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The Differential Equations of Conformable Curve in IR²

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ABSTRACT

In this paper, we get some characterizations of conformable curve in $\mathbb{R}^2$. We investigate the conformable curve in $\mathbb{R}^2$. We define the tangent vector of the curve using the conformable derivative and the arc parameter s. Then, we get the Frenet formulas with conformable frames. Moreover, we define the location vector of conformable curve according to Frenet frame in the plane $\mathbb{R}^2$.

Finally, we obtain the differential equation characterizing location vector and curvature of conformable curve in the plane $\mathbb{R}^2$.

Keywords: Conformable curve, Location vector, Conformable frame.

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Introduction

The fractional analysis phrase has been first appearing in a letter written by L' Hospital to Leibniz. In this letter, L' Hospital has asked Leibniz about a special structure that he used in his work for $\frac{d^n y}{dx^n}$. L' Hospital has asked how to take the derivative and what would be the result if the order of the derivative was a rational number, for instance, $n=\frac{1}{2}$ [1]. This question has created the first glint of fractional analysis. Most of the mathematical theories used in fractional analysis have been developed before the 20th century. However, to keep up with scientific developments, mathematicians have made a few changes to the structure of fractional calculus. Caputo has renewed the Riemann-Liouville fractional derivative and has introduced the Caputo derivative as a new derivative concept [2]. Many scientists, Khalil and his colleagues first came up with the definition of conformable derivative in 2014. Because of its similarity to the classical derivative definition, this derivative was the simplest of the fractional derivative definitions. Although the concepts of Riemann-Liouville and Caputo fractional derivatives are being widely used today, they are not as common as the conformable derivative because they have some deficiencies [3].The product and quotient rule, which could not be provided for the other fractional derivatives mentioned above, could be provided for this new definition of fractional derivative. In addition, a constant function has no Caputo fractional derivative among these fractional derivatives [4,5]. In a short time, many studies have been done on conformable derivatives. T. Abdejavad, J. Alzabut, F. Jarad, R.P. Agarval, A. Zbekler have studied Lyapunov type inequalities in the conformable derivative frame [6,7]. Moreover, further works have been done on the conformable derivative [8,9,10].

Finally, in this paper, the characterizations of a conformable curve in the plane $\mathbb{R}^2$ are expressed using the conformable derivative.

Geometric Preliminaries

Given a function $f:[0,\infty)\longrightarrow\mathbb{R}$. The conformable derivative of the function f of order α is defined by

$$T_\alpha f(x)=\lim_{h\to 0}\frac{f(x+hx^{1-\alpha})-f(x)}{h} \\ =x^{1-\alpha}f'(x) \tag{1}$$

for all $x>0,\alpha\in(0,1)$ [11]. The function $\gamma:(0,\infty)\longrightarrow\mathbb{R}^2$ is called a conformable curve in $\mathbb{R}^2$ if γ is α −differentiable,

Let $\gamma:(0,\infty)\longrightarrow\mathbb{R}^2$ be a conformable curve. The velocity vector of γ is determined by

$$\frac{T_\alpha\gamma(t)}{t^{1-\alpha}}, \tag{2}$$

for all $t\in(0,\infty)$.

Let $\gamma:(0,\infty)\longrightarrow\mathbb{R}^2$ be a conformable curve. Then the velocity function v of γ is defined by

$$v(t)=\frac{\|T_\alpha\gamma(t)\|}{t^{1-\alpha}} \tag{3}$$

for all $t\in(0,\infty)$.

Let $\gamma:(0,\infty)\longrightarrow\mathbb{R}^2$ be a conformable curve. The arc length function s of γ is defined by

$$s = \int_t^0 \|T_\alpha \gamma(t)\| \, dt \quad (4)$$

for all $t \in (0, \infty)$, it's said that γ is a unit speed.

Now, let us define the tangent vector of the curve using the conformable derivative and the arc parameter

$$s, e_1^\alpha(s) = T_\alpha \gamma(s) = \left(T_\alpha\big(x(s)\big), T_\alpha\big(y(s)\big)\right) = \left(\frac{d^\alpha x(s)}{ds^\alpha}, \frac{d^\alpha y(s)}{ds^\alpha}\right). \quad (5)$$

The norm of the tanget vector is $\|e_1^\alpha(s)\| = 1$. Furthermore,

$$e_2^\alpha(s) = \left(-T_\alpha\big(y(s)\big), T_\alpha(x(s))\right). \quad (6)$$

Here, for γ curve with the parameter s , $e_1^{(\alpha)}(s)$ and $e_2^{(\alpha)}(s)$ are the conformable unit tangent vector and unit normal vector of the curve γ, respectively, and the parameter s is the arc length. The Frenet-Serret formulas with conformable frames $e_1^\alpha(s), e_2^\alpha(s)$ are given as

$$\frac{de_1^{(\alpha)}(s)}{ds} = \mathrm{K}^{(\alpha)}(s) e_2^{(\alpha)}(s) \quad (7)$$

$$\frac{de_2^{(\alpha)}(s)}{ds} = -\mathrm{K}^{(\alpha)}(s) e_1^{(\alpha)}(s), \quad (8)$$

where $K^{(\alpha)}(s)$ is curvature of the unit speed curve $\alpha = \alpha(s)$.

Location Vector of a Conformable Curve in $\mathbb{R}^2$

In this chapter, we have used the proof method and terminology of see [12].
Let us take the conformable curve $\gamma = \gamma(s)$ into consideration in the plane $\mathbb{R}^2$. In this case, we can write the location vector of $\gamma(s)$ according to Frenet frame as

$$x = x(s) = \mu_1 e_1^\alpha(s) + \mu_2 e_2^\alpha(s), \quad (9)$$

here μ_1 and μ_2 are arbitrary functions connected to s.
If we differentiate the equality (9) and use Frenet equations, we get

$$\frac{d\mu_1}{ds} - \mu_2 \mathrm{K}^{(\alpha)} = s^{\alpha-1} \quad (10)$$

and

$$\frac{d\mu_2}{ds} + \mu_1 \mathrm{K}^\alpha = 0. \quad (11)$$

Then, by using (10) in (11), we get

$$\frac{d}{ds}\left[\frac{1}{\mathrm{K}^\alpha}\left(\frac{d\mu_1}{ds} - s^{\alpha-1}\right)\right] + \mu_1 K^{(\alpha)} = 0. \quad (12)$$

According to μ_1, this second order differential equation is a characterization obtained from the conformable curve $\gamma = \gamma(s)$.

In equation (12), by using change of variable

$$\varphi = \frac{1}{\mathrm{K}^{(\alpha)}}, \quad \theta = \int_0^s \mathrm{K}^{(\alpha)}.ds, \quad (13)$$

we obtain

$$\frac{d\varphi}{ds}\left(\frac{d\mu_1}{ds} - s^{\alpha-1}\right) + \varphi\left(\frac{d\mu_1}{ds} - \frac{ds^{\alpha-1}}{ds}\right) + \frac{\mu_1}{\varphi} = 0. \quad (14)$$

Now, if this differential equation is tried to be solved, we obtain

$$\frac{d\varphi}{ds} = \frac{d\varphi}{d\sigma}\frac{d\theta}{ds} = \frac{d\varphi}{d\theta}\frac{1}{\varphi} \quad (15)$$

$$\frac{d\mu_1}{ds} = \frac{d\mu_1}{d\theta}\frac{d\theta}{ds} = \frac{d\mu_1}{ds}\frac{1}{\varphi} \tag{16}$$

and

$$\frac{d^2\mu_1}{ds^2} = \frac{d}{ds}\left(\frac{d\theta}{ds}\right) = \frac{d}{ds}\left(\frac{d\mu_1}{ds}\frac{1}{\varphi}\right) = \frac{d}{d\theta}\left(\frac{d\mu_1}{d\theta}\frac{1}{\varphi}\right)\frac{1}{\varphi} = \frac{1}{\varphi^2}\left[\frac{d^2\mu_1}{d\theta^2} - \frac{1}{\varphi}\frac{d\mu_1}{d\theta}\frac{d\varphi}{d\theta}\right]. \tag{17}$$

Later, by using of (15), (16), (17) in equation (14), we get

$$\frac{d^2\mu_1}{d\theta^2} + \mu_1 = \varphi^2(\alpha-1)s^{\alpha-2} + s^{\alpha-1}. \tag{18}$$

Let us try to solve the differential equation (18). This equation's solution of homogeneous is

$$y_p = c_1\cos\theta + c_2\sin\theta. \qquad (18)_1$$

Due to variation of the parameters, we get this formula as following

$$y_p = v_1\cos\theta + v_2\sin\theta.$$

Here, functions v_1, v_2 are differentiable functions.
In that case, we acquire simply

$$y_p' = v_1'\cos\theta + v_2'\sin\theta - v_1.\sin\theta + v_2\cos\theta.$$

Additionally, because of

$$v_1'\cos\theta + v_2'\sin\theta = 0, \tag{19}$$

we obtain

$$y_p'' = -v_1\cos\theta - v_2\sin\theta - v_1'\sin\theta + v_2'\cos\theta,$$

and so we get

$$-v_1'\sin\theta + v_2'\cos\theta = \varphi^2(\alpha-1)s^{\alpha-2} - s^{\alpha-1}. \tag{20}$$

From the expressions (10) and (11), we acquire

$$v_1' = \sin\theta\varphi^2(\alpha-1)s^{\alpha-2} - s^{\alpha-1} \tag{21}$$
$$v_2' = \cos\theta\varphi^2(\alpha-1)s^{\alpha-2} - s^{\alpha-1}. \tag{22}$$

Afterward, if we integrate the expressions (21) and (22), respectively, we can acquire

$$v_1 = \left[\int \varphi^2\sin\theta(\alpha-1)s^{\alpha-2}d\theta - \int s^{\alpha-1}d\theta\right] \tag{23}$$
$$v_2 = \left[\int \varphi^2\cos\theta(\alpha-1)s^{\alpha-2}d\theta - \int s^{\alpha-1}d\theta\right]. \tag{24}$$

On the other side, if it is taken into account the equations $(18)_1$, (23), and (24), we can also get

$$\mu_1 = c_1\cos\theta + c_2\sin\theta + v_1e_1^\theta + v_2e_2^\theta$$

or

$$\mu_1 = c_1\cos\theta + c_2\sin\theta + \left[(\alpha-1)\int_0^s \varphi^2\sin\theta(\alpha-1)s^{\alpha-2}d\theta - \int_0^s s^{\alpha-1}d\theta\right] + + \left[(\alpha-1)\int_0^s \varphi^2\cos\theta(s^{\alpha-2}d\theta - \int_0^s s^{\alpha-1}d\theta\right].$$

On the other and, if $\frac{d\mu_1}{ds} = l(s)$ is being taken, from expression (10), we can express as follows:

$$\mu_2 = \frac{1}{\mathrm{K}^\alpha}[l(s) - s^{\alpha-1}]. \tag{25}$$

As a result, we give the following theorem:

Theorem 1: Let us assume that the curve $\gamma(s)$ is a unit speed conformable curve with α −Frenet frame in the plane $\mathbb{R}^2$. So, the location vector of the conformable curve $\gamma(s)$ is

$$\begin{aligned} x = x(s) = &\left\{c_1 \cos\theta + c_2 \sin\theta + \left[(\alpha-1)\int_0^s \varphi^2 \sin\theta s^{\alpha-2} d\theta - \int s^{\alpha-1} d\theta\right]\right. \\ &\left. + \left[(\alpha-1)\int_0^s \varphi^2 \cos\theta s^{\alpha-2} d\theta - \int_0^s s^{\alpha-1} d\theta\right]\right\} e_1^\alpha(s) \\ &+ \{\varphi[l(s) - s^{\alpha-1}]\} e_2^\alpha(s), \end{aligned} \tag{26}$$

where $\varphi = \frac{1}{\mathrm{K}^\alpha}, \theta = \int_0^s d\theta$.

Theorem 2: Let us assume that the curve $\gamma(s)$ is a unit-speed conformable curve with α −Frenet frame in the plane $\mathbb{R}^2$. Then the connection between the curvature of the α −Frenet frame conformable curve $\gamma(s)$ and the location vector can be written as follows

$$\frac{d}{ds}\left(\frac{1}{\mathrm{K}^\alpha}\left((\alpha-1)s^{\alpha-2}\frac{d\gamma}{ds} + s^{\alpha-1}\frac{d^2\gamma}{ds}\right) + \mathrm{K}^\alpha s^{\alpha-1}\frac{d\gamma}{ds} = 0\right). \tag{27}$$

Proof: Let us think $\gamma(s)$ be a unit speed conformable curve with α −Frenet frame in the plane $\mathbb{R}^2$. Then α −Frenet frame is provided by the following equations:

$$\frac{de_1^\alpha(s)}{ds^\alpha} = \mathrm{K}^\alpha e_2^\alpha(s) \tag{28}$$

and

$$\frac{de_2^\alpha(s)}{ds^\alpha} = -\mathrm{K}^\alpha e_1^\alpha(s). \tag{29}$$

By writting equation (28) in equation (29), we simply get

$$\frac{d}{ds}\left(\frac{1}{\mathrm{K}^\alpha}\frac{de_1^\alpha(s)}{ds^\alpha}\right) + \mathrm{K}^\alpha e_1^\alpha(s) = 0. \tag{30}$$

Besides, $e_1^\alpha(s) = T_\alpha\gamma(s) = s^{\alpha-1}\frac{d\gamma}{ds}$, by writting this expression in equation (30), we can obtain equations as follows:

$$\frac{d}{ds}\left(\frac{1}{\mathrm{K}^\alpha}\frac{d}{ds}\left(s^{\alpha-1}\frac{d\gamma}{ds}\right)\right) + \mathrm{K}^\alpha s^{\alpha-1}\frac{d\gamma}{ds} = 0$$

or

$$\frac{d}{ds}\left(\frac{1}{\mathrm{K}^\alpha}\left((\alpha-1)s^{\alpha-2}\frac{d\gamma}{ds} + s^{\alpha-1}\frac{d^2\gamma}{ds^2}\right) + \mathrm{K}^\alpha s^{\alpha-1}\frac{d\gamma}{ds} = 0\right).$$

As a result, the proof is being completed.

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Conflicts of interest

The authors stated that there is no conflict of interest.

References

[1] Nishimoto K., An essence of Nishimoto's Fractional Calculus (Calculus in the 21st century): Integrations and Differentiations of Arbitrary Order, Descartes Press Company, Koriyama, (1991).

[2] Weilber M., Efficient Numerical Methods for Fractional Differential Equations and their Analytical Background, Ph. D. Thesis, Von der Carl-Friedrich-Gaub-Fakultur Mathematic and Informatik der Te chnis-chen University, 2005.

[3] Khalil, R., Al Harani, M., Yousef A., Sababheh M., A new definition of fractional derivative, *J. Comput and Applied Mathematics*, 264 (2014) 65-70.

[4] Baleanu, D., Vacaru, S., Constant curvature coefficients and exact solutions in fractional gravity and geometric mechanics, *Open Physics*, *9*(5) (2011) 1267-1279.

[5] Baleanu, D., Vacaru, S. I., Fractional almost Kähler–Lagrange geometry, *Nonlinear Dynamics*, *64*(4) 365-373.

[6] Abdeljawad, T., Alzabut, J., Jarad, F., A generalized Lyapunov-type inequality in the frame of conformable derivatives, *Advances in Difference Equations*, 2017(1) 1-10.

[7] Abdeljawad, T., Agarwal, R. P., Alzabut, J., Jarad, F., Özbekler, A., Lyapunov-type inequalities for mixed non-linear forced differential equations within conformable derivatives, *Journal of Inequalities and Applications*, 1 (2018) 1-17.

[8] Atangana, A., Baleanu, D., Alsaedi, A., New properties of conformable derivative, *Open Mathematics*, 13(1) (2015).

[9] Anderson, D. R., Ulness, D. J., Newly Defined Comformable Derivatives Centered Polygonal Lacunary Functions View project Dynamic Equations on Times Scales View Project Newly Defined Comformable Derivatives, Advances in Dynamical Systems and Applications, 10(2) (2015) 109-137.

[10] Aminikhah H., Sheikhani A.H.R., Rezazadeh H., Sub-equation method fort he fractional regularized long-wave equations with comformable fractional derivatives, Sci. Iran, 23 (2016) 1048-1054.

[11] Gözütok U., Çoban H., Sağıroğlu Y., Frenet frame with respect to conformable derivative, *Filomat*., *33* (6) (2019).

[12] Mağden A., Yılmaz S., Ünlütürk Y., Characterizations of special time-like curves in Lorentzian plane $\mathbb{L}^2$, *International Journal of Geometric Methods In Modern Physics.*, *14* (10) (2017).