Properties of $J_p$-Statistical Convergence

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ABSTRACT

In this study, different characterizations of $J_p$-statistically convergent sequences are given. The main features of $J_p$-statistically convergent sequences are investigated and the relationship between $J_p$-statistically convergent sequences and $J_p$-statistically Cauchy sequences is examined. The properties provided by the set of bounded and $J_p$ statistical convergent sequences is shown. It is given that the statistical limit is unique. Furthermore, a sequence that $J_p$-statistically converges to the number $L$ has a subsequence that converges to the same number of $L$, is shown. The analogs of $J_p$ statistical convergent sequences is studied.

Keywords: Power series method, $J_p$-statistical convergence, $J_p$-statistical Cauchy

Introduction

Statistical convergence is a generalization of the concept of convergence in the Cauchy sense. The idea of statistical convergence was introduced under the name of "almost convergence" in the first edition [1] of Zygmund's monograph, published in 1935. The term "statistical convergence" was used by Fast [2] and Steinhaus [3] independently of each other. Also, statistical convergence was studied by Buck [4] in 1953 with the expression of "convergence in density".

Fridy [5] introduced the concept of the statistical Cauchy sequence and presented a characterization of statistical convergence without needing to know the statistical limit. Statistical convergence was considered as a regular summability method, and it was discussed in Schoenberg [6], Connor [7] and [8].

Although statistical convergence is a new field of study, it has become an active area of research in recent years (see Belen et al [9], [10], Burgin and Duman [11], Connor and Kline [12], Çakalli and Khan [13], Et and Şengül [14], Freedman and Sember [15], Miller [16], Salat [17], Savaş and Mohiuddine [18]). Many researchers have done and still do studies on statistical convergence ([19], [20], [21], [22]).

Ünver [23] defined the new density concept using the Abel method and presented a definition of a new version of statistical convergence via this density. Ünver and Orhan [24] gave a new density concept according to the power series method and the definitions of $F_p$-statistical convergence and strong $F_p$-convergence via this density. In the study, they gave a Krovkin-type approximation theorem. Belen et al. [25] defined the concepts of $F_p$-convergence respect to a power series method and strong $F_p$-convergence via a modulus function $f$. They examined the relationship between them. In addition, in the study, the concepts of $J_p$-statistical convergence and $f$-$J_p$-statistical convergence were given and the relationships between them were examined.

Now, let us remind the basic concepts used in this study.

Let $E \subseteq \mathbb{N}_0$, $E(n) = \{k \leq n : k \in E\}$ and $|E(n)|$ denote the cardinality of the set $E(n)$. If the limit $\delta(E) = \lim_{n \to \infty} \frac{|E(n)|}{n+1}$ exists, then the set $E \subseteq \mathbb{N}_0$ is said to have the usual density $\delta(E)$ [4]. The real number sequence $x = (x_k)$ is said to be statistically convergent to the number $L$, if the limit $\lim_{n \to \infty} \frac{1}{n+1} |\{k \leq n : |x_k - L| \geq \varepsilon\}| = 0$ for each $\varepsilon > 0$; i.e., $\delta(E_\varepsilon) = 0$ where $E_\varepsilon = \{k \leq n : |x_k - L| \geq \varepsilon\}$ and denoted by st-$\lim x = L$ [5].

Now let’s introduce the $J_p$-convergence given in Boss [26].

Let $\mathbb{N}_0$ be the set of non-negative integers. Let $(P_k)_{k \in \mathbb{N}_0}$ be a sequence of non-negative integers where $p_k > 0$, satisfying

$$P_n = \sum_{k=1}^{n} p_k \to \infty, (n \to \infty) \quad (1)$$

and

$$p(t) = \sum_{k=1}^{n} p_k t^k < \infty, \text{ (for } 0 < t < 1) \quad (2)$$

(2) (In other words, $p(t)$ has radius of convergence $R = 1$).

Let $x = (x_k)_{k \in \mathbb{N}_0}$ be a sequence of real numbers. In this case, the power series method $J_p$ is defined as follows:

If for every $0 < t < 1$, $p_t(x) = \sum_{k=1}^{n} p_k t^k x_k$ converges and $\lim_{t \to 1} \frac{p_t(x)}{p(t)} = L$, then $(x_k)$ is called $J_p$-convergent to
L, the sequence via the power series method and it is denoted as $x_k \to L$ ($J_p$). If $x_k \to L$ ($J_p$) as $x_k \to L$, the $J_p$-method is called regular. It is known that condition (1) or, equivalently, condition $p(t) \to \infty$ when $t \to 1^-$ guarantees the regularity of method $J_p$ (see, [4]). Therefore, assuming (1), we will consider only regular $J_p$-methods.

Let $E \subseteq \mathbb{N}_0$ be any set. If $\delta_{J_p}(E) = \lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E} p_k^r k^r = 0$ exists, then $\delta_{J_p}(E)$ is called the $J_p$-density of the set $E$. If $\lim_{t \to 1^-} \frac{1}{p(t)} \sum_{k \in E} p_k^r k^r = 0$ for every $\varepsilon > 0$, i.e., $\delta_{J_p}(E_\varepsilon) = 0$, then the number $L$ of the sequence $x = (x_k)$ is said to be $J_p$-statistically convergent. The set of all $J_p$-statistically convergent sequences will be denoted by $st_{J_p}$ [24].

In this study, some expected properties of the $J_p$-statistical convergent sequence space are examined.

### Main Results

In this section, we prove that if a sequence $x = (x_k)$ is $J_p$-statistical convergent then there is a subsequence of $x = (x_k)$ which is convergence to the same number in ordinary sense. Also, we show that the $J_p$-statistical limit is unique, and we give the relationship between $J_p$-statistical Cauchy sequences and $J_p$-statistical convergent sequences.

**Theorem 2.1** A real sequence $x = (x_k)$ is $J_p$-statistical convergent to a number $\ell$ if and only if there exists a subset $K = \{k \in \mathbb{N} : k = 1,2,\ldots\}$ such that $\delta_{J_p}(K) = 1$ and

$$\lim_{k \to \infty} x_k = \ell$$

Proof. Necessity. Let $x = (x_k)$ be $J_p$-statistical convergent to $\ell$.

$$K_+: = \{k \in \mathbb{N} : |x_k - \ell| \geq \frac{1}{r} \}$$

and

$$M_r := \{k \in \mathbb{N} : |x_k - \ell| < \frac{1}{r}, r = 1,2,\ldots\}$$

In this case, we get $\delta_{J_p}(K) = 0$ and

$$M_1 \supseteq M_2 \supseteq \cdots \supseteq M_i \supseteq M_{i+1} \supseteq \cdots \quad (3)$$

$$\delta_{J_p}(M_r) = 1. \quad (4)$$

Now, we have to show that $(x_k)$ converges to $\ell$ for $k \in M_r$. Assume that $(x_k)$ is not convergent to $\ell$. In this case, there is an $\varepsilon > 0$ for the infinitely many terms, such that $|x_k - \ell| \geq \varepsilon$.

Define $M_\varepsilon = \{k : |x_k - \ell| < \varepsilon\}$ and $\varepsilon > \frac{1}{r} (r = 1,2,\ldots)$. Hence

$$\delta_{J_p}(M_\varepsilon) = 0$$

and $M_r \subseteq M_\varepsilon$ from (3). So we have $\delta_{J_p}(M_r) = 0$, which is a contradiction with (4). Then $(x_k)$ is convergent to $\ell$.

Sufficiency. Suppose that there is a subset $K = \{k \in \mathbb{N} : k = 1,2,\ldots\}$ such that $\delta_{J_p}(K) = 1$ and

$$\lim_{k \to \infty} x_k = \ell$$

Therefore, for every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $|x_k - \ell| < \varepsilon, \forall k \geq N$ and $k \in K$.

Since $K_\varepsilon = \{k : |x_k - \ell| \geq \varepsilon\} \subseteq \mathbb{N} - \{k_{N+j} : j \in \mathbb{N} \text{ and } k_{N+j} \in K\}$

we have $\delta_{J_p}(K_\varepsilon) \leq 1 - 1 = 0$.

Thus, $x = (x_k)$ is statistically convergent to $\ell$.

**Theorem 2.2** Let the sequence $x = (x_k)$ be $J_p$-statistical convergent to a number $L$. In this case, there is a sequence $y$ that converges to the number $L$ and a sequence $z$ such that $x = y + z$. Now, we should show that $x_k \to L$ for $k \in \mathbb{N}_0$ and $n \geq N_j (j = 1,2,\ldots)$, we can find an increasing sequence of positive numbers $(N_j)$ such that $\delta_{J_p}(E_j) < \frac{1}{j}$. Now let’s define the $y$ and $z$ sequences as follows. Take $z_k = 0$ and $y_k = x_k$ when $N_0 < k \leq N_j$. For $\frac{1}{j} \geq 1$, let $N_j < k \leq N_{j+1}$, $z_k = 0$ and $y_k = x_k$ when $|x_k - L| < \frac{1}{j}$ and finally, when $|x_k - L| \geq \frac{1}{j}$, let $z_k = x_k - L$ and $y_k = L$. It is clear that we can write $x = y + z$. Now, we claim that the sequence $y$ is convergent to $L$. Let $\varepsilon > 0$ be given, let us choose $j$ such that $\varepsilon > \frac{1}{j}$ for $k \leq N_j$, if $|x_k - L| \geq \frac{1}{j}$ then $|y_k - L| = |L - L| = 0$

and if $|x_k - L| < \frac{1}{j}$ then $|y_k - L| = |x_k - L| < \frac{1}{j} < \varepsilon$

so $\lim_{k} x_k = L$ is obtained. Now, let us see $\lim_{k} x_k - \lim_{k} z_k = 0$. We should show that
\[
\lim_{t \to 1} \frac{1}{p(t)} \sum_{k \in E_{x}} p_{k} k^{t} = 0
\]
for \( E_{x} = \{ k \leq n : z_{k} \neq 0 \} \). Since
\[
\{ k \leq n : |z_{k}| \geq \varepsilon \} \subset \{ k \leq n : z_{k} \neq 0 \}
\]
for every \( \varepsilon > 0 \), we have
\[
\delta_{\text{Ip}}(\{ k \leq n : |z_{k}| \geq \varepsilon \}) \leq \delta_{\text{Ip}}(\{ k \leq n : z_{k} \neq 0 \}).
\]
Now if \( \delta > 0 \), \( j \in \mathbb{N} \) and \( \frac{1}{j} < \delta \) we have to show that
\[
\delta_{\text{Ip}}(\{ k \leq n : z_{k} \neq 0 \}) < \delta
\]
every \( n > N_{j} \). Let \( N_{j} < k \leq N_{j+1} \) then \( z_{k} \neq 0 \) is possible only with \( |x_{k} - L| \geq \frac{1}{j} \). So if \( N_{j} < k \leq N_{j+1} \) then
\[
\{ k \leq n : z_{k} \neq 0 \} = \{ k \leq n : |x_{k} - L| \geq \frac{1}{j} \}.
\]
Therefore, if \( N_{v} < k \leq N_{v+1} \) and \( v > j \) implies that
\[
\delta_{\text{Ip}}(\{ k \leq n : z_{k} \neq 0 \}) \leq \delta_{\text{Ip}}(\{ k \leq n : |x_{k} - L| \geq \frac{1}{v} \}) < \frac{1}{v} < \frac{1}{j} < \delta.
\]
Thus, the proof is complete.

**Corollary 2.1** If the sequence \( x = (x_{k}) \) is \( I_{p} \)-statistically convergent to the number \( L \), then \( \exists \{ n_{k} \} \subset (n_{n}) \ni x_{n_{k}} \to L \).

**Theorem 2.3** If \( x = (x_{k}) \) be a sequence such that \( \text{st}_{\text{Ip}} - \lim x = L \) and \( \text{st}_{\text{Ip}} - \lim x = K \). Let us choose \( L < K \). If we choose \( \varepsilon = \frac{K - L}{2} \) then
\[
(L - \varepsilon, L + \varepsilon) \cap (K - \varepsilon, K + \varepsilon) = \emptyset.
\]
Also, since \( \text{st}_{\text{Ip}} - \lim x = L \) and \( \text{st}_{\text{Ip}} - \lim x = K \)
\[
\delta_{\text{Ip}}(\{ k \leq n : |x_{k} - L| \geq \varepsilon \}) = 0
\]
\[
\delta_{\text{Ip}}(\{ k \leq n : |x_{k} - K| \geq \varepsilon \}) = 0
\]
then
\[
\delta_{\text{Ip}}(\{ k \leq n : |x_{k} - L| < \varepsilon \}) = 1
\]
\[
\delta_{\text{Ip}}(\{ k \leq n : |x_{k} - K| < \varepsilon \}) = 1.
\]
Hence, we get \( \{ k \leq n : |x_{k} - L| < \varepsilon \} \cap \{ k \leq n : |x_{k} - K| < \varepsilon \} = \emptyset \). This is a contradiction, as the sets are disjoint. Hence the theorem is proved.

The following theorem shows that the statistical convergence method is linear.

**Theorem 2.4** Let \( x = (x_{k}) \) and \( y = (y_{k}) \) be two real sequences.
(i) \( \text{st}_{\text{Ip}} - \lim x + y = L_{1} + L_{2} \) implies \( \text{st}_{\text{Ip}} - \lim x + y = L_{1} + L_{2} \).
(ii) \( \text{st}_{\text{Ip}} - \lim x = L_{1} \) and \( \alpha \in \mathbb{R} \) implies \( \text{st}_{\text{Ip}} - \lim (\alpha x) = \alpha L_{1} \).

**Proof.** (i) Let \( \text{st}_{\text{Ip}} - \lim x = L_{1} \) and \( \text{st}_{\text{Ip}} - \lim y = L_{2} \). For the set \( A_{1} = \{ k \leq n : |x_{k} - L_{1}| \geq \frac{\varepsilon}{2} \} \) since \( \delta_{\text{Ip}}(A_{1}) = 0 \), there is \( k \in \mathbb{N} \) such that \( |x_{k} - L_{1}| < \frac{\varepsilon}{2} \) for every \( k > k_{1} \) and \( k \in (\mathbb{N} - A_{1}) \) when \( \varepsilon > 0 \). For the set \( A_{2} = \{ k \leq n : |y_{k} - L_{2}| \geq \frac{\varepsilon}{2} \} \) since \( \delta_{\text{Ip}}(A_{2}) = 0 \), there is \( k_{2} \in \mathbb{N} \) such that \( |y_{k} - L_{2}| < \frac{\varepsilon}{2} \) for every \( k > k_{2} \) and \( k \in (\mathbb{N} - A_{2}) \) when \( \varepsilon > 0 \). Let define \( k_{0} = \max(k_{1}, k_{2}) \). Show \( |x_{k} + y_{k} - L_{1} - L_{2}| < \varepsilon \) for every \( k > k_{0} \) and \( k \in (\mathbb{N} - (A_{1} \cup A_{2})) \) and every \( k > k_{0} \). Since \( \delta_{\text{Ip}}(A_{1}) = 0 \) and \( \delta_{\text{Ip}}(A_{2}) = 0 \), then \( \delta_{\text{Ip}}(A_{1} \cup A_{2}) = 0 \). In that case for \( k > k_{0} \)
\[
|x_{k} + y_{k} - L_{1} - L_{2}| < |x_{k} - L_{1}| + |y_{k} - L_{2}|
\]
\[
\frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\]
and for every \( \varepsilon > 0 \)
\[
\delta_{\text{Ip}}(\{ k \leq n : |x_{k} + y_{k} - L_{1} - L_{2}| \geq \varepsilon \}) = 0.
\]
This gives \( \text{st}_{\text{Ip}} - \lim x + y = L_{1} + L_{2} \).

(ii) If \( \alpha = 0 \), we have nothing to prove. Let us assume that \( \alpha \neq 0 \).
\[
\delta_{\text{Ip}}(\{ k \leq n : |\alpha x_{k} - \alpha L_{1}| \geq \varepsilon \}) = \delta_{\text{Ip}}(\{ k \leq n : |\alpha| |x_{k} - L_{1}| \geq \varepsilon \})
\]
\[
\leq \delta_{\text{Ip}}(\{ k \leq n : |x_{k} - L_{1}| \geq \frac{\varepsilon}{|\alpha|} \}) = 0.
\]
So \( \text{st}_{\text{Ip}} - \lim (\alpha x) = \alpha L_{1} \) is obtained.

**Theorem 2.5** The space \( \text{st}_{\text{Ip}} \cap \ell_{\infty} \) is a closed subspace of the normed space \( \ell_{\infty} \).

**Proof.** Let \( x^{(n)} \in \text{st}_{\text{Ip}} \cap \ell_{\infty} \) and \( x^{(n)} \to x \in \ell_{\infty} \). Since \( x_{k} \in \text{st}_{\text{Ip}} \cap \ell_{\infty} \) there are real numbers \( a_{n} \) such that
\[
\text{st}_{\text{Ip}} - \lim x_{k}^{(n)} = a_{n} (n = 1, 2, \ldots).
\]
Since \( x^{(n)} \to x \), for every \( \varepsilon > 0 \), there is a number \( N = N(\varepsilon) \in \mathbb{N} \) such that
\[
|x^{(n)} - x^{(n)}| < \varepsilon / 3
\]
where \( p \geq n \geq N \). Here, \( || \) denotes the norm in a vector space. From Theorem 2.1, \( \mathbb{N} \) has a subset of \( K_{1} \) with \( \delta_{\text{Ip}}(K_{1}) = 1 \) and
\[
\lim_{k \to \infty} x^{(n)}_k = a_n. \quad (7)
\]

Since \( \delta_{J_p}(K_2) = 1 \), let us take \( k_1 \in K_1 \). From (7),
\[
| x^{(p)}_{k_1} - a_p | < \varepsilon / 3. \quad (8)
\]

Thus, for every \( p \geq n \geq N \) from (6), we have
\[
| a_p - a_n | \leq | a_p - x^{(p)}_{k_1} | + | x^{(p)}_{k_1} - x^{(n)}_{k_1} | + | x^{(n)}_{k_1} - a_n | \\
< \varepsilon / 3 + \varepsilon / 3 + \varepsilon / 3 = \varepsilon.
\]

Therefore \((a_n)\) is a Cauchy sequence and hence \((a_n)\) is convergent.

\[
limit_{n} a_n = a. \quad (9)
\]

We should show that \( x \) is \( J_p \)-statistical convergence to \( a \).

Since \( x^{(n)} \to x \), for every \( \varepsilon > 0 \), there is a \( N_1(\varepsilon) \) such that
\[
|x^{(n)} - x| < \varepsilon / 3
\]

where every \( j \geq N_1(\varepsilon) \). Also, from (9), for every \( \varepsilon > 0 \) there is a \( N_2(\varepsilon) \) \( \in \mathbb{N} \) such that
\[
|a_j - a| < \varepsilon / 3
\]

where every \( j \geq N_2(\varepsilon) \). Again, since \( st_{J_p}\lim x^{(n)} = a_m \), there is a set \( K \subseteq \mathbb{N} \) with \( \delta_{J_p}(K) = 1 \) and \( N_3(\varepsilon) \) \( \in \mathbb{N} \) for every \( \varepsilon > 0 \) such that
\[
|x^{(n)} - a_n| < \varepsilon / 3
\]

when \( j \in K \) and all \( j \geq N_3(\varepsilon) \). Let us say \( \max(N_1(\varepsilon), N_2(\varepsilon), N_3(\varepsilon)) = N_1(\varepsilon) \). In this case
\[
|x_j - a| \leq |x^{(n)}_j - x| + |x^{(n)}_j - a_n| + |a_n - a| \\
< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon
\]

is obtained for a given \( \varepsilon > 0 \) and all \( j \geq N_4(\varepsilon), j \in K \).

Therefore \( st_{J_p}\lim x = a \), i.e., \( x \in st_{J_p} \cap \ell_{\infty} \). So \( st_{J_p} \cap \ell_{\infty} \)

is a closed subspace of \( \ell_{\infty} \).

**Theorem 2.6** The space \( st_{J_p} \cap \ell_{\infty} \) is nowhere dense in \( \ell_{\infty} \).

**Proof.** Since every closed subspace of an arbitrary normed space \( S \) different from \( S \) is nowhere dense in \( S \) (Neubrun et al. 1968), it is sufficient to show that it is only \( st_{J_p} \cap \ell_{\infty} \neq \ell_{\infty} \). Let
\[
p_k = \begin{cases} 1, & k = n^2, n \in \mathbb{N}_0 \\ 0, & \text{otherwise.} \end{cases}
\]

and
\[
x_k = \begin{cases} 1, & k = n^2, n \in \mathbb{N}_0 \\ 0, & \text{otherwise.} \end{cases}
\]

Then \( x \) is not \( J_p \)-statistical convergent but bounded. Hence, \( st_{J_p} \cap \ell_{\infty} \neq \ell_{\infty} \).

**Definition 2.1** \( x = (x_k) \) is said to be \( J_p \)-statistical Cauchy sequence if for every \( \varepsilon > 0 \) there exists a \( N(\varepsilon) \) \( \in \mathbb{N} \) such that \( \delta_{J_p}(\{k \leq n: |x_k - x_n| < \varepsilon\}) = 1 \).

**Theorem 27** A sequence \( x = (x_k) \) is \( J_p \)-statistical convergent if and only if \( x = (x_k) \) is \( J_p \)-statistical Cauchy.

**Proof.** Let \( (x_k) \) be \( J_p \)-statistical convergent to \( L \). In this case, \( \delta_{J_p}(\{k \leq n: |x_k - L| \geq \varepsilon\}) = 0 \) for every \( \varepsilon > 0 \). Let us choose \( N \) as \( |x_N - L| \geq \varepsilon \) and define the sets as
\[
A_\varepsilon = \{k \leq n: |x_k - x_N| \geq \varepsilon\},
B_\varepsilon = \{k \leq n: |x_k - L| \geq \varepsilon\},
C_\varepsilon = \{k = N \leq n: |x_N - L| \geq \varepsilon\}
\]

In this case, it is clear that \( A_\varepsilon \subseteq B_\varepsilon \cup C_\varepsilon \). From here, \( \delta_{J_p}(A_\varepsilon) \leq \delta_{J_p}(B_\varepsilon) + \delta_{J_p}(C_\varepsilon) = 0 \) is obtained. So \( x \) is \( J_p \)-statistical Cauchy sequence. Conversely, let \( x \) be \( J_p \)-statistical Cauchy, but not \( J_p \)-statistical convergent. In this case, there exists \( N \) such that \( \delta_{J_p}(A_\varepsilon) = 0 \). Therefore,
\[
\delta_{J_p}(\{k \leq n: |x_k - x_N| < \varepsilon\}) = 1.
\]

Specifically, if \( |x_k - L| \leq \varepsilon / 2 \) we can write
\[
|x_k - x_N| \leq 2|x_k - L| < \varepsilon. \quad (10)
\]

Since \( x \) is not \( J_p \)-statistical convergent, \( \delta_{J_p}(B_\varepsilon) = 1 \). That is
\[
\delta_{J_p}(\{k \leq n: |x_k - L| < \varepsilon\}) = 0.
\]

Thus from (10),
\[
\delta_{J_p}(\{k \leq n: |x_k - x_N| < \varepsilon\}) = 0
\]

i.e., \( \delta_{J_p}(A_\varepsilon) = 1 \). This is a contradiction. So \( x \) is \( J_p \)-statistical convergent.

**Conclusion**

In this study, different characterizations of \( J_p \)-statistically convergent sequences are given. The main features of \( J_p \)-statistical convergent sequences are investigated and the relationship between \( J_p \)-statistical convergent sequences and \( J_p \)-statistical Cauchy sequences is examined.

**Acknowledgment**

The authors would like to thank the anonymous reviewers for their suggestions about paper.

**Conflicts of interest**

The author states that did not have conflict of interests.
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