2], u-type less order relation $\preccurlyeq_u$ and l-type less order relation $\preccurlyeq_l$ were given by Kuroiwa [3] and useful characterizations of these relations was given by Jahn and Ha [4] as the following definition.

Definiton 2.1. Let $A, B \in \mathcal{P}(Y)$. Then,

i. $A \preccurlyeq_l B :\Leftrightarrow B \subset A + K$,
ii. $A \preccurlyeq_u B :\Leftrightarrow A \subset B - K$,
iii. $A \preccurlyeq_s B :\Leftrightarrow B \subset A + K \text{ and } A \subset B - K$.

Proposition 2.2. [4] Let $A, B \in \mathcal{P}(Y)$. Then, the following assertion holds:

$A \preccurlyeq_s B$ and $B \preccurlyeq_s A \Leftrightarrow B + K = A + K, B - K = A - K$.

The certainly less order relation $\preccurlyeq_c$ is defined as follows:

Definition 2.3. [4] Let $A, B \in \mathcal{P}(Y)$. Then,
$A \preccurlyeq_c B :\Leftrightarrow (A = B)$ or $(A \neq B, \forall x \in A\ \forall y \in B : x \leq_K y)$.

The minmax less order relation ($\preccurlyeq_m$), the minmax certainly less order relation ($\preccurlyeq_{mc}$) and the minmax certainly nondominated order ($\preccurlyeq_{mn}$) relations are defined as:

Definition 2.4. [4] Let $A, B \in \mathcal{M}$.

i. $A \preccurlyeq_m B :\Leftrightarrow \min A \preccurlyeq_s \min B$ and $\max A \preccurlyeq_s \max B$,

ii. $A \preccurlyeq_{mc} B :\Leftrightarrow (A = B)$ or $(A \neq B, \min A \preccurlyeq_c \min B$ and $\max A \preccurlyeq_c \max B)$,

iii. $A \preccurlyeq_{mn} B :\Leftrightarrow (A = B)$ or $(A \neq B, \max A \preccurlyeq_s \min B)$,

where $\mathcal{M} = \{A \in \mathcal{P}(Y) \mid \min A \text{ and } \max A \text{ are nonempty}\}$.

Definition 2.5. [4] Let $A \in \mathcal{M}$. If the following equivalent conditions are satisfied

i. $\min A + K = A + K$ and $\max A - K = A - K$,
ii. $A \subset \min A + K$ and $A \subset \max A - K$,

then A is said to have the quasi domination property.

We denote the family of sets which have quasi domination property as $\mathcal{M}_0$.

The following relations are satisfied for $\preccurlyeq_s, \preccurlyeq_l, \preccurlyeq_u$.

Proposition 2.6. [4] Let $A, B \in \mathcal{P}(Y)$. Then,

i. $A \preccurlyeq_s B \Longrightarrow A \preccurlyeq_l B$,
ii. $A \preccurlyeq_s B \Longrightarrow A \preccurlyeq_u B$,
iii. $A \preccurlyeq_l B$ doesn't always imply $A \preccurlyeq_u B$ and vice versa.

Proposition 2.7. [4] Let $A, B \in \mathcal{M}_0$ with $A \neq B$. Then,

i. $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mc} B \Longrightarrow A \preccurlyeq_m B \Longrightarrow A \preccurlyeq_s B$,
ii. $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mn} B \Longrightarrow A \preccurlyeq_m B$,
iii. $A \preccurlyeq_{mn} B$ doesn't always imply $A \preccurlyeq_{mc} B$ and vice versa.

Indeed, quasi domination property is not required for relations $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mc} B \Longrightarrow A \preccurlyeq_m B$ and $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_{mn} B$ and $A \preccurlyeq_c B \Longrightarrow A \preccurlyeq_s B$.

Following set order relations $\preccurlyeq_{m_1}$and $\preccurlyeq_{m_2}$were introduced by Karaman et.al. [5]

Definition 2.8. Let $A, B \in \mathcal{P}(Y)$.

i. $A \preccurlyeq_{m_1} B :\Leftrightarrow (B \dot{-} A) \cap K \neq \emptyset$,
ii. $A \preccurlyeq_{m_2} B :\Leftrightarrow (A \dot{-} B) \cap (-K) \neq \emptyset$.

$\preccurlyeq_{m_1}$and $\preccurlyeq_{m_2}$ are partial order relations on the family of nonempty and bounded subsets of Y.

Proposition2.9. [4]: Let $A, B \in \mathcal{M}$. Then, following statements hold:

i. $A \preccurlyeq_m B$ and $B \preccurlyeq_m A \Leftrightarrow$ $\min A + K = \min B + K, \max A - K = \max B - K$, $\min A - K = \min B - K, \max A + K = \max B + K$.
ii. If K is pointed then,
$A \preccurlyeq_{mc} B$ and $B \preccurlyeq_{mc} A \Leftrightarrow \min A = \min B$ and $\max A = \max B$.
iii. If K is pointed and $A, B \in \mathcal{M}_0$ then,
$A \preccurlyeq_m B$ and $B \preccurlyeq_m A \Leftrightarrow \min A = \min B$ and $\max A = \max B$.

Remark 2.10: Quasi domination property was not used in the proof of Proposition 2.9 (iii) [Proposition 3.4, 4]. Since, we assume the pointedness of K, the proposition can be restated as:

Proposition 2.11: Let $A, B \in \mathcal{M}$. Then, following statements are equivalent:

i. $A \preccurlyeq_m B$ and $B \preccurlyeq_m A$,
ii. $A \preccurlyeq_{mc} B$ and $B \preccurlyeq_{mc} A$,
iii. $\min A = \min B$ and $\max A = \max B$.

By using these set order relations, minimal element of a family of sets is defined as the following way.

Definition 2.12. [4,5,12] Let $\mathcal{S} \subset \mathcal{P}(Y)$ and $* \in \{u, l, s, m, c, mn, mc, m_1, m_2\}$. The set $A \in \mathcal{S}$ is called a $*$-minimal element of $\mathcal{S}$ if for any $B \in \mathcal{S}$ such that $B \preccurlyeq_* A$ implies $A \preccurlyeq_* B$. The family of $*$-minimal elements of $\mathcal{S}$ is denoted by $* - \min \mathcal{S}$.

Relations Between Minimal Sets with Respect to Set Order Relations

In this section, we compare the set of minimal elements of a family of sets with respect to set orders mentioned in the previous section.

Proposition 3.1: Let $\mathcal{S} \subset \mathcal{M}$. Then, we have $mn - \min \mathcal{S} \subset c - \min \mathcal{S}$.

Proof: Let $A \in mn - \min \mathcal{S}$ and there exist $B \in \mathcal{S}$ such that $B \preccurlyeq_c A$. Since $\preccurlyeq_c$ is a partial order it suffices to show $A = B$. As $B \preccurlyeq_c A$ from Proposition 2.7 we have $B \preccurlyeq_{mn} A$. Since A is an mn-minimal element of $\mathcal{S}$ we get $A \preccurlyeq_{mn} B$. So, from the definition of $\preccurlyeq_{mn}$ we have $\max A \preccurlyeq_s \min B$ which gives

$$\min B \subset \max A + K, \tag{3.1}$$
$$\max A \subset \min B - K. \tag{3.2}$$

Now, we show that $A = B$. Assume the contrary. Inequality $B \preccurlyeq_c A$ implies that

$$b \leq_K a \text{ for all } a \in A \text{ and } b \in B. \quad (3.3)$$

Let $b_{min} \in \min B$. From (3.1) there exists $a_{max} \in \max A$ and $k_1 \in K$ such that $b_{min} = a_{max} + k_1$. Hence, $a_{max} \leq_K b_{min}$. From (3.3), we have $b_{min} \leq_K a_{max}$. Pointedness of K gives that $b_{min} = a_{max}$. So, $b_{min} \in \max A$. Thus, $\min B \subset \max A$. Similarly, the converse implication can be proved by using (3.2). So, $\min B = \max A$. This equality implies that $A = B$. Indeed, let $b \in B$. For any $y \in \max A\ (= \min B)$, from (3.3), we have $b \leq_K y$. Since $y \in \min B$ and K is pointed we obtain $b = y \in \min B$ which means $B \subset \min B$. As $\min B \subset B$, we get $\min B = B$. The equality $A = \max A$ can be shown similarly. Therefore, we obtain $B = \min B = \max A = A$. This contradicts with assumption. So $A = B$ and hence $A \in c - \min \mathcal{S}$.

Note that a c-minimal element does not have to be an mn-minimal element.

Example 3.2: Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B\big((0,0),1\big), C = B\left(\left(2,\frac{3}{2}\right),1\right)$ and $\mathcal{S} = \{A, C\}$. Then, we have $(0,1) \in A, \left(2,\frac{1}{2}\right) \in C$ and $(0,1) \nleq_K \left(2,\frac{1}{2}\right)$, $\left(2,\frac{1}{2}\right) \nleq_K (0,1)$ which gives $A \not\preccurlyeq_c C$ and $C \not\preccurlyeq_c A$. Hence, $c - \min \mathcal{S} = \{A, C\}$. In addition,

$$\min A = \{\, (x,y) \mid x^2 + y^2 = 1, x,y \leq 0 \,\},$$
$$\max A = \{\, (x,y) \mid x^2 + y^2 = 1, x,y \geq 0 \,\},$$
$$\min C = \left\{ (x,y) \,\middle|\, (x-2)^2 + \left(y - \frac{3}{2}\right)^2 = 1, x \leq 2, y \leq \frac{3}{2} \right\},$$
$$\max C = \left\{ (x,y) \,\middle|\, (x-2)^2 + \left(y - \frac{3}{2}\right)^2 = 1, x \geq 2, y \geq \frac{3}{2} \right\}.$$

Since $\max A \subset \min C - K$ and $\min C \subset \max A + K$ we have $A \preccurlyeq_{mn} C$. It is obvious that $C \not\preccurlyeq_{mn} A$. So, $C \notin mn - \min \mathcal{S}$.

Proposition 3.3: Let $\mathcal{S} \subset \mathcal{M}$. Then, we have $mc - \min \mathcal{S} \subset c - \min \mathcal{S}$.

Proof: Let $A \in mc - \min \mathcal{S}$ and there exist $B \in \mathcal{S}$ such that $B \preccurlyeq_c A$. Suppose $B \neq A$. Since $B \preccurlyeq_c A$ from Proposition 2.7 we have $B \preccurlyeq_{mc} A$. From mc-minimality of A we have $A \preccurlyeq_{mc} B$. So, by Proposition 2.11, we obtain

$$\min A = \min B, \max A = \max B.$$

Since $B \neq A$ and $B \preccurlyeq_c A$, inequality (3.3) holds. Let $a \in A$. Then, from (3.3) $y \leq a$ for any $y \in \max B = \max A$. As $y \in \max A$ and K is pointed we get $a = y \in \max A$. Hence, $A \subset \max A = \max B \subset B$.

Let $b \in B$. Then, from (3.3), $b \leq_K a$ for all $a \in \min A = \min B$. Hence from minimality of a and pointedness of K we obtain $b = a \in \min B$. Thus, $B \subset \min B = \min A \subset A$. So, $A = B$ which contradicts with the assumption.

Proposition 3.4: Let $\mathcal{S} \subset \mathcal{M}$. Then, $m - \min \mathcal{S} \subset mc - \min \mathcal{S}$.

Proof: Let $A \in m - \min \mathcal{S}$ and $B \preccurlyeq_{mc} A$ for some $B \in \mathcal{S}$. Then, from Proposition 2.7 we have $B \preccurlyeq_m A$. As A is m-minimal element we get $A \preccurlyeq_m B$. From Proposition 2.11 we obtain $A \preccurlyeq_{mc} B$. Hence, $A \in mc - \min \mathcal{S}$.

The converse inclusion in Proposition 3.4 is not true in general. The following example shows this fact.

Example 3.5: Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+, A = B\big((0,0),1\big), B = A \cap \{\, (x,y) \mid y \geq -x \,\}$. Then,

$\min A = \{\, (x,y) \mid x^2 + y^2 = 1, x,y \leq 0 \,\}$, $\min B = \operatorname{conv}\left\{\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)\right\}$,

$\max A = \max B = \{\, (x,y) \mid x^2 + y^2 = 1, x,y \geq 0 \,\}$.

Since $\min A \not\preccurlyeq_c \min B$ and $\min B \not\preccurlyeq_c \min A$ we have $A \not\preccurlyeq_{mc} B$ and $B \not\preccurlyeq_{mc} A$. So, $mc - \min \mathcal{S} = \{A, B\}$.

Furthermore, $\min A \subset \min B - K$, $\min B \subset \min A + K, \max A = \max B$. Then, we have $\min A \preccurlyeq_s \min B$ and $\max A \preccurlyeq_s \max B$, i.e. $A \preccurlyeq_m B$. As $\min B \not\subset \min A - K$ we get $B \not\preccurlyeq_m A$. Hence, $m - \min \mathcal{S} = \{A\}$.

This example also implies that $mc - \min \mathcal{S} \not\subset * - \min \mathcal{S}$ where $* \in \{s, l, m_1, m_2\}$. Now, we show this fact.

It is clear that $A \preccurlyeq_s B$, $B \not\preccurlyeq_s A, A \preccurlyeq_l B$, $B \not\preccurlyeq_l A$. So, $B \notin s - \min \mathcal{S}$ and $B \notin l - \min \mathcal{S}$.

Furthermore, $A \dot{-} B = \{(0,0)\}$ and $B \dot{-} A = \emptyset$. Thus, $A \preccurlyeq_{m_2} B$, $B \preccurlyeq_{m_1} A, B \not\preccurlyeq_{m_2} A$ and $A \not\preccurlyeq_{m_1} B$. Then it follows

$m_2 - \min \mathcal{S} = \{A\}$ and $m_1 - \min \mathcal{S} = \{B\}$.

Remark 3.6: Also note that an mc −minimal element does not have to be a u −minimal element. For example, if

$Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+, A = B\big((0,0),1\big), B = A \cap \{\, (x,y) \mid y \leq -x \,\}$, then it can be shown that $mc - \min \mathcal{S} = \{A, B\}$ and

$u - \min \mathcal{S} = \{B\}$, similar with Example 3.5.

The following corollary is a direct consequence of Proposition 3.3 and Proposition 3.4.

Corollary 3.7: Let $\mathcal{S} \subset \mathcal{M}$. Then, $m - \min \mathcal{S} \subset c - \min \mathcal{S}$.

Proposition 3.8: Let $\mathcal{S} \subset \mathcal{P}(Y)$. Then $s - \min \mathcal{S} \subset c - \min \mathcal{S}$.

Proof: Let $A \in s - \min \mathcal{S}$. Suppose there exists $B \in \mathcal{S}$ such that $B \preccurlyeq_c A$ and $A \neq B$. $B \preccurlyeq_c A$ implies $B \preccurlyeq_s A$ from Proposition 2.7. Since A is an s-minimal element, we obtain $A \preccurlyeq_s B$. Therefore,

$$B \subset A + K, \quad (3.4)$$
$$A \subset B - K. \quad (3.5)$$

Let $b \in B$. Then, from (3.4), there exists $a \in A$ and $k_1 \in K$ such that $b = a + k$. So, $a \leq_K b$. Since $B \preccurlyeq_c A$ and $B \neq A$, we have inequality (3.3) and hence $b \leq_K a$. From pointedness of K we obtain $b = a \in A$ which gives $B \subset A$. Conversely, let $a \in A$. Then from (3.5), there exists $b \in B$ and $k_2 \in K$ such that $a = b - k_2$. Thus, we have $a \leq_K b$. The inequality (3.3) implies $b \leq_K a$. Pointedness of K gives $a = b \in B$. Since $a \in A$ is arbitrary we obtain $A \subset B$. Therefore, $A = B$ which contradicts with the assumption. So, $A = B$ and $A \in c - \min \mathcal{S}$.

As seen in the following example an l −minimal or an u −minimal element does not have to be a $*$ −minimal element where $* \in \{c, mc, mn, m, s, u, m_1, m_2\}$ or

$* \in \{c, mc, mn, m, s, l, m_1, m_2\}$, respectively.

Example 3.9: Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+, A = [0,1] \times [0,1], B = \{(1,1)\}$ and $\mathcal{S} = \{A, B\}$. Then, $A \preccurlyeq_u B$ and $B \preccurlyeq_u A$. Hence, $u - \min\ \mathcal{S} = \{A, B\}$. Since $x \leq_K (1,1)$ for all $x \in A$, $A \preccurlyeq_c B$. Also, the relation $B \not\preccurlyeq_l A$ is obvious. So, from Proposition 2.6 and Proposition 2.7 we have $A \preccurlyeq_* B$ and $B \not\preccurlyeq_* A$,and then it follows $* - \min \mathcal{S} = \{A\}$ where $* \in \{c, mc, mn, m, s, l\}$.

In addition, $(0,0) \in A \dot{-} B = \{(-1,-1)\} + A$ and $B \dot{-} A = \emptyset$. Hence, we get $B \preccurlyeq_{m_1} A, A \preccurlyeq_{m_2} B, A \not\preccurlyeq_{m_1} B$ and $B \not\preccurlyeq_{m_2} A$.Thus, $m_1 - \min \mathcal{S} = \{B\}$ and $m_2 - \min \mathcal{S} = \{A\}$.

If $A = \{(0,0)\}, B = [0,1] \times [0,1]$ and $\mathcal{S} = \{A, B\}$ one can easily see $l - \min\ \mathcal{S} = \{A, B\}$, $* - \min \mathcal{S} = \{A\}$ where

$* \in \{c, mc, mn, m, s, u, m_1\}$ and $m_2 - \min \mathcal{S} = \{B\}$.

Also note that an m_1 −minimal or an m_2 −minimal element does not have to be a $*$ −minimal element where

$* \in \{c, mc, mn, m, s, l, u, m_2\}$ and
$* \in \{c, mc, mn, m, s, l, u, m_1\}$, respectively.

Example 3.10: Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B\big((0,0), 1\big), C = B((4,4), 2)$ and $\mathcal{S} = \{A, C\}$. Then, $A \dot{-} C = \emptyset$ and

$C \dot{-} A = B((4,4), 1)\} \subset K$. So, we have $A \not\preccurlyeq_{m_2} C$ and $C \not\preccurlyeq_{m_2} A$, respectively. Hence, $m_2 - \min \mathcal{S} = \{A, C\}$. In addition,

$$\min A = \{(x, y) \mid x^2 + y^2 = 1, x, y \leq 0\},$$
$$\max\ A = \{(x, y) \mid x^2 + y^2 = 1, x, y \geq 0\},$$
$$\min C = \{\, (x, y) \mid (x - 4)^2 + (y - 4)^2 = 4, x, y \leq 4\,\},$$
$$\max C = \{\, (x, y) \mid (x - 4)^2 + (y - 4)^2 = 4, x, y \geq 4\,\}.$$

Since $(x, y) \leq_K (a, b)$ for all $(x, y) \in A$ and $(a, b) \in C$ it follows $A \preccurlyeq_c C$. Also, $C \not\subset A - K$ and $A \not\subset C + K$, i.e. $C \not\preccurlyeq_u A$ and $C \not\preccurlyeq_l A$, respectively. Hence, from Proposition 2.6 and Proposition 2.7 we have $A \preccurlyeq_* C$ and $C \not\preccurlyeq_* A$ where

$* \in \{c, mc, mn, m, s, l, u\}$. Thus, $C \notin * - \min \mathcal{S}$ for $* \in \{c, mc, mn, m, s, l, u\}$.

Furthermore, since $A \dot{-} C = \emptyset$ and $C \dot{-} A = B((4,4), 1)\} \subset K$, we have $C \not\preccurlyeq_{m_1} A$ and $A \preccurlyeq_{m_1} C$, respectively. Thus,

$C \notin m_1 - \min \mathcal{S}$.

If $A = B\big((0,0), 2\big), C = B((4,4), 1)$, then similarly we have, $m_1 - \min \mathcal{S} = \{A, C\}$ and $C \notin * - \min \mathcal{S}$ for

$* \in \{c, mc, mn, m, s, l, u, m_2\}$.

Following example shows that a c-minimal element does not have to be a $*$-minimal element where $* \in \{\, mc, m, m_1, m_2, s, l, u\,\}$.

Example 3.11: Let $Y = \mathbb{R}^2$, $K = \mathbb{R}^2_+$, $A = B\big((1,1), 1\big), B = ([0,2] \times [0,2]) \backslash \{(x, y) \mid (x - 1)^2 + (y - 1)^2 > 1, x, y \leq 1\}$ and $\mathcal{S} = \{A, B\}$. Since, $(1,0), (0,1) \in A \cap B$, $(1,0) \not\leq_K (0,1)$ and $(0,1) \not\leq_K (1,0)$, we have $A \not\preccurlyeq_c B$ and $B \not\preccurlyeq_c A$. Hence, $c - \min \mathcal{S} = \{A, B\}$.

Also, $\min A = \min B = \{(x, y) \mid (x - 1)^2 + (y - 1)^2 = 1, x, y \leq 1\}$,

$\max A = \{(x, y) \mid (x - 1)^2 + (y - 1)^2 = 1, x, y \geq 1\}$ and $\max B = \{(2,2)\}$.

Now, we show this fact for each order relation.

i. It is obvious that $(x, y) \leq_K (2,2)$ for all $(x, y) \in \max A$. So, $\min A \preccurlyeq_c \min B$, $\max A \preccurlyeq_c \max B$, i.e. $A \preccurlyeq_{mc} B$. Also, $B \not\preccurlyeq_{mc} A$ because $(2,2) \not\leq_K (x, y)$ for any $(x, y) \in \max A$. Therefore, $B \notin mc - \min \mathcal{S}$.
ii. Since $\min A = \min B$ and $\max A \preccurlyeq_s \max B$, we have $A \preccurlyeq_m B$. Also, $\max B \not\preccurlyeq_s \max A$ which implies $B \not\preccurlyeq_m A$. So, $B \notin m - \min \mathcal{S}$.
iii. $A \subset B - K$ and $B \not\subset A - K$ imply $A \preccurlyeq_u B$ and $B \not\preccurlyeq_u A$, respectively. Thus, $B \notin\ u - \min \mathcal{S}$.
iv. From (iii) and (iv), we have $A \preccurlyeq_s B$ and $B \not\preccurlyeq_s A$. So, $B \notin s - \min \mathcal{S}$.
v. Since $B \dot{-} A = \{(0,0)\}, A \dot{-} B = \emptyset$, we get $A \preccurlyeq_{m_1} B$, $B \not\preccurlyeq_{m_1} A$ and $B \preccurlyeq_{m_2} A$, $A \not\preccurlyeq_{m_2} B$. Hence, $B \notin m_1 - \min \mathcal{S}$ and $A \notin m_2 - \min \mathcal{S}$.

This example also shows that an mn −minimal element does not have to be a $*$-minimal element where

$* \in \{\, mc, m, m_1, m_2, s, u\,\}$. Indeed, $\max A \not\preccurlyeq_s \min B$ and $\max B \not\preccurlyeq_s \min A$. Hence, $mn - \min \mathcal{S} = \{A, B\}$.

If we consider $A = B\big((1,1), 1\big), B = ([0,2] \times [0,2]) \backslash \{(x, y) \mid (x - 1)^2 + (y - 1)^2 > 1, x, y \geq 1\}$, it can easily be seen that $mn - \min \mathcal{S} = c - \min \mathcal{S} = \{A, B\}$ and $l - \min \mathcal{S} = \{B\}$.

Proposition 3.12: Let $\mathcal{S} \subset \mathcal{M}_0$. Then, following conditions are satisfied:

i. $m - \min \mathcal{S} \subset mn - \min \mathcal{S}$,
ii. $s - \min \mathcal{S} \subset m - \min \mathcal{S}$,
iii. $s - \min \mathcal{S} \subset mc - \min \mathcal{S}$,
iv. $s - \min \mathcal{S} \subset mn - \min \mathcal{S}$.

Proof:

i. Let $A \in m - \min \mathcal{S}$ and there exist $B \in \mathcal{S}$ such that $B \preccurlyeq_{mn} A$. If $A = B$ then, A is obviously an mn-minimal element. If $A \neq B$ then, $\max A \preccurlyeq_s \min B$. From Proposition 2.7, we have $B \preccurlyeq_m A$. Since $A \in m - \min \mathcal{S}$, we obtain $A \preccurlyeq_m B$. From Proposition 2.11, we get $\min A = \min B$ and $\max A = \max B$. Hence, we obtain $\max B = \max A \preccurlyeq_s \min B = \min A$ which implies $A \preccurlyeq_{mn} B$. So $A \in mn - \min \mathcal{S}$.

ii. Let $A \in s - \min \mathcal{S}$ and there exist $B \in \mathcal{S}$ such that $B \preccurlyeq_m A$. $B \preccurlyeq_m A$ implies $B \preccurlyeq_s A$ from Proposition 2.7. Since $A \in s - \min \mathcal{S}$, we have $A \preccurlyeq_s B$. Hence, we get $A + K = B + K$ and $A - K = B - K$ from Proposition 2.2. Also, quasi domination property of A and B implies $\min A + K = A + K = B + K = \min B + K$, $\max A - K = A - K = B - K = \max B - K$. Since, $B \preccurlyeq_m A$, we have $\min A = \min B$ and $\max A = \max B$. So, from Proposition 2.11, we obtain $A \preccurlyeq_m B$. Thus, $A \in m - \min \mathcal{S}$.

iii. Proof is obtained directly from (ii) and Proposition 3.4.

iv. It can be obtained directly from (i) and (ii).

If quasi domination property in Proposition 3.12 is omitted then, results may not be true as seen in the following examples.

Example 3.13: Let $K = \mathbb{R}^2_+$,
$A = \{ (x, y) \mid -x < y \leq 1 - x \} \cup \{ (x, y) \mid y = -x, x \leq 0 \}$, $B = \{ (x, y) \mid y = 2 - x \}$ and $\mathcal{S} = \{A, B\}$. Then,

$$\begin{aligned} \min A &= \{ (x, y) \mid y = -x, x \leq 0 \}, \\ \max A &= \{ (x, y) \mid y + x = 1 \}, \\ \min B = \max B &= B, \end{aligned}$$

And $A \notin \mathcal{M}_0$. As $\min B \not\subset \min A + K$, we have $\min A \not\preccurlyeq_s \min B$. So, $A \not\preccurlyeq_m B$. Also, it is clear that $B \not\preccurlyeq_m A$. It follows $A, B \in m - \min \mathcal{S}$. Also, it is obvious that $\max A \preccurlyeq_s \min B$ and $\max B \not\preccurlyeq_s \min A$. Hence, $A \preccurlyeq_{mn} B$ and $B \not\preccurlyeq_{mn} A$, respectively. Then, we obtain $B \notin mn - \min \mathcal{S}$.

Example 3.14: Let $K = \mathbb{R}^2_+$, $A = \{(-1,1)\} \cup \operatorname{int} B\big((1,0), 1\big)$, $B = \{(0,2)\}$ and $\mathcal{S} = \{A, B\}$. It can be easily seen that

$$\begin{aligned} &\min A = \max A = \{(-1,1)\}, \\ &\min B = \max B = B, \end{aligned}$$

and $A \notin \mathcal{M}_0$. Since $A \not\subset B - K$, we obtain $A \not\preccurlyeq_u B$ which implies $A \not\preccurlyeq_s B$. Also, as $A \not\subset B + K$ i.e. $B \not\preccurlyeq_l A$ and it follows $B \not\preccurlyeq_s A$. So, $A, B \in s - \min \mathcal{S}$.

However, we have $A \preccurlyeq_m B$ and $B \not\preccurlyeq_m A$ which imply $B \notin m - \min \mathcal{S}$.

Also, since $\max A \preccurlyeq_s \min B$ and $\max B \not\preccurlyeq_s \min A$ we obtain $A \preccurlyeq_{mn} B$ and $B \not\preccurlyeq_{mn} A$, respectively. Then, $B \notin mn - \min \mathcal{S}$.

Furthermore, since $(-1,1) \leq_K (0,2)$ and $(0,2) \not\leq_K (-1,1)$, we have $\min A \preccurlyeq_c \min B$, $\max A \preccurlyeq_c \max B$, $\min B \not\preccurlyeq_c \min A$ and $\max B \not\preccurlyeq_c \max A$. These relations imply $A \preccurlyeq_{mc} B$ and $B \not\preccurlyeq_{mc} A$. Hence,
$B \notin mc - \min \mathcal{S}$.

An mc −minimal element does not have to be an mn −minimal element. To show this fact Example 3.4 in [4] by Jahn and Ha can be used as follows.

Example 3.15 [4] Let $K = \mathbb{R}^2_+$, $A = \operatorname{conv}\{(-2,0), (-3,-1), (0,-2)\}$,
$B = \operatorname{conv}\{(4,2), (0,2), (4,-2)\}$ and
$\mathcal{S} = \{A, B\}$. Then as proved in [4], we have $A \preccurlyeq_{mn} B$ and $A \not\preccurlyeq_{mc} B$. It is obvious that $B \not\preccurlyeq_{mn} A$. Hence, $B \in mc - \min \mathcal{S}$ and $B \notin mn - \min \mathcal{S}$.

Next example shows that $m - \min \mathcal{S} \not\subset * - \min \mathcal{S}$ and $s - \min \mathcal{S} \not\subset * - \min \mathcal{S}$ where $* \in \{u, l, m_1, m_2\}$.

Example 3.16: Let $K = \mathbb{R}^2_+$, $A = [-1,1] \times [-1,1]$, $B = \{(0,0)\}$ and $\mathcal{S} = \{A, B\}$. Then,

$$\begin{aligned} &\min A = \{(-1,-1)\}, \\ &\max A = \{(1,1)\}, \\ &\min B = \max B = \{(0,0)\}. \end{aligned}$$

Since $(0,0) \not\leq_K (-1,-1)$ and $(1,1) \not\leq_K (0,0)$, we have $\min B \not\preccurlyeq_s \min A$ and $\max A \not\preccurlyeq_s \max B$, respectively. Then, it follows $B \not\preccurlyeq_m A$ and $A \not\preccurlyeq_m B$ and $m - \min \mathcal{S} = \{A, B\}$. However, as $B = \{(0,0)\} \subset A - K$ and $A \not\subset \{(0,0)\} - K = -K$ we obtain $B \preccurlyeq_u A$ and $A \not\preccurlyeq_u B$, respectively. Thus, $A \notin u - \min \mathcal{S}$. Also, since $B = \{(0,0)\} \subset A + K$ and $A \not\subset B + K$ we get $A \preccurlyeq_l B$ and $B \not\preccurlyeq_l A$. Hence, $B \notin l - \min \mathcal{S}$. Furthermore, since $B \not\preccurlyeq_l A$ and $A \not\preccurlyeq_u B$, we have $B \not\preccurlyeq_s A$ and $A \not\preccurlyeq_s B$. So, $s - \min \mathcal{S} = \{A, B\}$. In addition, it can be easily seen that $A \dot{-} B = A$ and $B \dot{-} A = \emptyset$. Then, it follows $B \preccurlyeq_{m_1} A$, $A \preccurlyeq_{m_2} B$, $A \not\preccurlyeq_{m_1} B$ and $B \not\preccurlyeq_{m_2} A$. So, $A \notin m_1 - \min \mathcal{S}$ and $B \notin m_2 - \min \mathcal{S}$.

The following example implies that an m −minimal element does not have to be an s −minimal element.

Example 3.17: Let $K = \mathbb{R}^2_+$, $A = \operatorname{conv}\{(1,0), (0,1)\}$, $B = [1,2] \times [0,1]$ and $\mathcal{S} = \{A, B\}$. We have

$$\begin{aligned} &\min B = \{(1,0)\}, \\ &\max B = \{(2,1)\}, \\ &\min A = \max A = A. \end{aligned}$$

So, it follows $\min A \not\preccurlyeq_s \min B$ and $\min B \not\preccurlyeq_s \min A$. Hence, $A \not\preccurlyeq_m B$ and $B \not\preccurlyeq_m A$. Then, we obtain $m - \min \mathcal{S} = \{A, B\}$. Also, we get $A \preccurlyeq_l B, A \preccurlyeq_u B, B \not\preccurlyeq_l A$ and $B \not\preccurlyeq_u A$ which imply $A \preccurlyeq_s B, B \not\preccurlyeq_s A$. Thus, $B \notin s - \min \mathcal{S}$.

The results in this article are summarized in Figures 1-9 below.

We have

$m_1 - \min \not\rightleftarrows * -\min$ where $* \in \{c, mc, mn, m, s, l, u, m_2\}$,

$m_2 - \min \not\rightleftarrows * -\min$ where $* \in \{c, mc, mn, m, s, l, u, m_1\}$,

$l - \min \not\rightleftarrows * -\min$ where $* \in \{c, mc, mn, m, s, u, m_1, m_2\}$,

$u - \min \not\rightleftarrows * -\min$ where $* \in \{c, mc, mn, m, s, l, m_1, m_2\}$.

So, these relations will be omitted in the figures.

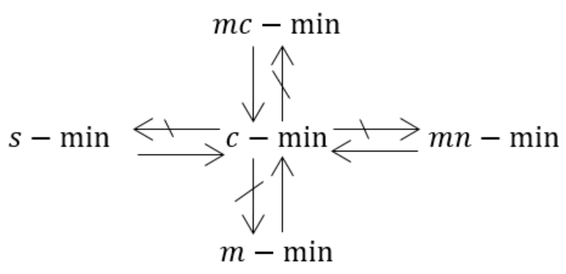

Figure 1. Comparison of c −minimal elements with others

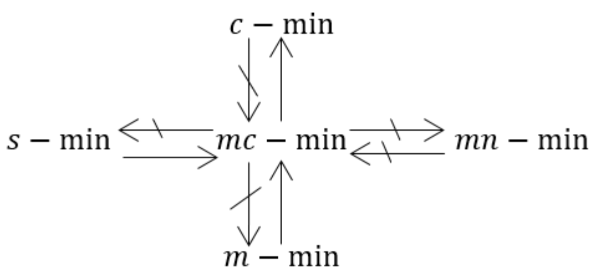

Figure 2. Comparison of mc −minimal elements with others under quasi domination assumption

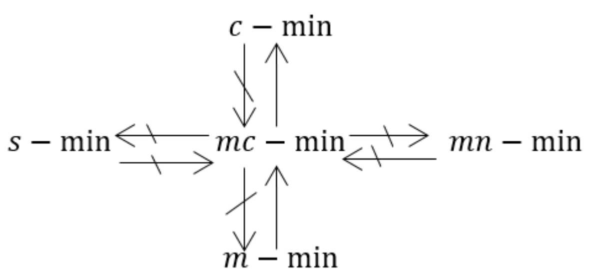

Figure 3. Comparison of mc −minimal elements with others without quasi domination assumption

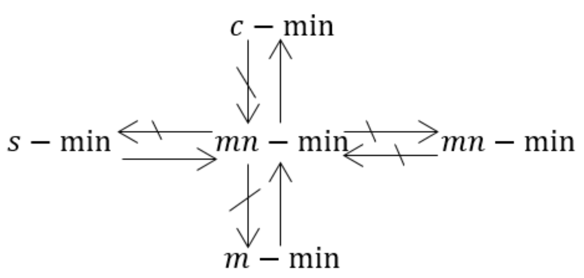

Figure 4. Comparison of mn −minimal elements with others under quasi domination assumption

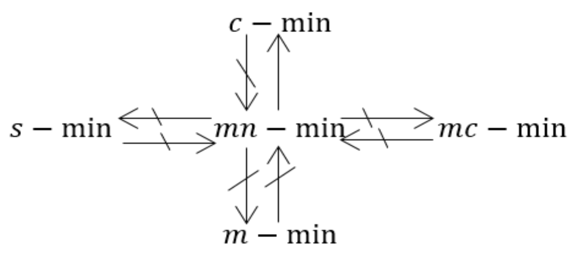

Figure 5. Comparison of mn −minimal elements with others without quasi domination assumption

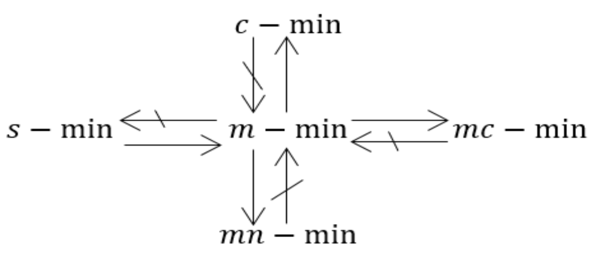

Figure 6. Comparison of m −minimal elements with others under quasi domination property assumption

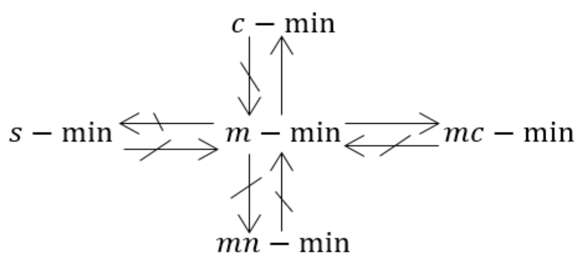

Figure 7. Comparison of m −minimal elements with others without quasi domination assumption

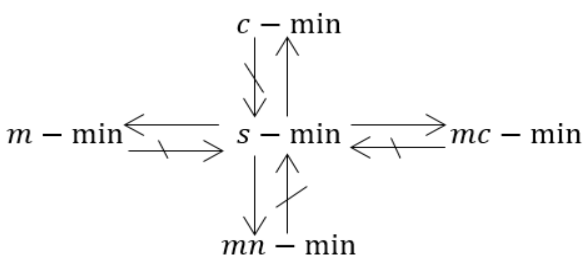

Figure 8. Comparison of s −minimal elements with others under quasi domination property assumption

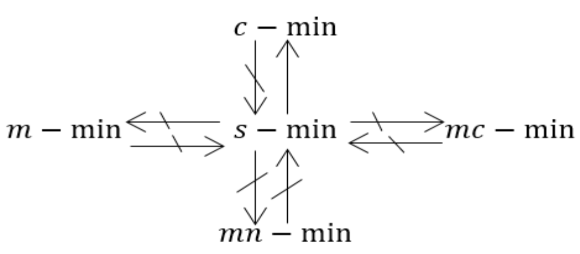

Figure 9: Comparison of s −minimal elements with others without quasi domination assumption

Conflicts of interest

There are no conflicts of interest in this work.

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