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Approximating of fixed points for Garsia-Falset generalized nonexpansive mappings

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Abstract — This paper studies the convergence of fixed points for Garsia-Falset generalized nonexpansive mappings. First, it investigates weak and strong convergence results for Garsia-Falset generalized nonexpansive mappings using the Temir-Korkut iteration in uniformly convex Banach spaces. This paper then exemplifies Garsia-Falset generalized nonexpansive mappings, which exceed the class of Suzuki generalized nonexpansive mappings. Moreover, it numerically compares this iteration's convergence speed with the well-known Thakur iteration of approximating the fixed point of Garsia-Falset generalized nonexpansive mapping. The results show that the Temir-Korkut iteration converges faster than the Thakur iteration converges. Finally, this paper discusses the need for further research.

Subject Classification (2020): 47H09, 47H10

1. Introduction

In mathematics and many disciplines, the concept of a fixed point is crucial. The conditions under which maps have solutions are provided by fixed point results. In particular, fixed-point approaches have been used in various disciplines, including biology, chemistry, economics, engineering, and informatics. Even if establishing the existence of a fixed point is an essential first step, the major and probably last stage in finding a solution is to find the exact value of the intended fixed point. An iterative procedure is one of the common ways to obtain the intended fixed point. In the last 65 years, many authors have been interested in these areas and established many iterative processes to approximate fixed points for nonexpansive mappings and a broader class of nonexpansive mappings.

Especially, some generalizations of nonexpansive mappings and the study of related fixed point theorems have been intensively carried out over the past decades [1–9]. A class of generalized nonexpansive mappings (GNMs) on a nonempty subset K of a Banach space X has been defined by Suzuki [5]. Such mappings were referred to as belonging to the class of mappings satisfying condition (C) (also referred as Suzuki GNM), which properly includes the class of nonexpansive mappings. Recently, fixed point theorems for Suzuki generalized nonexpansive mappings have been studied by a number of authors [10–14]. Every self-mapping Ψ on K providing condition (C) has an almost fixed point sequence for a nonempty bounded and convex subset K. Two new classes of GNMs that are wider

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than those providing the condition (C) were presented in 2011 by Garsia-Falset et al. [2], while retaining their fixed point properties. They investigated Garsia-Falset GNMs, named as condition (E). Later in 2019, for a general class of nonexpansive mappings that are not necessarily continuous on their domains, Pandey et al. [3] presented fixed point results and showed how several other classes of nonexpansive type mappings are appropriately included inside this class. Recently, Usurelu et al. [7] investigated the visualization of convergence behaviors of various iterative processes and some fixed point outcomes for this class of mappings.

Some related properties to the aforesaid topics are as follows: Let K be a nonempty subset of a Banach space X. A mapping $\Psi : K \to K$ is said to be nonexpansive if $\|\Psi u - \Psi v\| \leq \|u - v\|$, for all $u, v \in K$. A mapping $\Psi : K \to K$ is called quasi-nonexpansive if $\|\Psi u - p\| \leq \|u - p\|$, for all $u \in K$ and $p \in F(\Psi)$ (where $F(\Psi)$ denotes the set of all fixed points of Ψ). Therefore, the class of quasi-nonexpansive mappings is weaker than the class of nonexpansive mappings. Suzuki [5] presented the concept of GNMs, known as condition (C), in 2008. Let K be a nonempty convex subset of a Banach space X, a mapping $\Psi : K \to K$ satisfies condition (C) on K if, for all $u, v \in K$, $\frac{1}{2}\|u - \Psi u\| \leq \|u - v\| \Rightarrow \|\Psi u - \Psi v\| \leq \|u - v\|$. Suzuki [5] showed that the mapping satisfying condition (C) is weaker than nonexpansiveness and stronger than quasi-nonexpansiveness.

Recently, Garsia-Falset et al. [2] studied GNMs satisfying condition (E) that have a weaker property than Suzuki GNMs. Let K be a nonempty subset of a Banach space X. A mapping $\Psi : K \to X$ satisfies condition (E_μ) on K, if there exists $\mu \geq 1$ such that

$$\|u - \Psi v\| \leq \mu\|u - \Psi u\| + \|u - v\|$$

for all $u, v \in K$. Moreover, it is said that Ψ satisfies condition (E) on K, whenever Ψ satisfies condition (E_μ), for some $\mu \geq 1$. It is clearly seen that if $\Psi : K \to X$ is nonexpansive, then it satisfies condition (E_1) and from Lemma 7 in [5], we know that if $\Psi : K \to K$ satisfies condition (C) on K, then Ψ satisfies condition (E_3) (see [2]). Proposition 1 in [2], we know also that if $\Psi : K \to X$ a mapping which satisfies condition (E) on K has some fixed point, then Ψ is quasi-nonexpansive. Example 2 that is in [2] shows the converse is not true. Hence, the class of Garcia-Falset GNMs exceeds the class of Suzuki GNMs (also the class of nonexpansive mappings), however, it still remains stronger than quasi-nonexpansiveness.

The generalized $\alpha-$nonexpansive mappings (which includes $\alpha-$ nonexpansive mappings [1]) of nonexpansive type mappings are introduced by Pant and Shukla [4] in 2017. They have attained some fixed point results for this class of mappings. A mapping $\Psi : K \to K$ is called a generalized $\alpha-$nonexpansive mapping if, for each $u, v \in K$, there exists an $\alpha \in [0, 1)$ such that

$$\frac{1}{2}\|u - \Psi u\| \leq \|u - v\| \Rightarrow \|\Psi u - \Psi v\| \leq \alpha\|\Psi u - v\| + \alpha\|\Psi v - u\| + (1 - 2\alpha)\|u - v\|$$

In recent years, many iterations have been used to approximate fixed points of GNMs. In particular, with iteration development, a faster approach to the fixed point has gained importance. Iterations, such as the Thakur iteration [11] and the Temir-Korkut iteration [15], have recently been introduced and used to approximate the fixed points of GNMs. Usurelu et al. [7] studied the Thakur iteration in the new context of GNMs enriched with condition (E). The underlying setting of their method is a uniformly convex Banach space (UCBS). Thakur iteration: For $\{\zeta_n\}, \{\varsigma_n\}, \{\tau_n\} \in (0, 1)$ and arbitrary $u_1 \in K$ construct a sequence $\{u_n\}$ defined by

$$\begin{cases} z_n &= (1 - \tau_n)u_n + \tau_n \Psi u_n \\ y_n &= (1 - \varsigma_n)z_n + \varsigma_n \Psi z_n \\ u_{n+1} &= (1 - \zeta_n)\Psi z_n + \zeta_n \Psi y_n \end{cases} \tag{1.1}$$

For generalized $\alpha-$nonexpansive mappings in UCBS, Temir and Korkut [15] recently presented an iteration (named the Temir-Korkut iteration) and proved some convergence results using this iteration. Temir-Korkut iteration: For $\{\zeta_n\}, \{\varsigma_n\}, \{\tau_n\} \in (0,1)$ and arbitrary $u_1 \in K$ construct a sequence $\{u_n\}$ defined by

$$\begin{cases} z_n &= \Psi((1-\tau_n)u_n + \tau_n \Psi u_n) \\ y_n &= \Psi((1-\varsigma_n)\Psi u_n + \varsigma_n \Psi z_n) \\ w_n &= \Psi((1-\zeta_n)y_n + \zeta_n \Psi y_n) \\ u_{n+1} &= \Psi w_n \end{cases} \tag{1.2}$$

In this study, we use the Temir-Korkut iteration motivated by the above to prove weak and strong convergence results for Garsia-Falset GNMs, which is the generalization of Suzuki GNMs in the setting of UCBS. Moreover, we provide an example of Garsia-Falset GNM, which is not Suzuki GNM. In addition, we numerically show that the Temir-Korkut iteration converges to a fixed point of Garsia-Falset GNM faster than the Thakur iteration.

2. Preliminaries

This section recalls some basic notations to be used in main results.

Definition 2.1. [16] A Banach space X will be said to be uniformly convex if for each $\varepsilon \in (0,2]$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions $\|u\| = \|v\| = 1$, $\|u-v\| \geq \varepsilon$ imply $\frac{\|u+v\|}{2} \leq 1 - \delta(\varepsilon)$.

Definition 2.2. [17] A Banach space X is said to satisfy *Opial's condition* if, for each sequence $\{u_n\}$ in X, the condition $u_n \to u$ (weakly) as $n \to \infty$ and for all $v \in X$ with $v \neq u$ imply that

$$\liminf_{n\to\infty} \|u_n - u\| < \liminf_{n\to\infty} \|u_n - v\|$$

Definition 2.3. [18] Let $\{u_n\}$ be a bounded sequence in a Banach space X and $u \in X$. Then,

i. the asymptotic radius of $\{u_n\}$ at u is the number $r(u, \{u_n\}) = \limsup_{n\to\infty} \|u_n - u\|$.

ii. the asymptotic radius of $\{u_n\}$ relative to K is defined by $r(K, \{u_n\}) = \inf\{r(u, \{u_n\}) : u \in K\}$.

iii. the asymptotic center of $\{u_n\}$ relative to K is the set $A(K, \{u_n\}) = \{u \in K : r(u, \{u_n\}) = r(K, \{u_n\})\}$.

It is known that $A(K, \{u_n\})$ consists of exactly one-pointin UCBS.

Lemma 2.4. [19] Suppose that X is a UCBS and $0 < k \leq t_n \leq m < 1$ for all $n \in \mathbb{N}$. Let $\{u_n\}$ and $\{v_n\}$ be two sequences of X such that $\limsup_{n\to\infty}\|u_n\| \leq \kappa$, $\limsup_{n\to\infty}\|v_n\| \leq \kappa$ and $\limsup_{n\to\infty}\|t_n u_n + (1-t_n)v_n\| = \kappa$ hold for $\kappa \geq 0$. Then, $\lim_{n\to\infty} \|u_n - v_n\| = 0$.

Definition 2.5. [20] Let $\{u_n\}$ in K be a given sequence. $\Psi : K \to X$ with the nonempty fixed point set $F(\Psi)$ in K is said to satisfy condition (I) with respect to the $\{u_n\}$ if there is a nondecreasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$ and $\varphi(\kappa) > 0$ for all $\kappa \in (0,\infty)$ such that for all $n \in \mathbb{N}$, $\|u_n - \Psi u_n\| \geq \varphi(d(u_n, F(\Psi))$.

Theorem 2.6. [2] Let K be nonempty subset of a Banach space X. Let $\Psi : K \to X$ be a mapping. Then, $u = \Psi u$, if

i. there exists $\{u_n\}$ for Ψ in K such that $\lim_{n\to\infty} \|u_n - \Psi u_n\| = 0$ and $u_n \to u \in K$ (weakly).

ii. Ψ satisfies condition (E) on K,

iii. $(X, \|.\|)$ satisfies the Opial's condition.

3. Convergence of Garsia-Falset GNMs

This section proves weak and strong convergence theorems for (1.2) of Garsia-Falset GNMs in UCBS.

Lemma 3.1. Let K be a nonempty closed convex subset of a uniformly convex Banach space X, $\Psi : K \to K$ be a Garsia-Falset GNM with $F(\Psi) \neq \emptyset$. For arbitrary chosen $u_1 \in K$, let $\{u_n\}$ be a sequence generated by (1.2), then $\lim_{n\to\infty} \|u_n - p\|$ exists for any $p \in F(\Psi)$.

Proof.

Assume that $F(\Psi) \neq \emptyset$. Ψ is a quasi-nonexpansive because $\Psi : K \to K$ is a Garsia-Falset GNM. In order to prove, we follow Lemma 3.1 in [15]. By (1.2) and for any $p \in F(\Psi)$, because of Ψ quasi-nonexpansive mapping, then we have

$$\begin{aligned} \|z_n - p\| &= \|\Psi((1-\tau_n)u_n + \tau_n \Psi u_n) - p\| \\ &\leq \|(1-\tau_n)(u_n - p) + \tau_n(\Psi u_n - p)\| \\ &\leq (1-\tau_n)\|u_n - p\| + \tau_n\|u_n - p\| \\ &= \|u_n - p\| \end{aligned} \tag{3.1}$$

From (1.2) and (3.1), we have

$$\begin{aligned} \|y_n - p\| &= \|\Psi((1-\varsigma_n)\Psi u_n + \varsigma_n \Psi z_n) - p\| \\ &\leq \|(1-\varsigma_n)(\Psi u_n - p) + \varsigma_n(\Psi z_n - p)\| \\ &\leq (1-\varsigma_n)\|u_n - p\| + \varsigma_n\|u_n - p\| \\ &= \|u_n - p\| \end{aligned} \tag{3.2}$$

From (1.2) and (3.2), we get

$$\begin{aligned} \|w_n - p\| &= \|\Psi((1-\zeta_n)y_n + \zeta_n \Psi y_n) - p\| \\ &\leq \|(1-\zeta_n)(y_n - p) + \zeta_n(\Psi y_n - p)\| \\ &\leq (1-\zeta_n)\|u_n - p\| + \zeta_n\|u_n - p\| \\ &= \|u_n - p\| \end{aligned} \tag{3.3}$$

Moreover, from (1.2) and (3.3), we have

$$\|u_{n+1} - p\| = \|\Psi w_n - p\| \leq \|w_n - p\| \leq \|u_n - p\|$$

This implies that $\{\|u_n - p\|\}$ is bounded and non-increasing for all $p \in F(\Psi)$. Hence, $\lim_{n\to\infty} \|u_n - p\|$ exists.

Theorem 3.2. Let K be a nonempty closed convex subset of a UCBS X. For $\mu \geq 1$, let $\Psi : K \to K$ be a Garsia-Falset GNM. For arbitrarily chosen $u_1 \in K$, let $\{u_n\}$ be a sequence in K defined by (1.2) with $\{\zeta_n\}$, $\{\varsigma_n\}$ and $\{\tau_n\}$ real sequences in $(0, 1)$, then $F(\Psi) \neq \emptyset$ *if and only if* $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|u_n - \Psi u_n\| = 0$.

Proof.

Suppose $F(\Psi) \neq \emptyset$ and let $p \in F(\Psi)$. Then, from Lemma 3.1, $\lim_{n\to\infty} \|u_n - p\|$ exits and $\{u_n\}$ is bounded. In this part of the proof, we follow Theorem 3.1 in [15]. Put $\lim_{n\to\infty} \|u_n - p\| = \kappa$. Since a GNM satisfying the condition (E) is quasi-nonexpansive mapping and from (1.2), we have

$$\limsup_{n\to\infty} \|z_n - p\| \leq \limsup_{n\to\infty} \|u_n - p\| = \kappa$$

$$\limsup_{n\to\infty} \|w_n - p\| \leq \limsup_{n\to\infty} \|y_n - p\| \leq \limsup_{n\to\infty} \|u_n - p\| = \kappa$$

and

$$\limsup_{n\to\infty}\|\Psi u_n - p\| \leq \limsup_{n\to\infty}\|u_n - p\| = \kappa$$

Besides, $\|u_{n+1} - p\| \leq \|w_n - p\|$. Therefore, $\kappa \leq \liminf_{n\to\infty}\|w_n - p\|$. Thus, we have $\kappa = \lim_{n\to\infty}\|w_n - p\|$. Using here, we can get $\kappa = \liminf_{n\to\infty}\|w_n - p\| \leq \liminf_{n\to\infty}\|y_n - p\|$. Thus, we have $\kappa = \lim_{n\to\infty}\|y_n - p\|$. Moreover,

$$\begin{aligned} \|y_n - p\| &= \|\Psi((1-\varsigma_n)\Psi u_n + \varsigma_n \Psi z_n) - p\| \\ &\leq \|(1-\varsigma_n)(\Psi u_n - p) + \varsigma_n(\Psi z_n - p)\| \\ &\leq (1-\varsigma_n)\|\Psi u_n - p\| + \varsigma_n\|\Psi z_n - p\| \\ &\leq (1-\varsigma_n)\|u_n - p\| + \varsigma_n\|z_n - p\| \end{aligned}$$

Hence,

$$\|y_n - p\| - \|u_n - p\| \leq \frac{\|y_n - p\| - \|u_n - p\|}{\varsigma_n} \leq \|z_n - p\| - \|u_n - p\|$$

implies that $\|y_n - p\| \leq \|z_n - p\|$. Thus, we have $\kappa = \lim_{n\to\infty}\|z_n - p\|$. Again from (1.2), we have

$$\lim_{n\to\infty}\|(1-\tau_n)(u_n - p) + \tau_n(\Psi u_n - p)\| = \kappa$$

Thus, by Lemma 2.4, we have

$$\lim_{n\to\infty}\|u_n - \Psi u_n\| = 0$$

Conversely, suppose that $\{u_n\}$ is bounded $\lim_{n\to\infty}\|u_n - \Psi u_n\| = 0$. Let $p \in A(K, \{u_n\})$. Since Ψ satisfies condition (E) with $\mu \geq 1$, one has

$$\begin{aligned} r(\Psi p, \{u_n\}) = \limsup_{n\to\infty}\|u_n - \Psi p\| &\leq \limsup_{n\to\infty}(\mu\|\Psi u_n - u_n\| + \|u_n - p\|) \\ &\leq \limsup_{n\to\infty}\|u_n - p\| = r(p, \{u_n\}) \end{aligned}$$

This implies that for $\Psi p \in A(K, \{u_n\})$. For closed-bounded convex subsets of UCBSs, the asymptotic center consists of exactly one point. Therefore, $\Psi p = p$, i.e $F(\Psi) \neq \emptyset$, and the proof is complete.

Next, we prove the following strong convergence theorems of Garsia-Falset GNMs.

Theorem 3.3. Let X be a real UCBS and K a nonempty compact convex subset of X and $\Psi : K \to K$ be a Garsia-Falset GNM for $\mu \geq 1$. Let Ψ and $\{u_n\}$ be as in Theorem 3.2. If $F(\Psi) \neq \emptyset$, then $\{u_n\}$ defined by (1.2) converges strongly to a fixed point of Ψ.

Proof.

Suppose $F(\Psi) \neq \emptyset$. Thus by Theorem 3.2, we have $\lim_{n\to\infty}\|\Psi u_n - u_n\| = 0$. Since K is compact, there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that $u_{n_k} \to p$ as $k \to \infty$ for some $p \in K$. Since Ψ satisfies condition (E) with $\mu \geq 1$, then we have, for all $k \in \mathbb{N}$,

$$\|u_{n_k} - \Psi p\| \leq \mu\|\Psi u_{n_k} - u_{n_k}\| + \|u_{n_k} - p\|$$

Letting $k \to \infty$, we get $u_{n_k} \longrightarrow \Psi p$. Thus $\Psi p = p$, i.e. $p \in F(\Psi)$. Furthermore, $\lim_{n\to\infty}\|u_n - p\|$ exists for every $p \in F(\Psi)$, thus $\{u_n\}$ converges strongly to a fixed point of Ψ.

Moreover, we give below our second strong convergence theorem of Garsia-Falset GNMs satisfying condition (I).

Theorem 3.4. Let K be a nonempty closed convex subset of a UCBS X and $\Psi : K \to K$ be a Garsia-Falset GNM for $\mu \geq 1$. Let Ψ and $\{u_n\}$ be as in Theorem 3.2 and $F(\Psi) \neq \emptyset$. If Ψ satisfies condition (I), then $\{u_n\}$ defined by (1.2) converges strongly to a fixed point of Ψ.

Proof.

Using the same argument as in Theorem 3.5 in [7], we obtain the strong convergence theorem of Garsia-Falset GNMs satisfying condition (I).

Finally, we give the following weak convergence theorem of Garsia-Falset GNMs in a UCBS satisfying Opial's condition.

Theorem 3.5. Let K be a nonempty closed convex subset of a UCBS X endowed with Opial's condition and $\Psi : K \to K$ be a Garsia-Falset GNM for $\mu \geq 1$. Let Ψ and $\{u_n\}$ be as in Theorem 3.2 and $F(\Psi) \neq \emptyset$. Then, $\{u_n\}$ converges weakly to a fixed point of Ψ.

Proof.

Since $F(\Psi) \neq \emptyset$, let p be a fixed point of Ψ. By Theorem 3.2, the sequence $\{u_n\}$ is bounded and $\lim_{n\to\infty} \|\Psi u_n - u_n\| = 0$ and by Lemma 3.1, $\lim_{n\to\infty} \|u_n - p\|$ exists. Because X is uniformly convex, X is reflexive. Due to the reflexiveness of X, there exists a subsequence $\{u_{n_j}\}$ of $\{u_n\}$ such that $\{u_{n_j}\}$ converges weakly to some $\nu_1 \in X$. Since K is closed and convex subset of X, according to Mazur's Theorem, $\nu_1 \in K$. Hence, by Theorem 2.6, we obtain $\Psi\nu_1 = \nu_1$, consequently $\nu_1 \in F(\Psi)$. Arguing by contradiction, suppose that $\{u_n\}$ has two sub-sequences $\{u_{n_j}\}$ and $\{u_{n_k}\}$ converging weakly to ν_1 and ν_2, respectively. $\Psi\nu_2 = \nu_2$ is obtained in the same way which is used $\Psi\nu_1 = \nu_1$. After that, the uniqueness will be proved. By Lemma 3.1, $\lim_{n\to\infty} \|u_n - \nu_1\|$ and $\lim_{n\to\infty} \|u_n - \nu_2\|$ exist. Suppose that $\nu_1 \neq \nu_2$, afterward by the Opial's condition, we obtain

$$\begin{aligned} \lim_{n\to\infty} \|u_n - \nu_1\| &= \lim_{j\to\infty} \|u_{n_j} - \nu_1\| < \lim_{j\to\infty} \|u_{n_j} - \nu_2\| = \lim_{n\to\infty} \|u_n - \nu_2\| \\ &= \lim_{k\to\infty} \|u_{n_k} - \nu_2\| < \lim_{k\to\infty} \|u_{n_k} - \nu_1\| = \lim_{n\to\infty} \|u_n - \nu_1\| \end{aligned}$$

which is a contradiction. Hence, $\nu_1 = \nu_2$. Therefore, $\{u_n\}$ converges weakly to a fixed point of Ψ. This completes the proof.

4. Illustrative Example

This section exemplifies Garsia-Falset generalized nonexpansive mappings, which exceed the class of Suzuki generalized nonexpansive mappings.

Example 4.1. Let $K = [0,1] \subset \mathbb{R}$ endowed with usual norm in $\mathbb{R}$ and $\Psi : [0,1] \to [0,1]$ be a mapping defined by

$$\Psi u = \begin{cases} \frac{\arctan u}{2}, & u \neq 1 \\ \frac{5}{7}, & u = 1 \end{cases}$$

Here, Ψ is a Garsia-Falset GNM. We prove next that Ψ satisfies condition $(E_{\frac{7}{2}})$. For Ψ, satisfying condition $(E_{\frac{7}{2}})$ explicitly means to check if the following inequality holds:

$$\|u - \Psi v\| \leq \frac{7}{2}\|u - \Psi u\| + \|u - v\|, \text{ for all } u, v \in [0,1] \tag{4.1}$$

To verify that Ψ satisfies the condition $(E_{\frac{7}{2}})$, we consider the following cases.

Case I: Let $u = 1$ and $v \in [0,1)$. Then, (4.1) is written as follows for this particular case

$$\begin{aligned} \left|1 - \frac{\arctan v}{2}\right| &\leq \frac{7}{2}\left|1 - \frac{5}{7}\right| + |1 - v| \\ 1 - \frac{\arctan v}{2} &\leq 1 + 1 - v \end{aligned}$$

Then, we have

$$v - \frac{\arctan v}{2} \quad \leq \quad 1$$

which is ultimately equivalent to condition $f(v) = v - \frac{\arctan v}{2} \leq 1$. Since $f(v) = v - \frac{\arctan v}{2}$ is nondecreasing over the interval $[0, 1)$, then $Im(f) = [f(0), f(1)) = \big[f(0), 1 - \frac{\pi}{8}\big)$; therefore, this inequality is satisfied.

Case II: Let $u \in [0, 1)$ and $v = 1$. Then, (4.1) is written as follows for these particular assignments

$$\begin{aligned} \left|u - \frac{5}{7}\right| &\leq \frac{7}{2}\left|u - \frac{\arctan u}{2}\right| + |u - 1| \\ &\leq \frac{7u}{2} - \frac{7\arctan u}{4} + 1 - u \\ &\leq \frac{5u}{2} - \frac{7\arctan u}{4} + 1 \end{aligned}$$

Taking this time the function $g(u) = \frac{5u}{2} - \frac{7arctanu}{4} + 1$. Since the function $g(u) = \frac{5u}{2} - \frac{7arctanu}{4} + 1$ also is nondecreasing over the interval $[0, 1)$, then $Im(g) = [g(0), g(1)) = \Big[1, \frac{7}{2} - \frac{7\pi}{16}\Big)$. Hence, $g(u) \geq 1 > \left|u - \frac{5}{7}\right|$, for all $u \in [0, 1)$. Therefore, this inequality is satisfied.

Case III: Let $u, v \in [0, 1)$. One has

$$\|u - \Psi v\| \leq \|u - \Psi u\| + \|\Psi u - \Psi v\|$$

Substituting, we attain

$$\left|u - \frac{\arctan v}{2}\right| \leq \frac{7}{2}\left|u - \frac{\arctan u}{2}\right| + \frac{1}{2}\left|\arctan u - \arctan v\right|$$

From the mean value theorem, we have

$$\begin{aligned} \left|u - \frac{\arctan v}{2}\right| &\leq \frac{7}{2}\left|u - \frac{\arctan u}{2}\right| + \frac{1}{2}|u - v| \\ &\leq \frac{7}{2}\left|u - \frac{\arctan u}{2}\right| + |u - v| \end{aligned}$$

which is precisely (4.1), for $u, v \in [0, 1)$. Thus, this inequality is satisfied.

Case IV: Let $u = 1, v = 1$. Then, (4.1) which needs to be fulfilled is

$$\begin{aligned} \left|1 - \frac{5}{7}\right| &\leq \frac{7}{2}\left|1 - \frac{5}{7}\right| + |1 - 1| \\ \frac{2}{7} &\leq 1 \end{aligned}$$

and it is obviously satisfied.

Finally, if $u = 1, v = 0.8$ is taken, then

$$\frac{1}{2}\|u - \Psi u\| = \frac{1}{2}\left|1 - \frac{5}{7}\right| = \frac{1}{7} = 0.1428 < 0.2 = \|u - v\|$$

and

$$\|\Psi u - \Psi v\| = \left|\frac{5}{7} - \frac{\arctan(0.8)}{2}\right| = 0.37693 > 0.2 = \|u - v\|$$

Hence, Ψ does not satisfy condition (C).

Next, since an example of the Garsia-Falset GNM is provided, we will give it to a numerical reckoning for the iteration stated in the introduction. The observations are given in Table 1 and Figure 1.

4.1. Numerical Results

The convergence behaviors of (1.2) and (1.1) implementing Example 4.1 is compared. Let $\{\zeta_n\} = \{\varsigma_n\} = \{\tau_n\} = 0.75$, for all $n \geq 1$. 0 is the fixed point of the mapping defined in Example 4.1.

Table 1. Sequences generated by (1.2) and (1.1) for mapping Ψ of Example 4.1.

	Temir-Korkut iteration	Thakur iteration
u_1	0.80000000	0.80000000
u_2	0.01258820	0.15303068
u_3	0.00023817	0.03411124
u_4	0.00000451	0.00765877
u_5	0.00000009	0.00172020
u_6	0	0.00038637
u_7	0	0.00008678
u_8	0	0.00001949
u_9	0	0.00000438
u_{10}	0	0.00000098
u_{11}	0	0.00000022
u_{12}	0	0.00000005
u_{13}	0	0.00000001
u_{14}	0	0

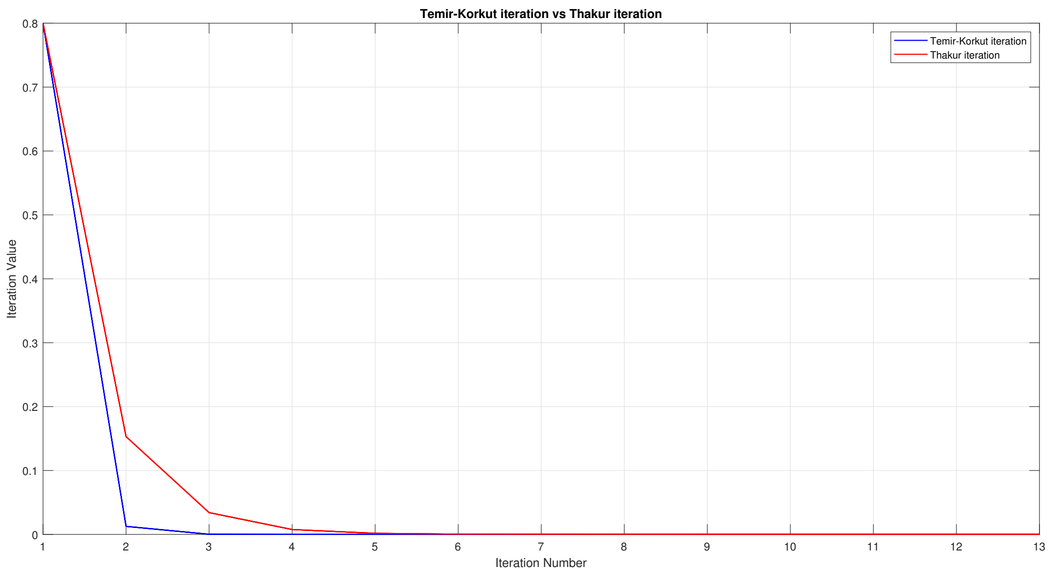

Figure 1. Convergences of the Temir-Korkut iteration and the Thakur iteration to the fixed point 0 of the mapping defined in Example 4.1.

5. Conclusion

This paper has studied the convergence of (1.2) to fixed points for the Garsia-Falset GNMs in UCBS. An illustrative numerical example has been presented, one of the crucial of this paper. Example 4.1 satisfies condition (E). However, this example does not satisfy condition (C). This is intended to show numerically that the Garcia-Falset GNMs class is actually wider than the Suzuki GNMs class.

Moreover, Table 1 and Figure 1, obtained from Example 4.1, can be observed that the Temir-Korkut iteration converges faster than the Thakur iteration. New iterations that converge faster than those presented herein can be developed in future studies. Additionally, more complex examples can be studied to compare the results obtained in this paper.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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