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Exactness of Proximal Group Homomorphisms

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ABSTRACT

This research introduces groups in proximity spaces which endowed with a proximity relation. Two penultimate choices for such relations are the Efremovic (EF) proximity relation and its extension, namely, the descriptive EF-proximity relation. There is a strong relationship between sets (groups) and set (group) descriptions. Therefore, in this paper we consider this relationship via exactness of descriptive homomorphisms between ordinary descriptive groups and meta-descriptive groups. The definition of a short exact sequence of descriptive homomorphisms is given. Then, results were obtained giving the relationships between the two short exact sequences.

Keywords: Proximal group, EF-proximity space, Descriptive EF-proximity space, Descriptive homomorphism.

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Introduction

The focus of this research is on algebraic structures in a descriptive EF-proximity space and exactness of proximal group homomorphisms which is an outgrowth of recent research [1, 2]. A descriptive proximity space [3, 4] is an extension of an Efremovič proximity space [5]. This extension is made possible by the introduction of feature vectors that describe each point in a proximity space. Sets A, B in a proximity space X are near, provided there is at least one pair of points $a \in A, b \in B$ with matching descriptions. The basic approach is to define binary operations on subsets in a space endowed with a proximity relation. By considering the features of points in a proximity space, it is then possible to define a descriptive proximity relation as well as descriptive binary operations. This leads to a study of groupoids in proximity spaces as well as other algebraic structures in proximity spaces such as semigroups and groups. In homological algebra, exact sequences play an important role. Exactness is a part of the fundamental concepts and is used, in particular, in the definition of some functors [6]. There is a strong relationship between sets (groups) and set (group) descriptions.

Preliminaries

Let X be a nonempty set of non-abstract points and let $\Phi = \{\phi_1, \phi_2, \cdots, \phi_n\}$ be a set of probe functions that represent features of each $x \in X$. In a discrete space, a non-abstract point has a location and features that can be measured [7]. This leads to a proximal view of sets of picture points in digital images [8]. A *probe function* $\Phi: X \longrightarrow \mathbb{R}$ represents a feature of a sample point in a picture. Let $\Phi(x) = (\phi_1, \phi_2, \cdots, \phi_n)$ denote a feature vector for x, which provides a description of each $x \in X$. To obtain a descriptive proximity relation (denoted by $\delta_{\{\Phi\}}$), one first chooses a set of probe functions. Let $A, B \in \mathcal{P}(X)$ and let $\mathcal{Q}(A)$ and $\mathcal{Q}(B)$ denote sets of descriptions of points in A and B, respectively (e.g., $\mathcal{Q}(A) = \{\Phi(a) | a \in A\}$).

The expression $A\delta_\Phi B$ reads A is *descriptively near* B. Similarly, $A\underline{\delta}_\Phi B$ reads A *is descriptively far from* B. In an ordinary metric closure space X, the closure of $A \subset X$ is defined by $cl(A) = \{x \in X | d(x, A) = 0\}$ [9]. For a set X endowed with a descriptive proximity δ_Φ, the descriptive closure of $A \subset X$ is defined by $cl_\Phi(A) = \{x \in X | \Phi(x) \in \mathcal{Q}(cl(A))\}$. The descriptive proximity of A and B is defined by $A\delta_\Phi B \Leftrightarrow \mathcal{Q}(cl(A)) \cap \mathcal{Q}(cl(B)) \neq \emptyset$. The *descriptive intersection* $\underset{\Phi}{\cap}$ of A and B is defined by $A \underset{\Phi}{\cap} B = \{x \in A \cup B | \Phi(x) \in \mathcal{Q}(A), \Phi(x) \in \mathcal{Q}(B)\}$. That is, $x \in A \cup B$ is in $A \underset{\Phi}{\cap} B$, provided $\Phi(x) = \Phi(a) = \Phi(b)$ for some $a \in A$, $b \in B$. Observe that A and B can be disjoint and yet $A \underset{\Phi}{\cap} B$ can be nonempty.

A *binary operation* on a set S is a mapping of $S \times S$ into S, where $S \times S$ is the set of all ordered pairs of elements of S. A *groupoid* is a system $S(*)$ consisting of a nonempty set S together with a binary operation " $*$ " on S.

Let $S(*)$ and $S'(\cdot)$ be groupoids. A mapping h of S into S' is called a homomorphism if $h(a * b) = h(a) \cdot h(b)$ for all $a, b \in S$. A one-to-one homomorphism h of S onto S' is called an *isomorphism* of S to S' [5].

Let we consider the groupoids $\mathcal{Q}(A)(*_1), \mathcal{Q}(B)(*_2)$, where $A \subseteq X, B \subseteq Y$. A mapping $h_\Phi: \mathcal{Q}(B) \to \mathcal{Q}(A)$ is called a *descriptive homomorphism* if it provides $h_\Phi(\Phi_B(b_1) *_2 \Phi_B(b_2)) = h_\Phi(\Phi_B(b_1)) *_1 h_\Phi(\Phi_B(b_2))$ for all $\Phi_B(b_1)$, $\Phi_B(b_2) \in \mathcal{Q}(B)$. A one-to-one descriptive homomorphism h_Φ is called a *descriptive monomorphism*, a descriptive homomorphism h_Φ of $\mathcal{Q}(B)$ onto $\mathcal{Q}(A)$ is called a *descriptive epimorphism* and one-to-one descriptive homomorphism h_Φ of $Q(B)$ onto $Q(A)$ is called a *descriptive isomorphism* [1].

Let $A(\cdot_1)$, $B(\cdot_2)$ be groupoids, $h: B \to A$ be a homomorphism and $\Phi_A: A \to Q(A), a \mapsto \Phi(a)$ be an object description. The object description Φ_A of A into $Q(A)$ is an object description homomorphism if $\Phi_A(a_1 \cdot_1 a_2) = \Phi_A(a_1) *_1 \Phi_A(a_2))$ for all $a_1, a_2 \in A$. Let we consider the descriptive homomorphism $h_\Phi: Q(B) \to Q(A)$ such that $h_\Phi(\Phi_B(b) = \Phi_A(h(b)$ [1].

$$\begin{array}{ccc} B_\Phi & \xrightarrow{h'} & A_\Phi \\ \downarrow \Phi_B & & \downarrow \Phi_A \\ Q(B) & \xrightarrow{h_\Phi} & Q(A) \end{array} \qquad (1)$$

Lemma 2.1: ([1]) $h_\Phi \circ \Phi_B = \Phi_A \circ h'$.

Theorem 2.2: ([1]) Let $(X, \delta_\Phi), (Y, \delta_\Phi)$ be descriptive EF-proximity spaces, $A(\cdot_1)$, $B(\cdot_2)$, $Q(B)(\circ_2)$ and $Q(A)(\circ_1)$ be groupoids and h be a homomorphism from $B(\cdot_2)$ to $A(\cdot_1)$. If there are a descriptive monomorphism h_Φ of $Q(B)$ to $Q(A)$ and an object description homomorphism Φ_A of A into $Q(A)$, then there is an object description homomorphism Φ_B of B into $Q(B)$.

Let $A_\Phi(*), B_\Phi(*), C_\Phi(*)$ be ordinary descriptive monoids, $h: B_\Phi \longrightarrow A_\Phi$, and $h': C_\Phi \longrightarrow B_\Phi$ be ordinary descriptive homomorphisms.

$$C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi \qquad (2)$$

A pair of ordinary descriptive homomorphisms (the diagram (2)) is said to be exact at B_Φ, provided $Imh' = Kerh$.

$$\cdots \xrightarrow{\boldsymbol{h_{n-1}}} (A_\Phi)_{n-1} \xrightarrow{\boldsymbol{h_n}} (A_\Phi)_n \xrightarrow{\boldsymbol{h_{n+1}}} (A_\Phi)_{n+1} \xrightarrow{\boldsymbol{h_{n+2}}} \cdots \qquad (3)$$

In general, a sequence of ordinary descriptive homomorphisms (the diagram (3)) is exact, provided each sequential pair h_n, h_{n+1} are exact at each $(A_\Phi)_n$ for $n \in \mathbb{N}$ [10].

$$\begin{array}{ccccc} C_\Phi & \xrightarrow{h'} & B_\Phi & \xrightarrow{h} & A_\Phi \\ \downarrow \Phi_C & & \downarrow \Phi_B & & \downarrow \Phi_A \\ Q(C) & \xrightarrow{h'_\Phi} & Q(B) & \xrightarrow{h_\Phi} & Q(A) \end{array} \qquad (4)$$

Lemma 2.3: ([10]) Let $h: B_\Phi \longrightarrow A_\Phi$ be an ordinary descriptive homomorphism, Φ_A, Φ_B be object descriptive homomorphisms and $h: Q(B) \longrightarrow Q(A)$ be a meta-descriptive homomorphism represented in the diagram (4). If h and Φ_A are descriptive monomorphisms, then so is Φ_B.

Theorem 2.4: ([10]) Let $A_\Phi(*), B_\Phi(*), C_\Phi(*)$ be ordinary descriptive monoids, $A_\Phi(*_\Phi), B_\Phi(*_\Phi), C_\Phi(*_\Phi)$ be meta-descriptive monoids, and $C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi$ be exact, represented in the diagram (4). If Φ_A, Φ_B are object descriptive monomorphisms, then $Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A)$ is exact.

Theorem 2.5: ([10]) In the diagram (4), let $A_\Phi(*), B_\Phi(*), C_\Phi(*)$ be ordinary descriptive monoids, $A_\Phi(*_\Phi), B_\Phi(*_\Phi)$ and $C_\Phi(*_\Phi)$ be meta-descriptive monoids. Then

i) If Φ_A, Φ_C are object descriptive monomorphisms, h'_Φ is a meta-descriptive monomorphism, and $C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi$ is exact, then Φ_B is an object descriptive monomorphism.

ii) If Φ_B is an object descriptive epimorphism, Φ_A is an object descriptive monomorphism and h'_Φ is a meta-descriptive monomorphism, then Φ_C is an object descriptive epimorphism.

Corollary 2.6: ([10]) In the diagram (4), let $A_\Phi(*), B_\Phi(*), C_\Phi(*)$ be ordinary descriptive monoids, $A_\Phi(*_\Phi), B_\Phi(*_\Phi)$ and $C_\Phi(*_\Phi)$ be meta-descriptive monoids. Then

i) If Φ_A, Φ_C are object descriptive monomorphisms, h'_Φ is a meta-descriptive monomorphism, and $C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi$ is exact, then $Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A)$ is exact.

$\boldsymbol{ii}$) If Φ_A, Φ_C are object descriptive monomorphisms, $e \longrightarrow C_\Phi \xrightarrow{h'} B_\Phi \xrightarrow{h} A_\Phi \longrightarrow e$ is short exact sequence, then $e_\Phi \longrightarrow Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A) \longrightarrow e_\Phi$ is a short exact sequence.

Exactness of Descriptive Group Homomorphisms

Theorem 3.1: Let $h: B_\Phi \longrightarrow A_\Phi$ be an ordinary descriptive homomorphism, Φ_A, Φ_B be object descriptive homomorphisms, and $h_\Phi: Q(B) \longrightarrow Q(A)$ be a meta-descriptive homomorphism in the diagram (1). If Φ_B is an object descriptive epimorphism and Φ_A is an object descriptive monomorphism, then $Imh = \Phi_A^{-1}(Imh_\Phi)$ and $Kerh_\Phi = \Phi_B(Kerh)$.

Proof: Since Φ_A is an object descriptive homomorphism, we get that $Imh = \Phi_A^{-1}(\Phi_A(Imh))$. Therefore, from Lemma 2.1,

$$Imh = \Phi_A^{-1}\big(Im(\Phi_A h)\big) = \Phi_A^{-1}\big(Im(h_\Phi \Phi_B)\big) = \Phi_A^{-1}(Imh_\Phi) \tag{5}$$

where Φ_B is an object descriptive epimorphism. Moreover, we obtain $Kerh_\Phi = h(h^{-1}(Kerh_\Phi)$. Thus $Kerh_\Phi = h(Ker(h_\Phi h))$, and so we have that $Kerh_\Phi = \Phi_B(Ker(\Phi_A h))$ by Lemma 2.1. Since Φ_A is an object descriptive monomorphism, we obtain $Kerh_\Phi = \Phi_B(Kerh)$.

Definition 3.2: Let $h: B_\Phi \longrightarrow A_\Phi$ and $h': C_\Phi \longrightarrow B_\Phi$ be ordinary descriptive homomorphisms.

$$e_\Phi \longrightarrow C_\Phi \xrightarrow{\quad h' \quad} B_\Phi \xrightarrow{\quad h \quad} A_\Phi \longrightarrow e_\Phi \tag{6}$$

The diagram (6) is said to be a short exact sequence if h' is a monomorphism and h is an epimorphism.

Theorem 3.3: In the diagram (4), let A_Φ, B_Φ, and C_Φ be ordinary descriptive groups and, $Q(A)$, $Q(B)$, and $Q(C)$ be meta-descriptive groups. Then

$\boldsymbol{i}$) If Φ_A, Φ_C are object descriptive epimorphisms, h is an ordinary descriptive epimorphism, and $Q(C) \xrightarrow{h'_\Phi} Q(B) \xrightarrow{h_\Phi} Q(A)$ is exact, then Φ_B is an object descriptive epimorphism.

$\boldsymbol{ii}$) If Φ_B is an object descriptive monomorphism, h is an ordinary descriptive epimorphism and h_Φ is a meta-descriptive monomorphism, then Φ_A is an object descriptive monomorphism.

Proof: $\boldsymbol{i}$) Let $\Phi_B \in Q(B), b \in B_\Phi$. In this case $h_\Phi(\Phi_B(b)) \in Q(A)$, and since Φ_A is an object descriptive epimorphism, $h_\Phi\big(\Phi_B(b)\big) = \Phi_A(a)$ for some $a \in A_\Phi$. Since h is an ordinary descriptive epimorphism, we get $a = h(b')$ for some $b' \in B_\Phi$. From Lemma 2.1, $h_\Phi\big(\Phi_B(b')\big) = \Phi_A\big(h(b')\big) = \Phi_A(a) = h_\Phi\big(\Phi_B(b)\big)$. Thus, $h_\Phi\left(\big(\Phi_B(b')\big)^{-1}\Phi_B(b)\right) = e_{Q(A)}$ and then $\big(\Phi_B(b')\big)^{-1}\Phi_B(b) \in Kerh_\Phi = Imh'_\Phi$ by exactness. Hence, we obtain $\big(\Phi_B(b')\big)^{-1}\Phi_B(b) = h'_\Phi(\Phi_C(c)), c \in C_\Phi$ since Φ_C is an object descriptive epimorphism. Because of $h'(c) \in B_\Phi$ and $b' \in B_\Phi$, we have that $b'h'(c) \in B_\Phi$ and from Lemma 2.2, $\Phi_B\big(b'h'(c)\big) = \Phi_B(b')\Phi_B\big(h'(c)\big) = \Phi_B(b')h'_\Phi\big(\Phi_C(c)\big) = \Phi_B(b')\big(\Phi_B(b')\big)^{-1}\Phi_B(b) = \Phi_B(b)$. Therefore, Φ_B is an object descriptive epimorphism.

$\boldsymbol{ii}$) Let $a \in Ker\Phi_A$. Since h is an ordinary descriptive epimorphism, there exists $b \in B$ such that $h(b) = a$. From Lemma 2.1, $h_\Phi\big(\Phi_B(b)\big) = \Phi_A\big(h(b)\big) = \Phi_A(a) = e_{Q(A)}$, and so $\Phi_B(b) = e_{Q(B)}$ by h_Φ is a meta-descriptive monomorphism. Therefore, we get $b = e_{Q(B)}$, since Φ_B is an object descriptive monomorphism. Consequently $a = h(b) = h\big(e_{B_\Phi}\big) = e_{A_\Phi}$. Thus $Ker\Phi_A = \{e_{A_\Phi}\}$.

$$\begin{array}{ccccccccc} e_\Phi & \longrightarrow & C_\Phi & \xrightarrow{\quad h' \quad} & B_\Phi & \xrightarrow{\quad h \quad} & A_\Phi & \longrightarrow & e_\Phi \\ & & \Phi_C \downarrow\uparrow \Phi_C^{-1} & & \Phi_B \downarrow\uparrow \Phi_B^{-1} & & \Phi_A \downarrow\uparrow \Phi_A^{-1} & & \\ e_{Q(C)} & \longrightarrow & Q(C) & \xrightarrow{\quad h'_\Phi \quad} & Q(B) & \xrightarrow{\quad h_\Phi \quad} & Q(A) & \longrightarrow & e_{Q(A)} \end{array} \tag{7}$$

Corollary 3.4: Let A_Φ, B_Φ, and C_Φ be ordinary descriptive groups, $Q(A)$, $Q(B)$, and $Q(C)$ be meta-descriptive groups, and in the diagram (7), each row be exact sequence.

i) If Φ_A and Φ_C are object descriptive monomorphisms, then Φ_B is an object descriptive monomorphism.

ii) If Φ_A and Φ_C are object descriptive epimorphisms, then Φ_B is an object descriptive epimorphism.

iii) If Φ_A and Φ_C are object descriptive isomorphisms, then Φ_B is an object descriptive isomorphism.

Two (short) exact sequences are called an isomorphic if there is a diagram of ordinary descriptive and meta-descriptive homomorphisms such that A_Φ, B_Φ, and C_Φ are object descriptive isomorphisms. In this case, it is easy to verify the diagram (7) with Φ_A^{-1}, Φ_B^{-1} and Φ_C^{-1}, is commutative.

In the diagram (7), exact sequence pairs are called ordinary-meta-descriptive homomorphism sequence or shortly called om-descriptive homomorphism sequence.

Proposition 3.5: Let A_Φ be ordinary descriptive group.

$$e_\Phi \longrightarrow (Kerh)_\Phi \xrightarrow{\ i\ } A_\Phi \xrightarrow{\ \pi\ } (A/Kerh)_\Phi \longrightarrow e_\Phi \tag{8}$$

Then, the diagram (8) is a short exact sequence.

Example 3.6: If $h: B_\Phi \longrightarrow A_\Phi$ is an ordinary descriptive homomorphism, $\Phi_{Kerh}: (Kerh)_\Phi \longrightarrow Q(Kerh)$ and $\Phi_B: (B)_\Phi \longrightarrow Q(B)$ are object descriptive monomorphisms, then from Proposition 3.5 and Corollary 2.6. (ii), we have

$$e_{Q(Kerh)} \longrightarrow Q(Kerh) \longrightarrow Q(B) \longrightarrow Q(B/Kerh) \longrightarrow e_{Q(B/Kerh)}. \tag{9}$$

The diagram (8) is a short exact sequence. Therefore, we have that the diagram (10).

$$\begin{array}{ccccccccc} e_\Phi & \longrightarrow & (Kerh)_\Phi & \xrightarrow{\ i\ } & B_\Phi & \xrightarrow{\ \pi\ } & (B/Kerh)_\Phi & \longrightarrow & e_\Phi \\ & & \downarrow \Phi_{Kerh} & & \downarrow \Phi_B & & \downarrow \Phi_{B/Kerh} & & \\ e_{Q(Kerh)} & \longrightarrow & Q(Kerh) & \xrightarrow{\ i\ } & Q(B) & \xrightarrow{\ h_\Phi\ } & Q(B/Kerh) & \longrightarrow & e_{Q(B/Kerh)} \end{array} \tag{10}$$

Conflicts of interest

There are no conflicts of interest in this work.

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