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DYNAMICAL BEHAVIOURS OF A DISCRETIZED MODEL WITH MICHAELIS-MENTEN HARVESTING RATE

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ABSTRACT. In this paper, we introduced nonstandard finite difference scheme (NSFD) for solving the continuos model with Michaelis-Menten harvesting rate. We have seen that the proposed scheme preserves local stability and positivity. Stability analysis of each fixed point of the discrete time model has been proven. Also, numerical comparisons were made between the nonstandard finite difference method and the other methods.

1. INTRODUCTION

Mathematical modelling is the most effective method to find solutions of the real world problems. Increasing number of studies on mathematical models in ecological models are very important issue to explain the dynamics of these [1,2]. Predator-prey interactions can be seen as the building blocks for ecological structures [3]. Predator-prey models have been studied since 1925 when mathematical model was propounded independently by Lotka and Volterra [4,5]. The Lotka-Volterra model is used in ecology and population dynamics of animal species. These models consider only four factors such as prey population size, predator population size, death rate of predator and conversion rate. These four rates are also linear. Actually, predator-prey interactions do not depend only these four factors. In general, some researchers have worked on various methods to solve nonlinear systems. El-Dib et. al. suggested a different scheme to modify the homotopy perturbation method with three expanded expansions [6]. They also finded attractive results for the accuracy of the method. In the model presented by Clark [7], the predator-prey harvesting depends on the predator density that grows logistically. Furthermore, the predator density is expressed with Holling Type I model with functional response [8]. Chaudhuri, Kar discussed a same situation but they used Holling Type II model with functional response [9]. Holling Type models with functional response have been introduced by Holling [10,11]. The functional response on these models depend on only the prey density. However, according to the Arditi and Ginzburg, the functional response may be depend on both prey density and predator density [12].

One of the functional response model is the ratio dependent model. The standard model that depend on prey shows “paradox of enrichment” and it is also called “biological control paradox”.

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Whereas, the ratio dependent models do not produce neither the enrichment paradox nor the biological control paradox [13, 14]. There are most interesting dynamics near the (0,0) point for this type models [15]. Xiao and Ruan [15] and Berezovskaya et. al. [16] observed that there are different types of topological structures around the root of the ratio dependent models. Also, Jost et. al [17] has denoted that the equilibrium point may either be a saddle point or an attractor for ratio dependent models. Though ratio dependent hypotheses cause hot discussions, a recent study by Jost and Arditi has showed that ratio dependent and prey dependent models are in good agreement with each other's time series [17]. In addition, Hsu [18] defend an idea that ratio dependent models are more sophisticated and flexible. Biologically, simulations and local stability analysis show that ratio dependent models produce richer and more suitable dynamics [19].

The aim of this article is to analyze the stability of equilibrium points in a ratio dependent system where predator density is subjected to harvesting with Michaelis-Menten type harvesting rate. Harvesting in the model can be two fold. The main purpose is the exploitation of the harvested reserve to increase the profit [20]. However, some researchers assumed harvesting from ecological perspective [21]. Michaelis-Menten type functional form of the catch rate $h(t)$ is given as:

$$h(t) = \frac{qE_y}{bE + ly},$$

where q is the catchability coefficient, E is the external effort dedicated to harvesting and b, l are positive constants [20].

We are considered the following ratio dependent system where predator density is subjected to harvesting with Michaelis-Menten type harvesting rate in [20]:

$$\begin{aligned} \frac{dx}{dt} &= \left(1 - \frac{x}{K}\right) rx - \frac{\alpha xy}{ay + x}, \\ \frac{dy}{dt} &= \frac{\alpha b_0 xy}{ay + x} - d_0 y - \frac{qE_y}{bE + ly}, \end{aligned} \tag{1}$$

where $x(t)$: prey population size (time dependent), $y(t)$: predator population size (time dependent), α: the maximum prey consumption rate, r: the internal growth rate of the prey, d_0: the death rate, a: the half-saturation constant, b_0: the conversion efficiency of the predator.

The parameters in equation (1) are supposed to be positive. Taking $x = Kx',\ y = \frac{Ky'}{a}, t = \frac{at'}{\alpha}$ in equation (1) we can rewrite the following system:

$$\frac{dx'}{dt'} = \alpha_0 x'(1 - x') - \frac{x'y'}{x' + y'}, \tag{2}$$

$$\frac{dy'}{dt'} = \frac{\beta_0 x' y'}{x' + y'} - \gamma_0 y' - \frac{E_0 y'}{E' + y'},$$

where $\alpha_0 = \frac{ar}{\alpha},\ \beta_0 = ab_0,\ E_0 = \frac{a^2 qE}{\alpha Kl},\ \gamma_0 = \frac{ad_0}{\alpha},\ E' = \frac{abE}{Kl}$.

We rewrite equation (2) by changing t', x', y' by t, x, y respectively:

$$\frac{dx}{dt} = \alpha_0 x(1 - x) - \frac{xy}{x + y},$$

$$\frac{dy}{dt} = \frac{\beta_0 xy}{x + y} - \gamma_0 y - \frac{E_0 y}{E' + y}. \tag{3}$$

For convenience, we change the independent variable t to $(x + y)t''$ and replacing t'' by t , equation (3) becomes:

$$\frac{dx}{dt} = \alpha_0 x(1 - x)(x + y) - xy,$$

$$\frac{dy}{dt} = \beta_0 xy - \gamma_0 y(x + y) - \frac{E_0 y(x + y)}{E' + y}. \tag{4}$$

In this work, the nonstandard finite difference scheme (NSFD) that defined by Mickens [22] has been developed for a ratio dependent model and stability analysis of the discretized system has been investigated using [23-25]. By using NSFD methods, it is aimed to find positive discrete solutions. However, numerical methods like Euler, Adams and Runge-Kutta can be used to examine the population dynamics. The most disadvantage of these methods are that their stability depends on the time step size. Whereas, nonstandard finite difference scheme preserves local stability of the equilibrium with arbitrary time step sizes [26]. By using nonstandard finite difference scheme, the arbitrary step size selection simplifys the solution of the problem [27]. In other advantage of these scheme is to protect the main features of the continuous model and therefore it gives dependable numerical results [3]. Bairagi et. al. [28] used standard Euler method and nonstandard finite difference method to compare their dynamic properties. Shokri et. al. [29] presented two NSFD methods based on Micken's rules to solve mathematical model of the Rosenzweig-MacArthur model. It can be seen that the new proposed method has special feature such as stability and positivity. Also, it is worth mentioning that nonstandard finite difference method has many computational advantages over conventional methods [30].

The article is arranged as follows: Part 2 deals with the existence and local stability analysis of equilibrium points. Numerical simulations are shown in part 3. The work ends with the summary of conclusion in part 4.

2. STABILITY AND EXISTENCE ANALYSIS OF THE EQUILIBRIUM POINTS

Definition 2.1. Nonstandard finite difference method is based on two basis. These are analyzed as follows: (i)

$$\frac{dz}{dt}=\frac{z_{k+1}-z_k}{\varphi(h)}$$

where $\varphi(h)=h+O(h^2)$.

(ii) both nonlinear and linear terms may need a nonlocal representation on the discrete computational case [25,31].

Constructing a numerical perspective for equation (4), we discretize the time variable at $t_n=nh$, t $(t\geq 0)$ where h $(h>0)$ is the time step size.

$x(t)$ and $y(t)$ at points t_n are shown with x_n and y_n. To present stability analysis of the discrete time model, the continuous nonlinear differential equation (4) will be discretizated respectively as follows:

$$x\to x_k,\quad y\to y_k,\quad x^2\to x_k x_{k+1}\,,\quad xy\to x_{k+1}y_k\,,$$

and

$$yx\to y_{k+1}x_k,\quad xy\to x_k y_k,\quad y^2\to y_{k+1}y_k.$$

Thus, positive solutions of the discretized model can be obtained. By using nonstandard finite difference scheme for equation (4), we can write the following discrete model with ratio dependent functional response:

$$\frac{x_{k+1}-x_k}{\varphi(h)}=-x_{k+1}y_k+\alpha_0(x_k+y_k)(1-x_{k+1})x_k\,,$$

$$\frac{y_{k+1}-y_k}{\varphi(h)}=\beta_0 x_k y_k-(y_k+x_k)\gamma_0 y_{k+1}-\frac{E_0 y_{k+1}(y_k+x_k)}{E'+y_k}, \tag{5}$$

where $\varphi(h)$ depends on the step size $\Delta t=h$ and it is called denominator function. It can be seen that how to find arbitrary choice of denominator functions in [24,25,26,27,31]. Let us indicate $h_1=\varphi(h)$.

If x_{k+1} and y_{k+1} are solved from equation (5), following iterations can be obtained:

$$x_{k+1}=\frac{(1+h_1\alpha_0 x_k+h_1\alpha_0 y_k)x_k}{[1+h_1\alpha_0 x_k(y_k+x_k)+h_1 y_k]},$$

$$y_{k+1}=\frac{(1+h_1\beta_0 x_k)y_k(E'+y_k)}{\left[\left(1+h_1\gamma_0(y_k+x_k)\right)(E'+y_k)+h_1E_0(y_k+x_k)\right]}. \tag{6}$$

It is easy to find that equation (6) has two boundary fixed points: $E_1^*=(0,0)$, $E_2^*=(1,0)$. Besides those boundary fixed points, equation (6) has also positive equilibrium $E_3^*=(x^*,y^*)$.

We take x_{k+1} is equivalent to x_k and then:

$$x_{k+1}=\frac{(1+h_1\alpha_0 x_k+h_1\alpha_0 y_k)x_k}{[1+h_1\alpha_0 x_k(y_k+x_k)+h_1 y_k]}=x_k\,,$$

$$\alpha_0 x_k+\alpha_0 y_k=\alpha_0 x_k x_k+\alpha_0 x_k y_k+y_k\,. \tag{7}$$

Similarly we take y_{k+1} is equivalent to y_k and then:

$$
y_{k+1} = \frac{(1 + h_1\beta_0 x_k)y_k(E' + y_k)}{\left[\left(1 + h_1\gamma_0(y_k + x_k)\right)(E' + y_k) + h_1 E_0(y_k + x_k)\right]},
$$

$$
\beta_0 x_k(E' + y_k) = (E' + y_k)\gamma_0(y_k + x_k) + E_0(y_k + x_k)\,. \tag{8}
$$

So fixed point (x^*, y^*) is satisfying:

$$
\alpha_0 x^* + \alpha_0 y^* = \alpha_0 x^* x^* + \alpha_0 x^* y^* + y^*,
$$

and

$$
\beta_0 x^*(E' + y^*) = (E' + y^*)\gamma_0(y^* + x^*) + E_0(y^* + x^*)\,.
$$

If we want to organize the local stability conditions, we can write the Jacobian matrix of equation (6) as the form:

$$
J(x, y) = \begin{pmatrix} \dfrac{\partial x_{k+1}}{\partial x_k} & \dfrac{\partial x_{k+1}}{\partial y_k} \\ \dfrac{\partial y_{k+1}}{\partial x_k} & \dfrac{\partial y_{k+1}}{\partial y_k} \end{pmatrix}
$$

where

$$
J_{11}(x_k, y_k) = \frac{[(1 + h_1\alpha_0 x_k + h_1\alpha_0 y_k) + h_1\alpha_0 x_k][1 + (x_k + y_k)h_1\alpha_0 x_k + h_1 y_k]}{[1 + (x_k + y_k)h_1\alpha_0 x_k + h_1 y_k]^2} - \frac{[2h_1\alpha_0 x_k + h_1\alpha_0 y_k]x_k(1 + h_1\alpha_0 x_k + h_1\alpha_0 y_k)}{[1 + (x_k + y_k)h_1\alpha_0 x_k + h_1 y_k]^2},
$$

$$
J_{12}(x_k, y_k) = \frac{[h_1\alpha_0 x_k][1+(x_k+y_k)h_1\alpha_0 x_k+h_1 y_k]-[h_1\alpha_0 x_k+h_1]x_k(1+h_1\alpha_0 x_k+h_1\alpha_0 y_k)}{[1+(x_k+y_k)h_1\alpha_0 x_k+h_1 y_k]^2},
$$

$$
J_{21}(x_k, y_k) = \frac{[h_1\beta_0 y_k(E' + y_k)]\left[\left(1 + h_1\gamma_0(y_k + x_k)\right)(E' + y_k) + h_1 E_0(y_k + x_k)\right]}{\left[\left(1 + h_1\gamma_0(y_k + x_k)\right)(E' + y_k) + h_1 E_0(y_k + x_k)\right]^2} - \frac{[h_1\gamma_0(E' + y_k) + h_1 E_0](1 + h_1\beta_0 x_k)y_k(E' + y_k)}{\left[\left(1 + h_1\gamma_0(y_k + x_k)\right)(E' + y_k) + h_1 E_0(y_k + x_k)\right]^2},
$$

and

$$
J_{22}(x_k, y_k) = \frac{[(1+h_1\beta_0 x_k)(E'+y_k)+y_k(1+h_1\beta_0 x_k)]}{\left[\left(1+h_1\gamma_0(y_k+x_k)\right)(E'+y_k)+h_1 E_0(y_k+x_k)\right]^2} \times \frac{\left[\left(1+h_1\gamma_0(y_k+x_k)\right)(E'+y_k)+h_1 E_0(y_k+x_k)\right]}{\left[\left(1+h_1\gamma_0(y_k+x_k)\right)(E'+y_k)+h_1 E_0(y_k+x_k)\right]^2} - \frac{\left[h_1\gamma_0(E'+y_k)+\left(1+h_1\gamma_0(y_k+x_k)\right)+h_1 E_0\right](1+h_1\beta_0 x_k)y_k(E'+y_k)}{\left[\left(1+h_1\gamma_0(y_k+x_k)\right)(E'+y_k)+h_1 E_0(y_k+x_k)\right]^2}.
$$

Lemma 2.1. We suppose that λ_1 and λ_2 are two roots of the Jacobian. Thus the following descriptions are given:

i) The fixed point of $J(x^*, y^*)$ is called stable (sink) fixed point, if $|\lambda_1| < 1$ and $|\lambda_2| < 1$.

ii) The fixed point of $J(x^*, y^*)$ is called unstable (source) fixed point, if $|\lambda_1| > 1$ and $|\lambda_2| > 1$.

iii) The fixed point of $J(x^*, y^*)$ is called unstable (saddle) fixed point, if $|\lambda_1| < 1$ and $|\lambda_2| > 1$ (or $|\lambda_1| > 1$ and $|\lambda_2| < 1$).

iv) The fixed point of $J(x^*, y^*)$ is called non-hyperbolic fixed point, if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ [32].

Theorem 2.1. $E_1^* = (0,0)$ is a nonhyperbolic fixed point.

Proof. The eigenvalues of Jacobian are $|\lambda_1| = 1$ and $|\lambda_2| = 1$ and then $E_1^* = (0,0)$ is called nonhyperbolic fixed point.

Now we search the other fixed points.

Using that $\alpha_0 x_k + \alpha_0 y_k = \alpha_0 x_k x_k + \alpha_0 x_k y_k + y_k$ and the fact that $y_k = 0$, we can obtain x_k as follows:

$$\alpha_0 x_k = \alpha_0 x_k x_k$$
$$x_k = 1$$

Theorem 2.2. Let $\alpha_0 > 0,\ \gamma_0 > 0,\ \beta_0 > 0,\ E_0 > 0,\ E' > 0$ and $h_1 > 0$.

i) $E_2^*(1,0)$ is a sink if $\beta_0 - \gamma_0 < \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$ and locally asymptotically stable.

ii) $E_2^*(1,0)$ is not a source.

iii) $E_2^*(1,0)$ is a saddle if $\beta_0 - \gamma_0 > \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$ hence unstable.

iv) $E_2^*(1,0)$ is non-hyperbolic if $\beta_0 - \gamma_0 = \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$.

Proof. The Jacobian of $E_2^*(1,0)$ is equal to $\begin{pmatrix} \frac{1}{1+h_1\alpha_0} & -\frac{h_1}{1+h_1\alpha_0} \\ 0 & \frac{(1+h_1\beta_0)E'}{((1+h_1\gamma_0)E'+h_1E_0)} \end{pmatrix}$.

The eigenvalues of Jacobian are $\lambda_1 = \frac{1}{1+h_1\alpha_0}$, $\lambda_2 = \frac{(1+h_1\beta_0)E'}{((1+h_1\gamma_0)E'+h_1E_0)}$.

i) $|\lambda_1| < 1$ since $h_1\alpha_0 > 0$ and $1 < 1 + h_1\alpha_0$ for $h_1 > 0,\ \alpha_0 > 0$.

$|\lambda_2| < 1$ since $\beta_0 - \gamma_0 < \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$.

ii) Since $|\lambda_1|$ is always less than 1 for $h_1 > 0$, $\alpha_0 > 0$, the second condition of Lemma 2.1 is not provided. Thus $E_2^*(1,0)$ is not a source.

iii) From the third condition of Lemma 1, $|\lambda_1| < 1$ and $\beta_0 - \gamma_0 > \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$, $E_2^*(1,0)$ is a saddle and hence unstable.

iv) From the fourth condition of Lemma 1, except in case of $\beta_0 - \gamma_0 = \frac{E_0}{E'}$ for $\beta_0 > \gamma_0$, one of the eigenvalues of $J(E_2^*(1,0))$ can not be -1 and 1.

Lemma 2.2. For $\lambda^2 - \operatorname{tr}\left(J(x^*, y^*)\right)\lambda + \det\left(J(x^*, y^*)\right) = 0$ and both roots satisfy $|\lambda_i| < 1,\ i = 1,2$ if the following conditions are satisfied [15].

1. $1 + \det\left(J(x^*, y^*)\right) + \operatorname{tr}\left(J(x^*, y^*)\right) > 0$,
2. $1 + \det\left(J(x^*, y^*)\right) - \operatorname{tr}\left(J(x^*, y^*)\right) > 0$,
3. $\det\left(J(x^*, y^*)\right) < 1$.

Theorem 2.3. $E_2^* = (1,0)$ is locally asymptotically stable if all situations of Lemma 2.2 hold.

Proof. Eigenvalues of $J\left(E_2^*(1,0)\right)$ are roots of

$$\lambda^2 + \det\left(J(E_2^*)\right)\lambda - \operatorname{tr}\left(J(E_2^*)\right)\lambda = 0$$

where

$$\operatorname{tr}(J(E_2^*)) = \frac{2E' + h_1\gamma_0E' + h_1E_0 + h_1\beta_0E' + h_1\alpha_0E' + h_1^2\alpha_0\beta_0E'}{(1+h_1\alpha_0)((1+h_1\gamma_0)E' + h_1E_0)}$$

and

$$\det(J(E_2^*)) = \frac{E' + h_1\beta_0E'}{(1+h_1\alpha_0)((1+h_1\gamma_0)E' + h_1E_0)}$$

$\operatorname{tr}\left(J(E_2^*)\right)(0) = 2$ and $\det\left(J(E_2^*)\right)(0) = 1$ which refers that there exist constants $\tilde{A}_{(E_2^*)} > 0$ such that $1 + \det\operatorname{tr}\left(J(E_2^*)\right)(h_1) + \operatorname{tr}\left(J(E_2^*)\right)(h_1) > 0$ for all $0 < h_1 < \tilde{A}_{(E_2^*)}$. After some calculations as below:

$$1 + \det\operatorname{tr}\left(J(E_2^*)\right)(h_1) + \operatorname{tr}\left(J(E_2^*)\right)(h_1) > 0 \leftrightarrow Ph_1^2 + Rh_1 + 4E' > 0$$

where $P = \alpha_0\gamma_0E' + \alpha_0E_0 + \alpha_0\beta_0E'$ and $R = 2E_0 + 2\gamma_0E' + 2\alpha_0E' + 2\beta_0E'$.
Therefore, $\tilde{A}_{(E_2^*)}$ can be selected as follows:

$$\tilde{A}_{(E_2^*)} = \left\{\begin{array}{cc} \dfrac{2\sqrt{|E'|}}{\sqrt{|P|}} & , \quad R = 0 \\ \dfrac{4E'}{|R|} & , \quad P = 0 \\ \min\left(\dfrac{|R|}{|P|}, \dfrac{2\sqrt{|E'|}}{|R|}\right) & , otherwise \end{array}\right\}$$

Suppose that E_2^* is a stable fixed point of system (5). Then,

$$1 + \det(J(E_2^*)) - \operatorname{tr}\left(J(E_2^*)\right) = \frac{h_1^2}{(1+h_1\alpha_0)\left((1+h_1\gamma_0)E' + h_1E_0\right)}(\alpha_0\gamma_0E' + \alpha_0E_0 + \alpha_0\beta_0E') > 0$$

and situation (2) of Lemma 2 holds. The last situation is equal to:

$$\det(J(E_2^*)) - 1 = \frac{h_1\beta_0E' - h_1\gamma_0E' - h_1E_0 - h_1\alpha_0E' - h_1^2\alpha_0\gamma_0E' - h_1^2\alpha_0E_0}{(1+h_1\alpha_0)((1+h_1\gamma_0)E' + h_1E_0)} < 0$$

$$h_1^2\left(\frac{-\alpha_0\gamma_0E' - \alpha_0E_0}{(1+h_1\alpha_0)((1+h_1\gamma_0)E' + h_1E_0)}\right) + h_1\left(\frac{\beta_0E' - \gamma_0E' - E_0 - \alpha_0E'}{(1+h_1\alpha_0)((1+h_1\gamma_0)E' + h_1E_0)}\right) < 0. \qquad (9)$$

The inequality (9) is true when $h_1 < A_{(E_2^*)}$

where $A_{(E_2^*)} = \frac{|-\alpha_0\gamma_0E' - \alpha_0E_0|}{|\beta_0E' - \gamma_0E' - E_0 - \alpha_0E'|}$.
Therefore, if $h_1 < \min(A_{(E_2^*)}, \tilde{A}_{(E_2^*)})$ conditions (1), (2) and (3) of Lemma 2.2 hold and E_2^* is a locally asymptotically stable.

Theorem 2.4. The fixed point $E_3^*(x^*, y^*)$ is locally asymptotically stable if whole situations of Lemma hold.

Proof. Using the equations $\alpha_0 x^* + \alpha_0 y^* = \alpha_0 x^* x^* + \alpha_0 x^* y^* + y^*$ and $\beta_0 x^*(E' + y^*) = (E' + y^*)\gamma_0(y^* + x^*) + E_0(y^* + x^*)$, the Jacobian $J(x^*, y^*) = J_{ij}(x^*, y^*)_{2\times 2}$ can be obtained as :

$$J_{11}(x^*, y^*) = \frac{1 + 2h_1\alpha_0 x^* + h_1\alpha_0 y^* - 2h_1\alpha_0 x^{*2} - h_1\alpha_0 x^* y^*}{1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*},$$

$$J_{12}(x^*, y^*) = \frac{h_1\alpha_0 x^* - h_1\alpha_0 x^{*2} - h_1 x^*}{1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*},$$

$$J_{21}(x^*, y^*) = \frac{h_1\beta_0 y^*(E' + y^*) - h_1\gamma_0 y^*(E' + y^*) - h_1 E_0 y^*}{[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]},$$

and

$$J_{22}(x^*, y^*) = \frac{E' + y^* + h_1\beta_0 x^*(E' + 2y^*) - h_1\gamma_0 y^*(E' + 2y^* + x^*) - h_1 E_0 y^*}{[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}.$$

Eigenvalues of $J(x^*, y^*)$ are roots of the equation in Lemma :

$$\begin{aligned} tr\big(J(E_3^*)\big) = {} & \frac{2E' + 2y^* + 2h_1\beta_0 x^* E' + 3h_1\beta_0 x^* y^* + 3h_1\alpha_0 x^* E' + 3h_1\alpha_0 x^* y^* + 3h_1^2\alpha_0\beta_0 x^{*2} E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & + \frac{4h_1^2\alpha_0\beta_0 x^{*2} y^* + 2h_1^2\alpha_0\beta_0 x^* y^* E' + 3h_1^2\alpha_0\beta_0 x^* y^{*2} + 2h_1\beta_0 y^* E' + 2h_1\alpha_0 y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & - \frac{2h_1\alpha_0 x^{*2} E' + 2h_1\alpha_0 x^{*2} y^* + 2h_1^2\alpha_0\beta_0 x^{*3} E' + 2h_1^2\alpha_0\beta_0 x^{*3} y^* + h_1\alpha_0 x^* y^* E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & - \frac{h_1\alpha_0 x^* y^{*2} + {h_1}^2\alpha_0\beta_0 x^{*2} y^* E' + h_1\alpha_0\beta_0 x^{*2} y^{*2} + h_1\gamma_0 y^* E' + 2h_1\gamma_0 y^{*2} + h_1\gamma_0 x^* y^*}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & - \frac{h_1 E_0 y^* + h_1^2\alpha_0\gamma_0 x^* y^* E' + 2h_1^2\alpha_0\gamma_0 x^* y^{*2} + {h_1}^2\alpha_0\gamma_0 x^{*2} y^* + {h_1}^2\alpha_0 E_0 x^* y^* + h_1^2\alpha_0\gamma_0 y^{*2} E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & - \frac{2h_1^2\alpha_0\gamma_0 y^{*3} + h_1^2\alpha_0\gamma_0 y^{*2} + {h_1}^2\alpha_0 E_0 y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \end{aligned}$$

and

$$\begin{aligned} \det\big(J(E_3^*)\big) = {} & \frac{E' + y^* + h_1\beta_0 x^* E' + 2h_1\beta_0 x^* y^* - h_1\gamma_0 y^* E' - 2h_1\gamma_0 y^{*2} - h_1^2\gamma_0 x^* y^* - h_1^2\gamma_0 x^* y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & + \frac{2h_1\alpha_0 x^* E' - h_1 E_0 y^* + 2h_1\alpha_0 x^* y^* + 2h_1^2\alpha_0\beta_0 x^{*2} E' + 4h_1^2\alpha_0\beta_0 x^{*2} y^* - h_1^2 x^* E_0 y^*}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & + \frac{h_1\alpha_0 y^* E' - h_1^2\alpha_0\gamma_0 x^* y^* E' - 4h_1^2\alpha_0\gamma_0 x^* y^{*2} - 2h_1^2\alpha_0\gamma_0 x^{*2} y^* - h_1^2\alpha_0 E_0 x^* y^* - h_1^2\alpha_0 x^* y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & + \frac{h_1\alpha_0 y^{*2} + {h_1}^2\alpha_0\beta_0 x^* y^{*2} - h_1\alpha_0\gamma_0 y^{*2} E' - 2h_1\alpha_0\gamma_0 y^{*3} - {h_1}^2\alpha_0 E_0 y^{*2} - 2h_1\alpha_0 x^{*2} E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \\ & + \frac{3h_1^2\alpha_0\gamma_0 x^{*2} y^{*2} - 2h_1^2\alpha_0 x^{*2} y^* - 2{h_1}^2\alpha_0\beta_0 x^{*3} E' - 4{h_1}^2\alpha_0\beta_0 x^{*3} y^* + h_1^2\alpha_0\gamma_0 x^{*2} y^* E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]} \end{aligned}$$

$$+\frac{h_1^2\alpha_0 E_0 x^{*2}y^* + 2{h_1}^2\alpha_0\gamma_0 x^{*3}y^* - h_1\alpha_0 x^* y^* E' - h_1^2\alpha_0\beta_0 x^{*2}y^{*2} + h_1^2\alpha_0\gamma_0 y^{*2}x^*E'}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$
$$+\frac{2h_1^2\alpha_0\gamma_0 x^* y^{*3} + h_1^2\alpha_0\gamma_0 x^{*2}y^{*2} + {h_1}^2\alpha_0 E_0 x^* y^{*2} + {h_1}^2\beta_0 x^* y^* E' + {h_1}^2\beta_0 x^* y^{*2} - h_1\alpha_0 x^* y^* E'}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$

$\mathrm{tr}(J(E_3^*))(0) = 2$ and $\det(J(E_3^*))(0) = 1$ which refers that there exist constants $\widetilde{K}_{(E_3^*)} > 0$ such that $1 + \mathrm{tr}(J(E_3^*))(h_1) + \det(J(E_3^*))(h_1) > 0$ for all $0 < h_1 < \widetilde{K}_{(E_3^*)}$.
We require to find the constants $\widetilde{K}_{(E_3^*)}$.

$$1 + \mathrm{tr}(J(E_3^*))(h_1) + \det(J(E_3^*))(h_1) > 0 \leftrightarrow Bh_1^2 + Ch_1 + 4E' + 4y^* > 0$$

where

$$\begin{aligned}B =\ & 6\alpha_0\beta_0 x^{*2}E' + 9\alpha_0\beta_0 x^{*2}y^* + 3\alpha_0\beta_0 x^* y^* E' + 5\alpha_0\beta_0 x^* y^{*2} - 4\alpha_0\beta_0 x^{*3}E' - 6\alpha_0\beta_0 x^{*3}y^* \\ & -\alpha_0\beta_0 x^{*2}y^*E' - 2\alpha_0\gamma_0 x^* y^* E' - 6\alpha_0\gamma_0 x^* y^{*2} - 3\alpha_0\gamma_0 x^{*2}y^* - 2\alpha_0 E_0 x^* y^* - 2\alpha_0\gamma_0 y^{*2}E' \\ & -2\alpha_0\gamma_0 y^{*3} - \alpha_0\gamma_0 x^* y^{*2} - 2\alpha_0 E_0 y^{*2} + \alpha_0\gamma_0 x^{*2}y^*E' + 4\alpha_0\gamma_0 x^{*2}y^{*2} + \alpha_0 E_0 x^{*2}y^* + \\ & \beta_0 x^* y^{*2} + 2\alpha_0\gamma_0 x^{*3}y^* - \alpha_0\beta_0 x^{*2}y^{*2} + \alpha_0\gamma_0 y^{*2}x^*E' + 2\alpha_0\gamma_0 x^* y^{*3} + \alpha_0 E_0 x^* y^{*2} + \beta_0 x^* y^* E' - E_0 x^* y^* - \\ & \gamma_0 x^* y^* E' - \gamma_0 x^* y^{*2}\end{aligned}$$

and

$$\begin{aligned}C =\ & 4\beta_0 x^* E' + 6\beta_0 x^* y^* + 6\alpha_0 x^* E' + 6\alpha_0 x^* y^* + 4\alpha_0 y^* E' + 4\alpha_0 y^{*2} - 4\alpha_0 x^{*2}E' - 4\alpha_0 x^{*2}y^* - 2\alpha_0 x^* y^* E' \\ & - 2\alpha_0 x^* y^{*2} - \alpha_0\beta_0 x^{*2}y^{*2} - 2\gamma_0 y^* E' - 4\gamma_0 y^{*2} - 2\gamma_0 x^* y^* - 2E_0 y^* - 2\alpha_0\gamma_0 y^{*3}\end{aligned}$$

Thus, $\widetilde{K}_{(E_3^*)}$ can be selected as follows:

$$\widetilde{K}_{(E_3^*)} = \left\{\begin{array}{ll} \dfrac{2\sqrt{|E'|} + 2\sqrt{|y^*|}}{\sqrt{|B|}} & , \quad C = 0 \\ \dfrac{4E' + 4y^*}{|C|} & , \quad B = 0 \\ \min\left(\dfrac{|C|}{|B|}, \dfrac{2\sqrt{|E'|} + 2\sqrt{|y^*|}}{|C|}\right) & , otherwise \end{array}\right\}$$

Suppose that E_3^* is a stable fixed point of system (6). Then,

$$1 + \det(J(E_3^*)) - \mathrm{tr}(J(E_3^*)) = h_1^2\,\frac{(\alpha_0\beta_0 x^{*2}y^* - \alpha_0\beta_0 x^* y^* E' - \alpha_0\beta_0 x^* y^{*2} - 2\alpha_0\beta_0 x^{*3}y^* + \alpha_0\beta_0 x^{*2}y^*E'}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$

$$+\frac{2\alpha_0\gamma_0 y^{*3} - \alpha_0\gamma_0 x^* y^{*2} - \alpha_0\gamma_0 x^{*2}y^* + \alpha_0\gamma_0 x^{*2}y^*E' + 4\alpha_0\gamma_0 x^{*2}y^{*2}}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$
$$+\frac{\alpha_0 E_0 x^{*2}y^* + 2\alpha_0\gamma_0 x^{*3}y^* - \alpha_0\beta_0 x^{*2}y^{*2} + \alpha_0\gamma_0 y^{*2}x^*E' + 2\alpha_0\gamma_0 x^* y^{*3}}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$
$$+\frac{\beta_0 x^* y^* E' + \beta_0 x^* y^{*2} + \alpha_0 E_0 x^* y^{*2} - \gamma_0 x^* y^* E' - \gamma_0 x^* y^{*2} - E_0 x^* y^*)}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]}$$
$$+h_1\,\frac{(\alpha_0 x^* y^{*2} + \alpha_0\beta_0 x^{*2}y^{*2} - 2\alpha_0\gamma_0 y^{*3} - \alpha_0 x^* y^{*2})}{(1+h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E'+y^*)+h_1\beta_0 x^*(E'+y^*)]} > 0$$

and condition (2) holds. The last condition is equal to:

$$\det\big(J(E_3^*)\big) - 1 = \frac{h_1\beta_0 x^* y^* + h_1\alpha_0 x^* E' + h_1\alpha_0 x^* y^* + h_1^2\alpha_0\beta_0 x^{*2}E' + 3h_1^2\alpha_0\beta_0 x^{*2}y^* - {h_1}^2 E_0 x^* y^*}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-h_1\gamma_0 y^* E' - 2h_1\gamma_0 y^{*2} - h_1\gamma_0 x^* y^* - h_1 E_0 y^* - h_1^2\alpha_0\gamma_0 x^* y^* E' - 4h_1^2\alpha_0\gamma_0 x^* y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-2h_1^2\alpha_0\gamma_0 x^{*2}y^* - h_1^2\alpha_0 E_0 x^* y^* - h_1\alpha_0\gamma_0 y^{*2}E' - 2h_1\alpha_0\gamma_0 y^{*3} - {h_1}^2\alpha_0 E_0 y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-2h_1\alpha_0 x^{*2}y^* - 2h_1^2\alpha_0\beta_0 x^{*3}E' - 4h_1^2\alpha_0\beta_0 x^{*3}y^* + {h_1}^2\alpha_0\gamma_0 x^{*2}y^* E' + 3{h_1}^2\alpha_0\gamma_0 {x^*}^2 y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{+h_1^2\alpha_0 E_0 x^{*2}y^* + 2{h_1}^2\alpha_0\gamma_0 x^{*3}y^* - h_1\alpha_0 x^* y^* E' - h_1^2\alpha_0\beta_0 x^{*2}y^{*2} + h_1^2\alpha_0\gamma_0 y^{*2}x^* E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{+2h_1^2\alpha_0\gamma_0 x^* y^{*3} + h_1^2\alpha_0\gamma_0 x^{*2}y^{*2} + {h_1}^2\alpha_0 E_0 x^* y^{*2} + {h_1}^2\beta_0 x^* y^* E' + {h_1}^2\beta_0 x^* y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-2h_1\alpha_0 x^{*2}E' - h_1\alpha_0 x^* y^{*2} - {h_1}^2\gamma_0 x^* y^* E' - {h_1}^2\gamma_0 x^* y^{*2} - {h_1}^2\alpha_0\beta_0 x^* y^* E'}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\det\big(J(E_3^*)\big) - 1 < 0$$

$$\frac{h_1^2(\alpha_0\beta_0 x^{*2}E' + 3\alpha_0\beta_0 x^{*2}y^* - E_0 x^* y^* - \alpha_0\gamma_0 x^* y^* E' - 4\alpha_0\gamma_0 x^* y^{*2} - 2\alpha_0\gamma_0 x^{*2}y^* - \alpha_0 E_0 x^* y^*}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-2\alpha_0\beta_0 x^{*3}E' - 4\alpha_0\beta_0 x^{*3}y^* + \alpha_0\gamma_0 x^{*2}y^* E' + 3\alpha_0\gamma_0 x^{*2}y^{*2} + \alpha_0 E_0 x^{*2}y^* + 2\alpha_0\gamma_0 x^{*3}y^* - \alpha_0\beta_0 x^{*2}y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{2\alpha_0\gamma_0 x^* y^{*3} + \alpha_0\gamma_0 x^{*2}y^{*2} + \alpha_0 E_0 x^* y^{*2} + \beta_0 x^* y^* E' + \beta_0 x^* y^{*2} - \gamma_0 x^* y^* E' - \gamma_0 x^* y^{*2}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-\alpha_0 E_0 y^{*2} + \alpha_0\gamma_0 x^* y^{*2}E' - \alpha_0\beta_0 x^* y^* E')}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$+h_1\frac{(\beta_0 x^* y^* + \alpha_0 x^* E' + \alpha_0 x^* y^* - \gamma_0 y^* E' - 2\gamma_0 y^{*2} - \gamma_0 x^* y^* - E_0 y^* - \alpha_0\gamma_0 y^{*2}E' - 2\alpha_0\gamma_0 y^{*3}}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

$$\frac{-\alpha_0 x^* y^* E' - 2\alpha_0 x^{*2}E' - \alpha_0 x^* y^{*2} - 2\alpha_0 x^{*2}y^*)}{(1 + h_1\alpha_0 x^* + h_1\alpha_0 y^*)[(E' + y^*) + h_1\beta_0 x^*(E' + y^*)]}$$

The above inequality is true when $h_1 < K_{(E_3^*)}$ where $K_{(E_3^*)} = \frac{|M|}{|N|}$ and

$$\begin{aligned} M = {} & \alpha_0\beta_0 x^{*2}E' + 3\alpha_0\beta_0 x^{*2}y^* - E_0 x^* y^* - \alpha_0\gamma_0 x^* y^* E' - 4\alpha_0\gamma_0 x^* y^{*2} - 2\alpha_0\gamma_0 x^{*2}y^* - \alpha_0 E_0 x^* y^* - 2\alpha_0\beta_0 x^{*3}E' \\ & - 4\alpha_0\beta_0 x^{*3}y^* + \alpha_0\gamma_0 x^{*2}y^* E' + 3\alpha_0\gamma_0 x^{*2}y^{*2} + \alpha_0 E_0 x^{*2}y^* + 2\alpha_0\gamma_0 x^{*3}y^* - \alpha_0\beta_0 x^{*2}y^{*2} \\ & + 2\alpha_0\gamma_0 x^* y^{*3} + \alpha_0\gamma_0 x^{*2}y^{*2} + \alpha_0 E_0 x^* y^{*2} + \beta_0 x^* y^* E' + \beta_0 x^* y^{*2} - \gamma_0 x^* y^* E' - \gamma_0 x^* y^{*2} \\ & - \alpha_0 E_0 y^{*2} + \alpha_0\gamma_0 y^{*2}E' - \alpha_0\beta_0 x^* y^* E' \end{aligned}$$

$$\begin{aligned} N = {} & \beta_0 x^* y^* + \alpha_0 x^* E' + \alpha_0 x^* y^* - \gamma_0 y^* E' - 2\gamma_0 y^{*2} - \gamma_0 x^* y^* - E_0 y^* - \alpha_0\gamma_0 y^{*2}E' - 2\alpha_0\gamma_0 y^{*3} - \alpha_0 x^* y^* E' \\ & - 2\alpha_0 x^{*2}E' - \alpha_0 x^* y^{*2} - 2\alpha_0 x^{*2}y^* \end{aligned}$$

Therefore, if $h_1 < \min(K_{(E_3^*)}, \widetilde{K}_{(E_3^*)})$, all conditions of Lemma hold and E_3^* is a stable.

3. NUMERICAL SIMULATIONS

In this part, parameters are chosen: $\alpha_0 = 0.5$, $\beta_0 = 0.16$, $\gamma_0 = 0.1$, $E' = 0.05$ and $E_0 = 0.001$ [20]. These parameters provide the conditions of Theorem 2.2, 2.3, 2.4, 2.5. Table 1 and Table 2 present the effect of time step size on Theta method (for $\theta = 1$) and NSFD scheme for $E_0 = 0.001$ and $E_0 = 0.004$, respectively. It is seen from Table 1- 2; the nonstandard discretization is more effective than the classical method for bigger step-size. In Figure 1(a), it can be seen that, $E_1^*(0,0)$ is a nonhyperbolic, $E_2^*(1,0)$is a saddle and the interior fixed point $E_3^*(x^*,y^*)$ is an unstable focus for $E_0 = 0.001$ at the equation (6). If the parameter value of E_0 is increased from 0.001 to 0.0022, not changing other parameter values, some trajectories go to (0, 0) and some creates limit cycle (Figure 1(b)). If we again change the parameter E_0 from 0.0022 to 0.004, keeping other parameter values unchanged, $E_1^*(0,0)$and $E_3^*(x^*,y^*)$ are stable. That's way, the system exhibits bistability (Figure 1(c)) and some trajectories go to (0, 0) and some converge to the interior fixed point. If we again increase the parameter value of E_0 from 0.004 to 0.005, the system exhibits tristability (Figure 1(d)). Here, some trajectories go to (0, 0), some converge to (1, 0) and some converge to the interior equilibrium, attached on the initial values. A further increment in E_0 from 0.005 to 0.01 keeping other parameter values unspoilt, $E_1^*(0,0)$ and $E_2^*(1,0)$ can be fined stable fixed point and the system exhibits bistability (Figure 1(e)). It is observed that some trajectories go to (0, 0) and some converge to (1, 0). If we increase the parameter value of E_0 from 0.01 to 0.05, not changing other parameter values, all trajectories converges to (1, 0) (Figure 1(f)). Because, only the fixed point $E_2^*(1,0)$ is stable and the system exhibits monostability. In Figure 2, the graphs of x and y solutions are drawn respectively for values of $E_0 = 0.001$, $E_0 = 0.0022$, $E_0 = 0.004$, $E_0 = 0.005$, $E_0 = 0.01$, $E_0 = 0.05$ in different initial conditions. Also, the phase portrait of equation (6) corresponding to different E_0 and the same initial conditions (0.3, 0.159) is drawn in Figure 3. In Figure 4, the phase portrait of the equation (6) using NSFD for different E_0 at the same initial conditions (0.3, 0.159) is drawn. In Figure 5, the phase portrait of the equation (4) using RK4 for (a) : $E_0 = 0.001$, (b) : $E_0 = 0.0022$, (c) : $E_0 = 0.05$, (d) : $E_0 = 0.01$at the same initial conditions (0.3, 0.159) is simulated. The phase portrait drawn using NSFD is compatible with the phase portrait drawn using RK4 (Runge-Kutta 4th order method). As in [33,34] it was seen that this method gives accurate and convergence results for very small h. In all calculations with NSFD, the denominator function is selected as

$$h_1 = \varphi(h) = (e^{\alpha_0 h} - 1)/\alpha_0 \text{ and } h = 0.01$$

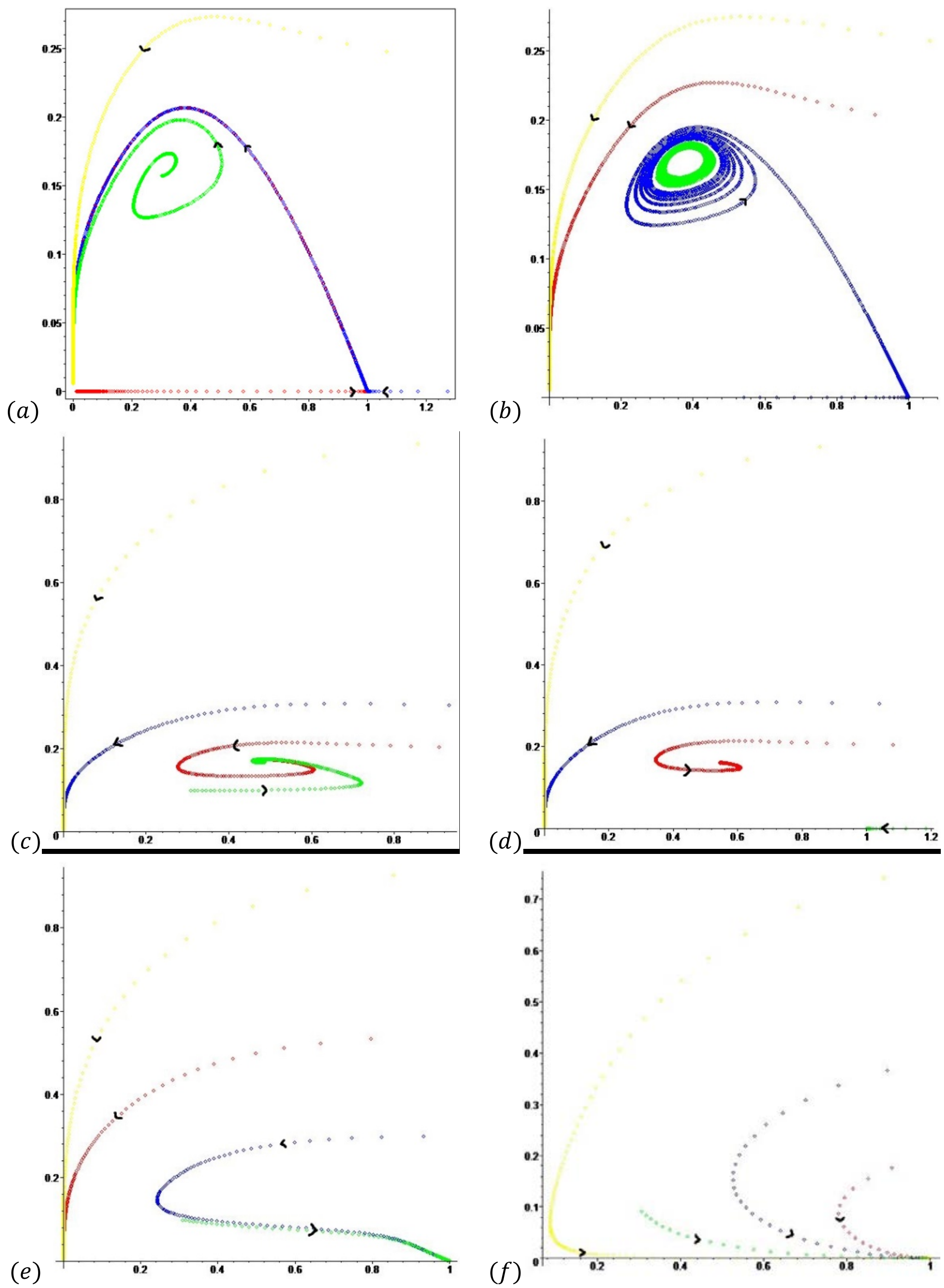

FIGURE 1. The phase portraits for different E_0 . (a) E_0 =0.001, (b) E_0 = 0.0022, (c) E_0 =0.004, (d) E_0 =0.005, (e) E_0 =0.01, (f) E_0 =0.05.

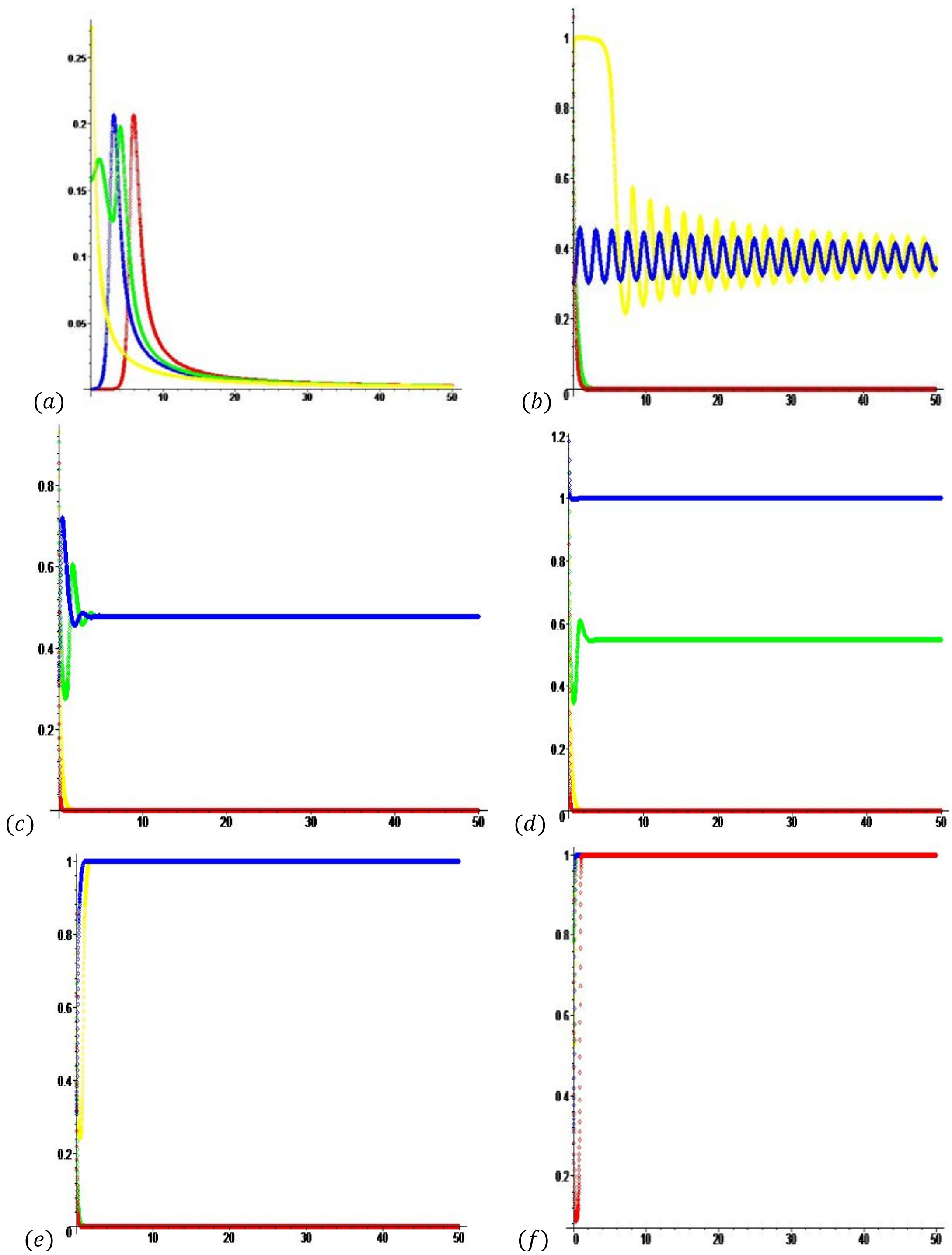

FIGURE 2. x solutions for different initial conditions. (a) E_0 =0.001, (b) E_0 = 0.0022, (c) E_0 =0.004, (d) E_0 =0.005, (e) E_0 =0.01, (f) E_0 =0.05.

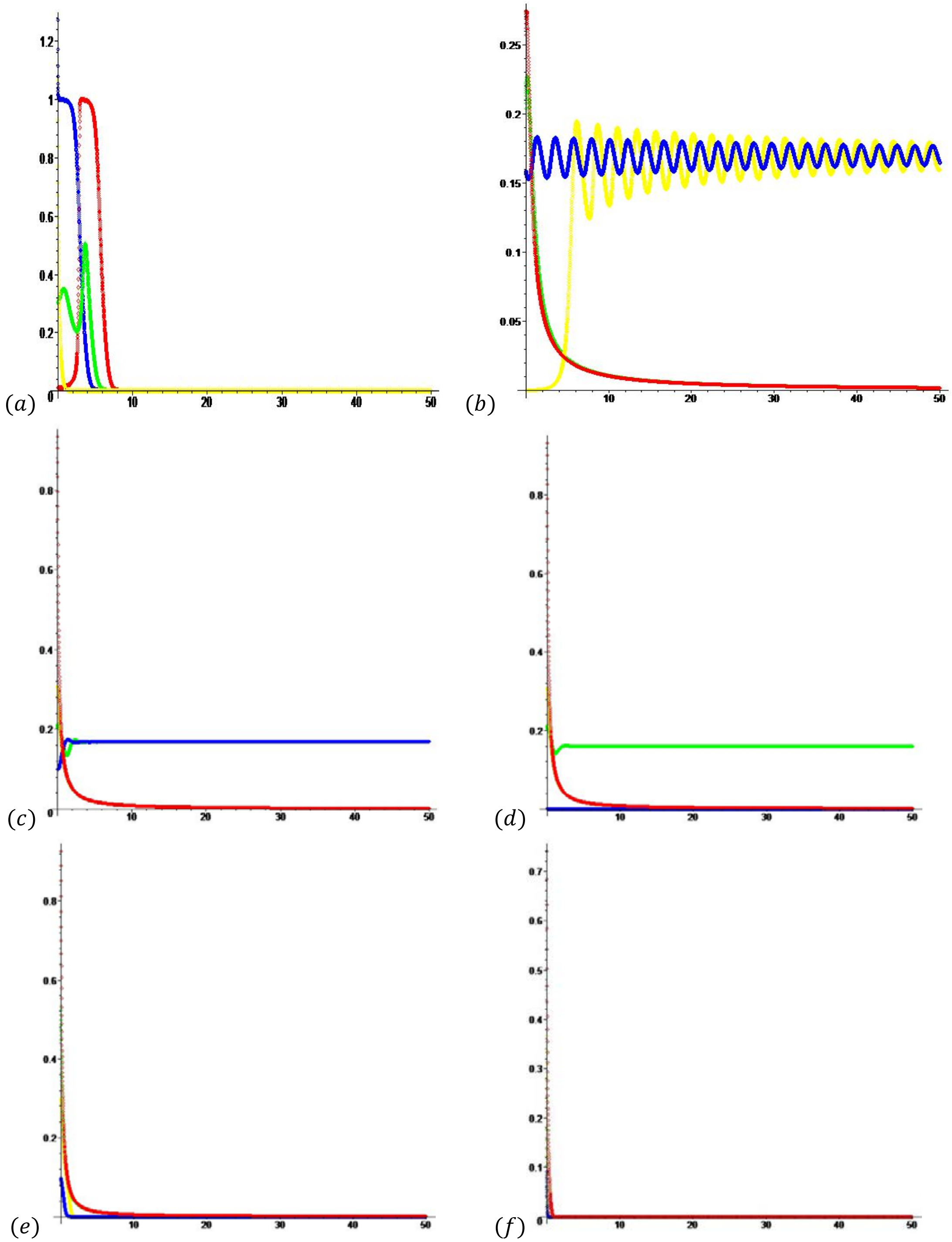

FIGURE 3. y solutions for different initial conditions. (a) E_0 =0.001, (b) E_0 = 0.0022, (c) E_0 =0.004, (d) E_0 =0.005, (e) E_0 =0.01, (f) E_0 =0.05.

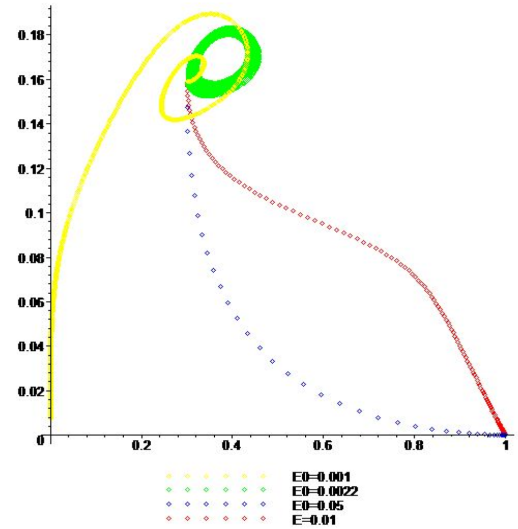

FIGURE 4. The phase portrait of equation (6) using NSFD for different E_0 at the same initial conditions $(0.3, 0.159)$

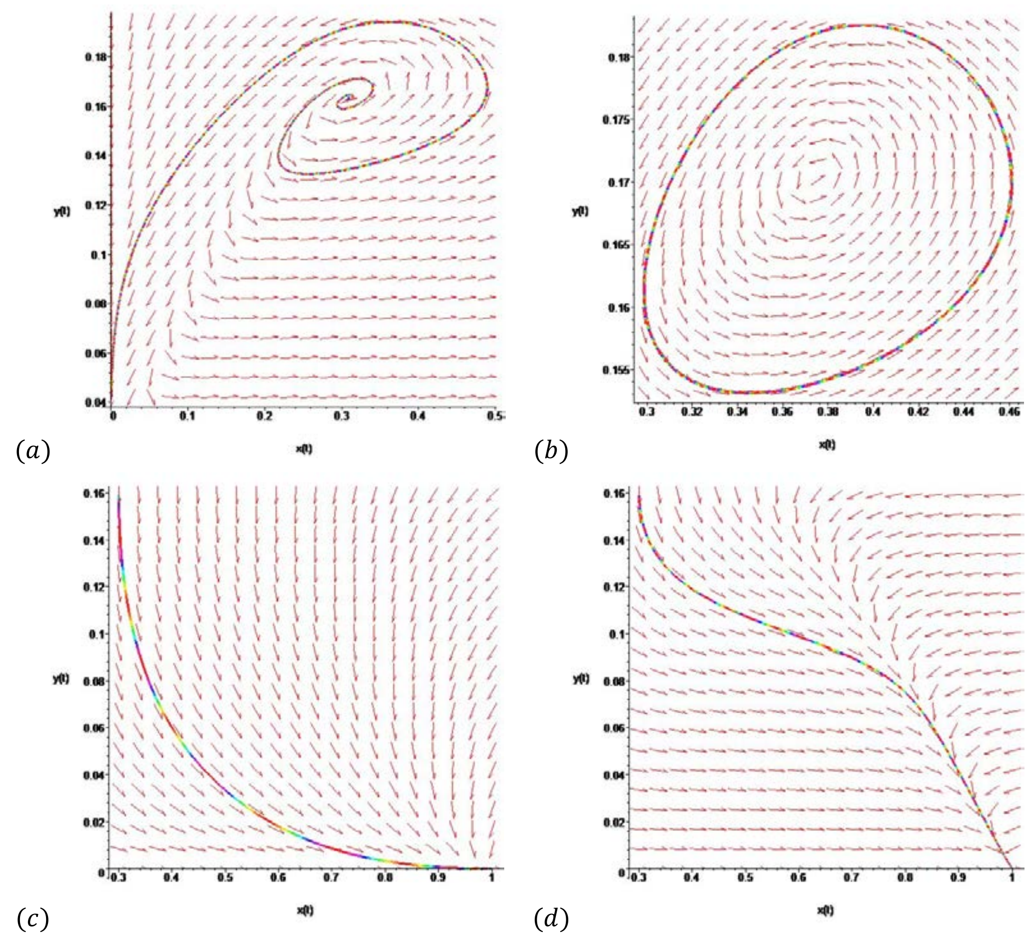

FIGURE 5. The phase portrait of equation (4) using RK4 for (a) E_0 = 0.001, (b) E_0 =0.0022, (c) E_0= 0.05, (d) E_0 = 0.01 at the same initial conditions (0.3, 0.159)

h	**Theta method (θ=1)**	**NSFD scheme**
0.000000001	Convergence	Convergence
0.00001	Convergence	Convergence
0.001	Convergence	Convergence
0.01	Convergence	Convergence
0.03	Convergence	Convergence
0.09	Convergence	Convergence
0.1	Divergence	Convergence
0.11	Divergence	Convergence
0.15	Divergence	Divergence

TABLE 1. Effect of time step sizes on the numerical methods for $E_0 = 0.001$

h	**Theta method (θ=1)**	**NSFD scheme**
0.000000001	Convergence	Convergence
0.00001	Convergence	Convergence
0.001	Convergence	Convergence
0.01	Convergence	Convergence
0.03	Convergence	Convergence
0.1	Convergence	Convergence
0.3	Convergence	Convergence
0.35	Divergence	Convergence
0.4	Divergence	Divergence

TABLE 2. Effect of time step sizes on the numerical methods for $E_0 = 0.004$

4. CONCLUSIONS

In this study, we used nonstandard finite difference scheme to discretize the system with Michaelis-Menten harvesting rate. It was shown that solutions are positive for all positive initial values. The stability of equilibrium points was investigated to prove their stability features are same for both the continuous system and the discrete system. The qualitative results were given in Table 1-2 to show the effectiveness of NSFD schemes. Also, numerical simulations of the model are presented. According to us, NSFD schemes can also be studied for fractional order competitive system in the future.

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