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Dual-Gaussian Pell and Pell-Lucas Numbers

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ABSTRACT

In this study, we define a new type of Pell and Pell-Lucas numbers which are called dual-Gaussian Pell and dual-Gaussian Pell-Lucas numbers. We also give the relationship between negadual-Gaussian Pell and Pell-Lucas numbers and dual-complex Pell and Pell-Lucas numbers. Also, some sum and product properties of Pell and Pell-Lucas numbers are given. Moreover, we obtain the Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity and some sum formulas for these new type numbers. Some algebraic properties of dual-Gaussian Pell and Pell-Lucas numbers are investigated. Furthermore, we give the matrix representation of dual-Gaussian Pell and Pell-Lucas numbers.

Keywords: Dual-Gaussian numbers, Dual-Gaussian Pell numbers, Dual-Gaussian Pell-Lucas numbers.

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Introduction

Complex numbers, Hyperbolic numbers and Dual numbers arise in many areas such as coordinate transformation, matrix modeling, displacement analysis, rigid body dynamics, velocity analysis, static analysis, dynamic analysis, transformation, mechanics, kinematics, physics, mathematics, and geometry. Horadam [1] introduced the concept, the complex Fibonacci numbers, called the Gaussian Fibonacci numbers $GF_n = F_n + iF_{n-1}$ where $F_n \in \mathbb{R}, i^2 = -1$ and F_n, nth Fibonacci numbers. Fjelstad and Gal [2] defined the hyperbolic numbers $H = h + jh^*$ where $h, h^* \in \mathbb{R}, j^2 = 1$ and $j \neq \pm 1$. Clifford [3] described the dual numbers $D = d + \varepsilon d^*$ where $d, d^* \in \mathbb{R}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. Messelmi [4] expressed the dual-complex numbers $Z = z + \varepsilon z^*$ where $z, z^* \in \mathbb{C}, \varepsilon^2 = 0$ and $\varepsilon \neq 0$. There are several studies in the literature that are concerned with these numbers [5-8].

Fjelstad and Gal [2] inspected the extensions of the hyperbolic complex numbers to *n*-dimensions and they gave *n*-dimensional dual complex numbers in algebra and analysis. Matsuda [9] et al. inspected the two-dimensional rigid transformation which is more concise and efficient than the standard matrix presentation, by modifying the ordinary dual number construction for the complex numbers. Akar et al. [10] introduced arithmetical operations on dual-hyperbolic numbers. They investigated dual hyperbolic number and hyperbolic complex number valued functions. Majernik [11] gave three types of the four-component number systems which are formed by using the complex, binary and dual two-component numbers. Aydın [12] formulated, if $z_1 = x_1 + ix_2$ and $z_2 = y_1 + iy_2$ any dual-complex number by $w = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2$.

Moreover, addition, subtraction, multiplication and division of dual-complex numbers and was defined by

$$w_1 \pm w_2 = (z_1 + \varepsilon z_2) \pm (z_3 + \varepsilon z_4) = (z_1 \pm z_3) + \varepsilon(z_2 \pm z_4)$$

$$w_1 \times w_2 = (z_1 + \varepsilon z_2) \times (z_3 + \varepsilon z_4) = (z_1 z_3) + \varepsilon(z_1 z_4 + z_2 z_3)$$

and

$$\frac{w_1}{w_2} = \frac{z_1 + \varepsilon z_2}{z_3 + \varepsilon z_4} = \frac{(z_1 + \varepsilon z_2)(z_3 - \varepsilon z_4)}{(z_3 + \varepsilon z_4)(z_3 - \varepsilon z_4)} = \frac{z_1}{z_3} + \varepsilon \frac{z_2 z_3 - z_1 z_4}{{z_3}^2}.$$

Table 1. Multiplication scheme of dual-complex numbers

$\times$	$\mathbf{1}$	$\boldsymbol{i}$	$\boldsymbol{\varepsilon}$	$\boldsymbol{i\varepsilon}$
1	1	i	ε	$i\varepsilon$
i	i	-1	$i\varepsilon$	$-\varepsilon$
ε	ε	$i\varepsilon$	0	0
$i\varepsilon$	$i\varepsilon$	$-\varepsilon$	0	0

The conjugations can operate on dual-complex numbers as follows:

$$w = x_1 + ix_2 + \varepsilon y_1 + i\varepsilon y_2$$

$$w^{*1} = (x_1 - ix_2) + (\varepsilon y_1 - i\varepsilon y_2), complex\ conjugation$$

$$w^{*2} = (x_1 + ix_2) - (\varepsilon y_1 + i\varepsilon y_2), dual\ conjugation$$

$$w^{*3} = (x_1 - ix_2) - (\varepsilon y_1 - i\varepsilon y_2), coupled\ conjugation$$

$$w^{*4} = (x_1 - ix_2)\left(1 - \varepsilon \frac{y_1 + iy_2}{x_1 + ix_2}\right), dual - complex\ conjugation$$

$$w^{*5} = (y_1 + iy_2) - (\varepsilon x_1 + i\varepsilon x_2), anti - dual\ conjugation$$

Therefore, the norm of dual-complex numbers is defined as

$$N_w^{*1} = \|w \times w^{*1}\| = \sqrt{|z_1|^2 + 2\varepsilon Re(z_1 z_2^*)}$$

$$N_w^{*2} = \|w \times w^{*2}\| = \sqrt{|z_1|^2}$$

$$N_w^{*3} = \|w \times w^{*3}\| = \sqrt{|z_1|^2 - 2i\varepsilon Im(z_1 z_2^*)}$$

$$N_w^{*4} = \|w \times w^{*4}\| = \sqrt{|z_1|^2}$$

$$N_w^{*5} = \|w \times w^{*5}\| = \sqrt{z_1 z_2 + \varepsilon(z_2^2 - z_1^2)}$$

Beneficial point is the number sequences that have been studied over many years. For $n \in \mathbb{N}_0$, Pell and Pell-Lucas numbers are defined by the recurrence relations, respectively. $P_{n+2} = 2P_{n+1} + P_n$, $P_0 = 0$, $P_1 = 1$ and $Q_{n+2} = 2Q_{n+1} + Q_n$, $Q_0 = 2$, $Q_1 = 2$. Besides the nth Pell and Pell-Lucas number are formulized as $P_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$ and $Q_n = \alpha^n + \beta^n$, where $\alpha = 1 + \sqrt{2}$, $\beta = 1 - \sqrt{2}$. These formulas are called as Binet's formula [13, 14].

Many researchers studied several areas of this number sequence. Halıcı and Çürük [15] examined the dual numbers and investigated the characteristic properties of them. They also gave equations about conjugates and some important features of these newly defined numbers. Azak and Güngör [16] defined the dual complex Fibonacci and Lucas numbers and gave the well-known properties for these numbers. Aydın [17] defined dual-complex k-Pell numbers, dual-complex k-Pell quaternions and also gave some algebraic properties of them.

In the following sections, the dual-Gaussian Pell and the dual-Gaussian Pell-Lucas numbers will be defined. In this work, a variety of algebraic properties of dual-Gaussian Pell and dual-Gaussian Pell-Lucas numbers are presented in a unified manner. Some identities will be given for dual-Gaussian Pell and dual-Gaussian Pell-Lucas numbers such as Binet's formula, generating function, d'Ocagne's identity, Catalan's identity, Cassini's identity, and some sum formulas. The dual-Gaussian Pell and the dual-Gaussian Pell-Lucas numbers' properties will also be obtained using matrix representation.

Dual-Gaussian Pell and Pell-Lucas numbers

Definition 2.1: For $n \in \mathbb{N}_0$, the dual-Gaussian Pell and the dual-Gaussian Pell-Lucas numbers are defined by

$$DGP_{n+3} = P_{n+3} + iP_{n+2} + \varepsilon P_{n+1} + i\varepsilon P_n$$

$$DGQ_{n+3} = Q_{n+3} + iQ_{n+2} + \varepsilon Q_{n+1} + i\varepsilon Q_n$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers. ε, denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), i denotes the imaginary unit ($i^2 = -1$) and $i\varepsilon$ denotes the imaginary dual unit.

$DGP_0 = i - 2\varepsilon + 5i\varepsilon$, $DGP_1 = 1 + \varepsilon - 2i\varepsilon$ and

$DGP_2 = 2 + i + i\varepsilon, \ldots$

$DGQ_0 = 2 - 2i + 6\varepsilon - 14i\varepsilon$, $DGQ_1 = 2 + 2i - 2\varepsilon + 6i\varepsilon$ and $DGQ_2 = 6 + 2i + 2\varepsilon - 2i\varepsilon, \ldots$

Let DGQ_{n+3} and DGQ_{m+3} be two dual-Gaussian Pell-Lucas numbers. The addition, substraction and multiplication of the dual-Gaussian Pell-Lucas numbers are given by

$$DGQ_{n+3} \pm DGQ_{m+3} = (Q_{n+3} \pm Q_{m+3}) + i(Q_{n+2} \pm Q_{m+2}) + \varepsilon(Q_{n+1} \pm Q_{m+1}) + i\varepsilon(Q_n \pm Q_m)$$

$$DGQ_{n+3} \times DGQ_{m+3} = (Q_{n+3}Q_{m+3} - Q_{n+2}Q_{m+2}) + i(Q_{n+3}Q_{m+2} + Q_{n+2}Q_{m+3}) + \varepsilon(Q_{n+3}Q_{m+1} - Q_{n+2}Q_m + Q_{n+1}Q_{m+3} - Q_nQ_{m+2}) + i\varepsilon(Q_{n+3}Q_m + Q_{n+2}Q_{m+1} + Q_{n+1}Q_{m+2} + Q_nQ_{m+3}).$$

There exist five different conjugations. Dual-Gaussian Pell-Lucas numbers can operate as follows:

$$DGQ_{n+3} = Q_{n+3} + iQ_{n+2} + \varepsilon Q_{n+1} + i\varepsilon Q_n$$

$$DGQ_{n+3}^{*1} = (Q_{n+3} - iQ_{n+2}) + (\varepsilon Q_{n+1} - i\varepsilon Q_n), complex\ conjugation$$

$$DGQ_{n+3}^{*2} = (Q_{n+3} + iQ_{n+2}) - (\varepsilon Q_{n+1} + i\varepsilon Q_n), dual\ conjugation$$

$$DGQ_{n+3}^{*3} = (Q_{n+3} - iQ_{n+2}) - (\varepsilon Q_{n+1} - i\varepsilon Q_n), coupled\ conjugation$$

$$DGQ_{n+3}^{*4} = (Q_{n+3} - iQ_{n+2})\left(1 - \varepsilon\frac{Q_{n+1} + iQ_n}{Q_{n+3} + iQ_{n+2}}\right), dual - complex\ conjugation$$

$$DGQ_{n+3}^{*5} = (Q_{n+1} + iQ_n) - (\varepsilon Q_{n+3} + i\varepsilon Q_{n+2}), anti - dual\ conjugation$$

Similarly, the properties for dual-Gaussian Pell numbers are obtained.

Lemma 2.2: Let P_n and Q_n be the Pell and the Pell-Lucas numbers, respectively. The following relations are satisfied

$$Q_{n+1}^2 + Q_n^2 = 8P_{2n+1}$$
$$Q_{n+1}^2 - Q_n^2 = 8P_{2n+1} - 4(-1)^n$$
$$Q_{2n+2} + Q_{2n} = 8P_{2n+1}$$
$$Q_{2n+2} - Q_{2n} = 2Q_{2n+1}$$
$$Q_{n+r}Q_n = Q_{2n+r} + Q_r(-1)^n$$
$$Q_mQ_{n+r} + Q_{m+r}Q_n = 2Q_{m+n+r} + (-1)^nQ_{m-n}Q_r$$
$$Q_mQ_{n+r} - Q_{m+r}Q_n = (-8)(-1)^nP_{m-n}P_r$$

Proof: The proofs are carried out with the help of the Binet's formula.

Proposition 2.3: DGQ_n be a dual-Gaussian Pell-Lucas number. The following properties hold.

$$DGQ_{n+3} + DGQ_{n+3}^{*1} = 2Q_{n+3} + 2\varepsilon Q_{n+1}$$
$$DGQ_{n+3} \times DGQ_{n+3}^{*1} = 8P_{2n+5} + 16\varepsilon P_{2n+3}$$
$$DGQ_{n+3} + DGQ_{n+3}^{*2} = 2Q_{n+3} + 2iQ_{n+2}$$
$$DGQ_{n+3} \times DGQ_{n+3}^{*2} = [8P_{2n+5} - 4(-1)^n] + 2i[Q_{2n+5} + 2(-1)^n]$$
$$DGQ_{n+3} + DGQ_{n+3}^{*3} = 2Q_{n+3} + 2i\varepsilon Q_n$$
$$DGQ_{n+3} \times DGQ_{n+3}^{*3} = 8P_{2n+5} + 32i\varepsilon(-1)^n$$

Similarly, the proposition for dual-Gaussian Pell numbers is obtained.

Definition 2.4: For $n \in \mathbb{N}_0$, DCP_n and DCQ_n the dual-complex Pell and the dual-complex Pell-Lucas numbers, the negadual-Gaussian Pell and the negadual-Gaussian Pell-Lucas numbers are defined by

$$DGP_{-n} = (-1)^{n+1}DGP_n^{*1}$$
$$DGQ_{-n} = (-1)^nDGQ_n^{*1}$$

where P_n and Q_n, are the *nth* Pell and Pell-Lucas numbers. Also, DCP_n and DCQ_n, are the dual-complex Pell and dual-complex Pell-Lucas numbers. ε, denotes the pure dual unit ($\varepsilon^2 = 0, \varepsilon \neq 0$), i denotes the imaginary unit ($i^2 = -1$) and $i\varepsilon$ denotes the imaginary dual unit.

$$DGQ_{-n} = Q_{-n} + iQ_{-n-1} + \varepsilon Q_{-n-2} + i\varepsilon Q_{-n-3}$$

When the equality is established,

$$DGQ_{-n} = (-1)^nQ_n + i(-1)^{n+1}Q_{n+1} + \varepsilon(-1)^{n+2}Q_{n+2} + i\varepsilon(-1)^{n+3}Q_{n+3}$$
$$DGQ_{-n} = (-1)^n[Q_n - iQ_{n+1} + \varepsilon Q_{n+2} - i\varepsilon Q_{n+3}]$$
$$DGQ_{-n} = (-1)^nDCQ_n^{*1}$$

Similarly, DGP_{-n} is found.

Theorem 2.5: Let DGP_n and DGQ_n be the dual-Gaussian Pell and the dual-Gaussian Pell-Lucas numbers, respectively. The following relations are satisfied

$$2(DGP_{n+1} + DGP_n) = DGQ_{n+1}$$
$$2(DGP_{n+1} - DGP_n) = DGQ_n$$
$$DGP_{n+1} + DGP_{n-1} = DGQ_n$$
$$DGP_{n+1} - DGP_{n-1} = 2DGP_n$$
$$DGP_{n+2} + DGP_{n-2} = 6DGP_n$$
$$DGP_{n+2} - DGP_{n-2} = 2DGQ_n$$

$DGQ_{n+1} + DGQ_n = 4DGP_{n+1}$

$DGQ_{n+1} - DGQ_n = 4DGP_n$

$DGQ_{n+1} + DGQ_{n-1} = 8DGP_n$

$DGQ_{n+1} - DGQ_{n-1} = 2DGQ_n$

$DGQ_{n+2} + DGQ_{n-2} = 6\,DGQ_n$

$DGQ_{n+2} - DGQ_{n-2} = 16\,DGP_n$

Proof:

$$\begin{aligned}2(DGP_{n+1} + DGP_n) &= 2(P_{n+1} + iP_n + \varepsilon P_{n-1} + i\varepsilon P_{n-2} + P_n + iP_{n-1} + \varepsilon P_{n-2} + i\varepsilon P_{n-3})\\ &= 2(P_{n+1} + P_n) + 2i(P_n + P_{n-1}) + 2\varepsilon(P_{n-1} + P_{n-2}) + 2i\varepsilon(P_{n-2} + P_{n-3})\\ &= Q_{n+1} + iQ_n + \varepsilon Q_{n-1} + i\varepsilon Q_{n-2} = DGQ_{n+1}\end{aligned}$$

The other steps of the theorem can be proved by a similar method.

Theorem 2.6: (Generating Function Formula) Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. Generating function formula for this numbers is as follows

$$m(t) = \frac{(i - 2\varepsilon + 5i\varepsilon) + t(1 - 2i + 5\varepsilon - 12i\varepsilon)}{1 - 2t - t^2}$$

$$h(t) = \frac{(2 - 2i + 6\varepsilon - 14i\varepsilon) + t(-2 + 6i - 14\varepsilon + 34i\varepsilon)}{1 - 2t - t^2}.$$

Proof: Let $h(t)$ be the generating function for dual-Gaussian Pell-Lucas numbers as

$h(t) = \sum_{n=0}^{\infty} DGQ_n t^n$. Using $h(t)$, $2th(t)$ and $t^2h(t)$, we get the following equations,

$th(t) = \sum_{n=0}^{\infty} DGQ_n t^{n+1}$, $t^2h(t) = \sum_{n=0}^{\infty} DGQ_n t^{n+2}$. After the needed calculations, the generating function for dual-

Gaussian Pell-Lucas numbers is obtained as

$$h(t) = \frac{DGQ_0 + DGQ_1 t - 2DGQ_0 t}{1 - 2t - t^2}$$

$$h(t) = \frac{(2 - 2i + 6\varepsilon - 14i\varepsilon) + t(-2 + 6i - 14\varepsilon + 34i\varepsilon)}{1 - 2t - t^2}.$$

Similarly, generating function formula for dual-Gaussian Pell numbers is obtained.

Theorem 2.7: (Binet's Formula) Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. Binet's formula for this number is as follows

$$DGP_n = \frac{\hat{\alpha}\alpha^{n-3} - \hat{\beta}\beta^{n-3}}{\alpha - \beta}$$

$$DGQ_n = \hat{\alpha}\alpha^{n-3} + \hat{\beta}\beta^{n-3}$$

where $\hat{\alpha} = \alpha^3 + i\alpha^2 + \varepsilon\alpha^1 + i\varepsilon$, $\alpha = 1 + \sqrt{2}$ and $\hat{\beta} = \beta^3 + i\beta^2 + \varepsilon\beta^1 + i\varepsilon$, $\beta = 1 - \sqrt{2}$.

Proof:

$$\begin{aligned}DGQ_n &= Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + i\varepsilon Q_{n-3}\\ &= (\alpha^n + \beta^n) + i(\alpha^{n-1} + \beta^{n-1}) + \varepsilon(\alpha^{n-2} + \beta^{n-2}) + i\varepsilon(\alpha^{n-3} + \beta^{n-3})\\ &= \alpha^{n-3}(\alpha^3 + i\alpha^2 + \varepsilon\alpha^1 + i\varepsilon) + \beta^{n-3}(\beta^3 + i\beta^2 + \varepsilon\beta^1 + i\varepsilon)\end{aligned}$$

$$DGQ_n = \hat{\alpha}\alpha^{n-3} + \hat{\beta}\beta^{n-3}.$$

Similarly, Binet's formula for dual-Gaussian Pell numbers is obtained.

Theorem 2.8: (d'Ocagne's Identity) Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. d'Ocagne's identity for this number is as follows

$$DGP_m\,DGP_{n+1} - DGP_{m+1}\,DGP_n = 8(-1)^{n+1}P_{m-n} - 2i(-1)^n P_{m-n} - \varepsilon[6(-1)^n(P_{m-n}) + (-1)^n(P_{m-n-2} + P_{m-n+2})] - 12i\varepsilon(-1)^n P_{m-n}$$

$$DGQ_m\,DGQ_{n+1} - DGQ_{m+1}\,DGQ_n = 8[(-1)^n P_{m-n+1} + (-1)^m P_{n-m}] + 16i(-1)^n P_{m-n}$$
$$+8\varepsilon[(-1)^m(P_{n-m-2} + P_{n-m+2}) - (-1)^n(P_{m-n-2} + P_{m-n+2})]$$

Proof:

$$DGQ_m\,DGQ_{n+1} - DGQ_{m+1}\,DGQ_n = (Q_m + iQ_{m-1} + \varepsilon Q_{m-2} + i\varepsilon Q_{m-3})(Q_{n+1} + iQ_n + \varepsilon Q_{n-1} + i\varepsilon Q_{n-2}) - (Q_{m+1} + iQ_m + \varepsilon Q_{m-1} + i\varepsilon Q_{m-2})(Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + i\varepsilon Q_{n-3}) = 8[(-1)^n P_{m-n+1} + (-1)^m P_{n-m}] + 16i(-1)^n P_{m-n} + 8\varepsilon[(-1)^m(P_{n-m-2} + P_{n-m+2}) - (-1)^n(P_{m-n-2} + P_{m-n+2})]$$

Similarly, d'Ocagne's identity for dual-Gaussian Pell numbers is obtained.

Theorem 2.9: (Catalan's Identity) Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. Catalan's identity for this number is as follows

$$DGP_n^2 - DGP_{n+r}\,DGP_{n-r} = P_r^2[(-1)^{n-r} + (-1)^{n+r}] + iP_r(-1)^{n-r}[P_{r-1} - P_{r+1}]$$
$$+\varepsilon P_r[(-1)^{n-1}(P_{-r-2} + P_{-r+2}) + (-1)^{n-r}(P_{r+2} + P_{-r+2})]$$
$$+i\varepsilon P_r[(-1)^{n-r-1}(P_{r+3} + P_{r-1}) + (-1)^{n-r}(P_{r-3} + P_{r+1})]$$

$$DGQ_n^2 - DGQ_{n+r}\,DGQ_{n-r} = 4(-1)^n[2P_r^2 - P_r] + 16i(-1)^{n-r}[P_r^2]$$
$$-8\varepsilon P_r[(-1)^{n-r}(P_{r+2} + P_{r-2}) + (-1)^{n-1}(P_{-r+2} + P_{-r-2})]$$
$$+8i\varepsilon(-1)^{n-r}P_r[(P_{r+3} + P_{r-1}) - (P_{r-3} + P_{r+1})]$$

Proof:

$$\begin{aligned}
DGQ_n^2 - DGQ_{n+r}\,DGQ_{n-r}
&= (Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + i\varepsilon Q_{n-3})(Q_n + iQ_{n-1} + \varepsilon Q_{n-2} + i\varepsilon Q_{n-3})\\
&\quad - (Q_{n+r} + iQ_{n+r-1} + \varepsilon Q_{n+r-2} + i\varepsilon Q_{n+r-3})(Q_{n-r} + iQ_{n-r-1} + \varepsilon Q_{n-r-2} + i\varepsilon Q_{n-r-3})\\
&= 4(-1)^n[2P_r^2 - P_r] + 16i(-1)^{n-r}[P_r^2]\\
&\quad - 8\varepsilon P_r[(-1)^{n-r}(P_{r+2} + P_{r-2}) + (-1)^{n-1}(P_{-r+2} + P_{-r-2})]\\
&\quad + 8i\varepsilon(-1)^{n-r}P_r[(P_{r+3} + P_{r-1}) - (P_{r-3} + P_{r+1})]
\end{aligned}$$

Similarly, Catalan's identity for dual-Gaussian Pell numbers is obtained.

Theorem 2.10: (Cassini's Identity) Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. Cassini's identity for this number is as follows

$$DGP_n^2 - DGP_{n+1}\,DGP_{n-1} = -2(-1)^n + 2i(-1)^n - 12\varepsilon(-1)^n + 12i\varepsilon(-1)^n$$
$$DGQ_n^2 - DGQ_{n+1}\,DGQ_{n-1} = 4(-1)^n - 16i(-1)^n + 96(-1)^n\varepsilon - 96i\varepsilon(-1)^n$$

Proof: If $r = 1$ is taken in the Catalan's identity, Cassini's identity is obtained. Similarly, Cassini's identity for dual-Gaussian Pell numbers is obtained.

Theorem 2.11: Let DGP_n and DGQ_n be the dual-Gaussian Pell and Pell-Lucas numbers. In this case

$$\sum_{k=1}^{n} DGP_k = \left(\frac{Q_{n+1}-1}{2}\right) + i\left(\frac{Q_n-1}{2}\right) + \varepsilon\left(\frac{Q_{n-1}+1}{2}\right) + i\varepsilon\left(\frac{Q_{n-2}-3}{2}\right)$$
$$\sum_{k=1}^{n} DGP_{2k-1} = \left(\frac{P_{2n}}{2}\right) + i\left(\frac{P_{2n-1}-1}{2}\right) + \varepsilon\left(\frac{P_{2n-2}+2}{2}\right) + i\varepsilon\left(\frac{P_{2n-3}-5}{2}\right)$$
$$\sum_{k=1}^{n} DGP_{2k} = \left(\frac{P_{2n+1}-1}{2}\right) + i\left(\frac{P_{2n}}{2}\right) + \varepsilon\left(\frac{P_{2n-1}-1}{2}\right) + i\varepsilon\left(\frac{P_{2n-2}+2}{2}\right)$$
$$\sum_{k=1}^{n} DGQ_k = (2P_{n+1} - 2) + i(2P_n) + \varepsilon(2P_{n-1} - 2) + i\varepsilon(2P_{n-2} + 4)$$
$$\sum_{k=1}^{n} DGQ_{2k-1} = \left(\frac{Q_{2n}-1}{2}\right) + i\left(\frac{Q_{2n-1}+3}{2}\right) + \varepsilon\left(\frac{Q_{2n-2}-5}{2}\right) + i\varepsilon\left(\frac{Q_{2n-3}+15}{2}\right)$$
$$\sum_{k=1}^{n} DGQ_{2k} = \left(\frac{Q_{2n+1}-1}{2}\right) + i\left(\frac{Q_{2n}-1}{2}\right) + \varepsilon\left(\frac{Q_{2n-1}+3}{2}\right) + i\varepsilon\left(\frac{Q_{2n-2}-5}{2}\right)$$

Proof:

$$\sum_{k=1}^{n} DGQ_k = \sum_{k=1}^{n}(Q_k + iQ_{k-1} + \varepsilon Q_{k-2} + i\varepsilon Q_{k-3})$$

$$= \sum_{k=1}^{n} Q_k + i\sum_{k=0}^{n-1} Q_k + \varepsilon\sum_{k=-1}^{n-2} Q_k + i\varepsilon\sum_{k=-2}^{n-3} Q_k$$
$$= (2P_{n+1} - 2) + i(2P_n) + \varepsilon(2P_{n-1} - 2) + i\varepsilon(2P_{n-2} + 4)$$

Other sums are proven through the same method. Similarly, Sums are proven for dual-Gaussian Pell numbers is obtained.

Theorem 2.12: Let DGP_n and DGQ_n be the dual-Gaussian Pell-Lucas numbers. For $n \geq 1$be integer. Then, the matrix representations of these sequences with both negative and positive indices are as follows

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} DGP_2 & DGP_1 \\ DGP_1 & DGP_0 \end{bmatrix} = \begin{bmatrix} DGP_{n+2} & DGP_{n+1} \\ DGP_{n+1} & DGP_n \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} DGP_0 \\ DGP_1 \end{bmatrix} = \begin{bmatrix} DGP_n \\ DGP_{n+1} \end{bmatrix}$$
$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} DGQ_2 & DGQ_1 \\ DGQ_1 & DGQ_0 \end{bmatrix} = \begin{bmatrix} DGQ_{n+2} & DGQ_{n+1} \\ DGQ_{n+1} & DGQ_n \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}^n \begin{bmatrix} DGQ_0 \\ DGQ_1 \end{bmatrix} = \begin{bmatrix} DGQ_n \\ DGQ_{n+1} \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} DGP_2 & DGP_1 \\ DGP_1 & DGP_0 \end{bmatrix} = \begin{bmatrix} DGP_{-n+2} & DGP_{-n+1} \\ DGP_{-n+1} & DGP_{-n} \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} DGP_0 \\ DGP_1 \end{bmatrix} = \begin{bmatrix} DGP_{-n} \\ DGP_{-n-1} \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} DGQ_2 & DGQ_1 \\ DGQ_1 & DGQ_0 \end{bmatrix} = \begin{bmatrix} DGQ_{-n+2} & DGQ_{-n+1} \\ DGQ_{-n+1} & DGQ_{-n} \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}^n \begin{bmatrix} DGQ_0 \\ DGQ_1 \end{bmatrix} = \begin{bmatrix} DGQ_{-n} \\ DGQ_{-n-1} \end{bmatrix}$$

Proof:

For the prove, we utilize induction principle on *n*. The equality holds for *n* = 1. Now assume that the equality is true for *n* >1. Then, we can verify for $n + 1$ as follows

$$\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^{n+1} \begin{bmatrix} DGP_2 & DGP_1 \\ DGP_1 & DGP_0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}^n \begin{bmatrix} DGP_2 & DGP_1 \\ DGP_1 & DGP_0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} DGP_{n+2} & DGP_{n+1} \\ DGP_{n+1} & DGP_n \end{bmatrix} = \begin{bmatrix} DGP_{n+3} & DGP_{n+2} \\ DGP_{n+2} & DGP_{n+1} \end{bmatrix}$$

Thus, the first step of the theorem can be proved easily. Similarly, the other steps of the proof are seen by induction on *n*.

Conclusions

This study presents the dual-Gaussian Pell-Lucas and Pell numbers. We obtained these new numbers not defined in the literature before. These number sequences have great importance as they are used in quantum physics, applied mathematics, kinematic, differential equations and cryptology. Since this study includes some new results, it contributes to literature by providing essential information concerning these new numbers. Therefore, we hope that this new number system and properties that we have found will offer a new perspective to the researchers.

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Conflicts of interest

There are no conflicts of interest in this work.

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