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ROLE OF IDEALS ON σ-TOPOLOGICAL SPACES

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Abstract. In this writeup, we have discussed the role of ideals on σ-topological spaces. Using this idea, we have also studied and discussed two operators $()^{*\sigma}$ and ψ_σ. We have extended this concept to a new generalized set and investigated some basic properties of these concepts using $()^{*\sigma}$ and ψ_σ operators.

1. Introduction

In topological space, the idea of ideal was known by Kuratowski [7] and Vaidyanathswamy [13]. After that, in the ideal topological space, local function was introduced and studied by Vaidyanathswamy. Njåstad [12] has introduced compatability of the topology with the help of an ideal. In [5, 6] Janković and Hamlett introduced further the characteristics of ideal topological spaces and ψ-operator was introduced by them in 1990. A new type of topology from original ideal topological space was also introduced. In this new topological space, a Kuratowski-closure operator was defined using the local function. Also from ψ-operator, they proved that interior operator can be deduced in the new topological space. In 2007, using ψ-operator Modak and Bandhyopadhyay in [8] introduced generalized open sets. The idea of ideal m-space was introduced by Al-Omari and Noiri in [1, 2] and they also investigated two operators identical with ψ-operator and local function in 2012. Their

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extensive works related to this topic can be found in [3, 4].

The idea of σ-topological space have been introduced and studied here. In this paper, ideal σ-topological space has been introduced and two set operators σ-local and ψ_σ and their properties have been studied. Finally σ-codense ideal, σ-compatible ideal and $\psi_\sigma - C$ set using ψ_σ operator have been introduced. Further investigation of various properties of that knowledge have been studied.

2. Preliminaries

Related to this paper, we have discussed some definitions, examples and results in this article.

Definition 1. *A family γ of subsets of a set T is called σ-topology if the following conditions are satisfied:*
(i) $\emptyset, T \in \gamma$.
(ii) γ is closed under countable union.
(iii) γ is closed under finite intersection.
The couple (T, γ) is said to be a σ-topological space. The member of γ is called σ-open set in (T, γ) and the complement of σ-open set is called σ-closed set.

Note 1. *Every topology on a non-empty set T is a σ-topology but every σ-topology on T may not be a topology. For an example, let $T = \mathbb{R}$, set of all real numbers and $\gamma = \{\emptyset, \mathbb{R}\} \cup \{S \subset \mathbb{R} : S \text{ is countable}\}$. Then γ is σ-topology on T. But $\bigcup\limits_{p \in \mathbb{R}\setminus\mathbb{Q}} \{p\} \notin \gamma$, i.e, γ is not closed under arbitrary union. Hence γ is not a topology on $T = \mathbb{R}$.*

Definition 2. *A non-empty family J of subsets of T is called an ideal on T, if*
(i) $M \in J$ and $N \subset M$ implies $N \in J$ (heredity).
(ii) $M \in J$ and $N \in J$ imply $M \cup N \in J$ (finite additivity).

Definition 3. *Let (T, γ) be a σ-topological space and $M \subset T$. The σ-interior and σ-closure of M in (T, γ) are defined as respectively:*

$$\cup\{V : V \subset M \text{ and } V \in \gamma\} \text{ and } \cap\{C : M \subset C \text{ and } T \setminus C \in \gamma\}$$

The σ-interior and σ-closure of M in (T, γ) are denoted as $Int^\sigma(M)$ and $Cl^\sigma(M)$ respectively.

Theorem 1. *Let (T, γ) be a σ-topological space and M, N be two subsets of T, then*
(i) $p \in Cl^\sigma(M)$ if and only if for any σ-open set V containing p, $V \cap M \neq \emptyset$.
(ii) If $M \subset N$ then $Cl^\sigma(M) \subset Cl^\sigma(N)$.

Proof. (i) Let $p \in Cl^\sigma(M)$. If possible let there exists a σ-open set V containing p such that $V \cap M = \emptyset$. This implies $M \subset T \setminus V$. Since $T \setminus V$ is σ-closed in T

containing M, so $Cl^{\sigma}(M) \subset T \setminus V$. This implies $Cl^{\sigma}(M) \cap V = \emptyset$, which contradicts the fact that $p \in Cl^{\sigma}(M) \cap V$. Thus if $p \in Cl^{\sigma}(M)$, then for any σ-open set V containing p, $V \cap M \neq \emptyset$.
Conversely, let for any σ-open set V containing p, $V \cap M \neq \emptyset$. If possible let $p \notin Cl^{\sigma}(M)$. Then $p \in T \setminus Cl^{\sigma}(M) = V$(say). This implies $V \cap Cl^{\sigma}(M) = \emptyset$ and hence $V \cap M = \emptyset$, as $M \subset Cl^{\sigma}(M)$, which contradicts our assumption. Hence $p \in Cl^{\sigma}(M)$.
(ii) Let $p \in Cl^{\sigma}(M)$. Then for any σ-open set V containing p, $V \cap M \neq \emptyset$. This implies $V \cap N \neq \emptyset$, since $M \subset N$. Thus $p \in Cl^{\sigma}(N)$. Hence $Cl^{\sigma}(M) \subset Cl^{\sigma}(N)$. $\square$

Theorem 2. *Let (T, γ) be a σ-topological space and $M \subset T$, then $Int^{\sigma}(M) = T \setminus Cl^{\sigma}(T \setminus M)$.*

Proof. $Cl^{\sigma}(T \setminus M) = Cl^{\sigma}(M^c) = \cap\{F : M^c \subset F, F^c \in \gamma\}$ where $M^c = T \setminus M$ and $F^c = T \setminus F$. This implies $\{Cl^{\sigma}(T \setminus M)\}^c = \cup\{F^c : M \supset F^c, F^c \in \gamma\}$. Thus $T \setminus Cl^{\sigma}(T \setminus M) = Int^{\sigma}(M)$. Hence the result. $\square$

Definition 4. *Let (T, γ) be a σ-topological space and $M \subset T$. Then M is called a σ-neighbourhood of $p \in T$, if there exists $V \in \gamma$ such that $p \in V \subset M$.*

Definition 5. *Let (T, γ) be a σ-topological space and J be an ideal on T. Then the triplicate (T, γ, J) is called an ideal σ-topological space.*

Definition 6. *Let (T, γ, J) be an ideal σ-topological space. Then*

$M^*(J, \gamma) = \{p \in T : M \cap V \notin J$ *for every* $V \in \gamma(p)\}$, *where* $\gamma(p) = \{V \in \gamma : p \in V\}$

is said to be the σ-local function of M with respect to J and γ.
When there is no confusion, we will write M^J or simply $M^{*\sigma}$ or $M^*(J, \gamma)$ and call it the "σ-local function of M".

Example 1. *Let $T = \{p, q, r\}$, $\gamma = \{\emptyset, T, \{p\}, \{p, q\}, \{p, r\}\}$ and $J = \{\emptyset, \{p\}\}$. Take $M = \{p, q\}$. Then $M^{*\sigma} = \{t \in T : M \cap V \notin J$ for every $V \in \gamma(t)\} = \{q\}$.*

Theorem 3. *Let (T, γ) be a σ-topological space with I and J ideals on T and let M and N be subsets of T. Then*
(i) $\emptyset^{*\sigma} = \emptyset$.
(ii) $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$.
(iii) If $M \subset N$ *then* $M^{*\sigma} \subset N^{*\sigma}$.
(iv) If $I_1 \in I$ *then* $I_1^{*\sigma} = \emptyset$.
(v) $I \subset J$ *implies* $M^{*\sigma}(J) \subset M^{*\sigma}(I)$.
(vi) $M^{*\sigma} \cup N^{*\sigma} = (M \cup N)^{*\sigma}$.
(vii) $(\bigcup_i M_i)^{*\sigma} = \bigcup_i (M_i^{*\sigma})$.
(viii) $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.
(ix) $M^{*\sigma} \setminus N^{*\sigma} = (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$.
(x) For any $O \in \gamma, O \cap (O \cap M)^{*\sigma} \subset O \cap M^{*\sigma}$.

(xi) For any $I_1 \in I, (M \cup I_1)^{*\sigma} = M^{*\sigma} = (M \setminus I_1)^{*\sigma}$.
(xii) $M^{*\sigma}(I \cap J) = M^{*\sigma}(I) \cup N^{*\sigma}(J)$.
(xiii) $\gamma \cap I = \{\emptyset\}$ *if and only if* $T = T^{*\sigma}$.
(xiv) $M^{*\sigma} \subset Cl^{\sigma}(M)$.

Proof. (i) Here $\emptyset^{*\sigma} = \{p \in T : \emptyset \cap V \notin I$ for every $V \in \gamma(\mathrm{p})\}$. But $\emptyset \cap V = \emptyset \in I$ for every $V \in \gamma(p)$. Thus $\emptyset^{*\sigma}$ contains no element of T. Therefore $\emptyset^{*\sigma} = \emptyset$.
(ii) Let $p \in (M^{*\sigma})^{*\sigma}$. Then for every $V \in \gamma(p), V \cap M^{*\sigma} \notin I$ and hence $V \cap M^{*\sigma} \neq \emptyset$. Let $y \in V \cap M^{*\sigma}$. Then $V \in \gamma(y)$ and $y \in M^{*\sigma}$. This implies $V \cap M \notin I$ and hence $p \in M^{*\sigma}$. Therefore $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$.
(iii) Let $p \in M^{*\sigma}$. Then for every $V \in \gamma(p), V \cap M \notin I$. Since $M \subset N$, therefore $V \cap M \subset V \cap N$. Since $V \cap M \notin I$, so $V \cap N \notin I$. This implies $p \in N^{*\sigma}$ and so $M^{*\sigma} \subset N^{*\sigma}$.
(iv) Since $I_1 \in I$. Then for every $V \in \gamma$, $V \cap I_1 \subset I_1 \in I$ and by heredity, $V \cap I_1 \in I$. So $I_1^{*\sigma} = \{p \in T : I_1 \cap V \notin I$ for every $V \in \gamma(\mathrm{p})\}$ implies $I_1^{*\sigma} = \emptyset$.
(v) Let $p \in M^{*\sigma}(J)$. Then for every $V \in \gamma(p)$, $M \cap V \notin J$ implies $M \cap V \notin I$ (since $I \subset J$). So $p \in M^{*\sigma}(I)$. Hence $M^{*\sigma}(J) \subset M^{*\sigma}(I)$.
(vi) We know $M \subset M \cup N$ and $N \subset M \cup N$. This implies $M^{*\sigma} \subset (M \cup N)^{*\sigma}$ and $N^{*\sigma} \subset (M \cup N)^{*\sigma}$ (by Theorem 3 (iii)). So $M^{*\sigma} \cup N^{*\sigma} \subset (M \cup N)^{*\sigma}$. For reverse inclusion, let $p \notin (M^{*\sigma} \cup N^{*\sigma})$. Then $p \notin M^{*\sigma}$ and $p \notin N^{*\sigma}$. So there exist V, $O \in \gamma(p)$ such that $V \cap M \in I$ and $O \cap N \in I$. This implies $(V \cap M) \cup (O \cap N) \in I$ since I is additive.
Now

$$\begin{aligned}(V \cap M) \cup (O \cap N) &= [(V \cap M) \cup O] \cap [(V \cap M) \cup N] \\ &= (V \cup O) \cap (M \cup O) \cap (V \cup N) \cap (M \cup N) \\ &\supset (V \cap O) \cap (M \cup N)\end{aligned}$$

This implies $(V \cap O) \cap (M \cup N) \in I$ (since I is hereditary). Since $V \cap O \in \gamma(p)$, $p \notin (M \cup N)^{*\sigma}$. Contrapositively $p \in (M \cup N)^{*\sigma}$ implies $p \in M^{*\sigma} \cup N^{*\sigma}$. Thus $(M \cup N)^{*\sigma} \subset M^{*\sigma} \cup N^{*\sigma}$. Hence we get $M^{*\sigma} \cup N^{*\sigma} = (M \cup N)^{*\sigma}$.
(vii) Proof is obvious and hence omitted.
(viii) We know $M \cap N \subset M$ and $M \cap N \subset N$. This implies $(M \cap N)^{*\sigma} \subset M^{*\sigma}$ and $(M \cap N)^{*\sigma} \subset M^{*\sigma}$ (by Theorem 3 (iii)). So $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.

Independent Proof: If possible let $(M \cap N)^{*\sigma}$ not be a subset of $M^{*\sigma} \cap N^{*\sigma}$. Then there exists $p \in (M \cap N)^{*\sigma}$ but $p \notin M^{*\sigma} \cap N^{*\sigma}$. Now $p \in (M \cap N)^{*\sigma}$ implies $V \cap (M \cap N) \notin I$ for every $V \in \gamma(p)$, i.e., $(V \cap M) \cap (V \cap N) \notin I$ for every $V \in \gamma(p)$. This implies $V \cap M \notin I$ and $V \cap N \notin I$ for every $V \in \gamma(p)$. So $p \in M^{*\sigma}$ and $p \in N^{*\sigma}$ which implies $p \in M^{*\sigma} \cap N^{*\sigma}$ which contradicts the fact that $p \notin M^{*\sigma} \cap N^{*\sigma}$. Hence $(M \cap N)^{*\sigma} \subset M^{*\sigma} \cap N^{*\sigma}$.

(ix) We know $M = (M \setminus N) \cup (M \cap N)$. This implies

$$\begin{aligned} M^{*\sigma} &= [(M \setminus N) \cup (M \cap N)]^{*\sigma} \\ &= (M \setminus N)^{*\sigma} \cup (M \cap N)^{*\sigma} \ (by\ Theorem\ 3\ (vii)) \\ &\subset (M \setminus N)^{*\sigma} \cup N^{*\sigma} \ (by\ Theorem\ 3\ (iii)) \end{aligned}$$

This implies $M^{*\sigma} \setminus N^{*\sigma} \subset (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$.

Again $M \setminus N \subset M$. Then $(M \setminus N)^{*\sigma} \subset M^{*\sigma}$ and hence $(M \setminus N)^{*\sigma} \setminus N^{*\sigma} \subset M^{*\sigma} \setminus N^{*\sigma}$. Thus we obtain $M^{*\sigma} \setminus N^{*\sigma} = (M \setminus N)^{*\sigma} \setminus N^{*\sigma}$

(x) We have $O \cap M \subset M$. This implies $(O \cap M)^{*\sigma} \subset M^{*\sigma}$ (by Theorem 3 (iv)). So $O \cap (O \cap M)^{*\sigma} \subset O \cap M^{*\sigma}$.

(xi) We have $M \subset (M \cup I_1)$. This implies $M^{*\sigma} \subset (M \cup I_1)^{*\sigma}$. Let $p \in (M \cup I_1)^{*\sigma}$. Then for every $V \in \gamma(p)$, $V \cap (M \cup I_1) \notin I$. This implies $V \cap M \notin I$. If not, suppose that $V \cap M \in I$. Since $V \cap I_1 \subset I_1 \in I$, by heredity $V \cap I_1 \in I$ and hence by finite additivity $(V \cap M) \cup (V \cap I_1) \in I$. This implies $V \cap (M \cup I_1) \in I$, a contradiction. Consequently $p \in M^{*\sigma}$. Therefore $(M \cup I_1)^{*\sigma} \subset M^{*\sigma}$. So $(M \cup I_1)^{*\sigma} = M^{*\sigma}$.

Also $M \setminus I_1 \subset M$ implies $(M \setminus I_1)^{*\sigma} \subset M^{*\sigma}$. For the converse, let $p \in M^{*\sigma}$, we claim that $p \in (M \setminus I_1)^{*\sigma}$. If not, there exists $V \in \gamma(p)$ such that $V \cap (M \setminus I_1) \in I$. This implies $I_1 \cup (V \cap (M \setminus I_1)) \in I$, since $I_1 \in I$ (by finite additivity). Thus $I_1 \cup (V \cap M) \in I$. So $V \cap M \in I$, a contradiction to the fact that $p \in M^{*\sigma}$. Hence $M^{*\sigma} \subset (M \setminus I_1)^{*\sigma}$. So $M^{*\sigma} = (M \setminus I_1)^{*\sigma}$. Consequently $(M \cup I_1)^{*\sigma} = M^{*\sigma} = (M \setminus I_1)^{*\sigma}$.

(xii) We have $I \cap J \subset I$ and $I \cap J \subset J$. This implies $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(I)$ and $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(J)$ (by Theorem 3 (v)). So $M^{*\sigma}(I \cap J) \supset M^{*\sigma}(I) \cup M^{*\sigma}(J)$. For reverse, let $p \in M^{*\sigma}(I \cap J)$. Then for every $V \in \gamma(p), V \cap M \notin I \cap J$. Thus $V \cap M \notin I$ or $V \cap M \notin J$. This implies $p \in M^{*\sigma}(I)$ or $p \in M^{*\sigma}(J)$. These imply $p \in M^{*\sigma}(I) \cup M^{*\sigma}(J)$ and hence $M^{*\sigma}(I) \cup M^{*\sigma}(J) \supset M^{*\sigma}(I \cap J)$. So $M^{*\sigma}(I \cap J) = M^{*\sigma}(I) \cup M^{*\sigma}(J)$.

(xiii) From definition $T^{*\sigma} \subset T$.

For reverse inclusion let $p \in T$. If possible let $p \notin T^{*\sigma}$. Then there exists $V \in \gamma(p)$ such that $V \cap T \in I$. This implies $V \in I$, a contradiction. Hence $T \subset T^{*\sigma}$. Thus $T = T^{*\sigma}$.

Conversely, suppose that $T = T^{*\sigma}$ holds. If possible let $V \in \gamma \cap I$ and $p \in V$. Then $V \cap T \subset V \in \gamma \cap I$. This implies $V \cap T \in \gamma \cap I$ and hence $V \cap T \in I$. Thus $p \notin T^{*\sigma}$, a contradiction.

(xiv) Let $p \in M^{*\sigma}$. Then for every $V \in \gamma(p)$, $V \cap M \notin I$. This implies $V \cap M \neq \emptyset$, for all $p \in M^{*\sigma}$. Thus $p \in Cl^{\sigma}(M)$. Hence $M^{*\sigma} \subset Cl^{\sigma}(M)$. □

Result 1. *Let (T, γ) be a σ-topological space with J an ideal on T and $M \subset T$. Then $V \in \gamma$, $V \cap M \in J$ implies $V \cap M^{*\sigma} = \emptyset$.*

Proof. If possible let $V \cap M^{*\sigma} \neq \emptyset$ and let $p \in V \cap M^{*\sigma}$. This implies $p \in V$ and for all $N_p \in \gamma(p)$ such that $N_p \cap M \notin J$. Since $p \in V \in \gamma$ then $V \cap M \notin J$, which is a contradiction. Hence the result. $\square$

Result 2. *Let (T, γ) be a σ-topological space with J an ideal on T. Then $(M \cup M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$ for all $M \in \wp(T)$.*

Proof. Let $p \notin M^{*\sigma}$. Then there exists $V_p \in \gamma(p)$ such that $V_p \cap M \in J$. This implies $V_p \cap M^{*\sigma} = \emptyset$. This implies $V_p \cap (M \cup M^{*\sigma}) = (V_p \cap M) \cup (V_p \cap M^{*\sigma}) = V_p \cap M \in J$. Thus $p \notin (M \cup M^{*\sigma})^{*\sigma}$. Hence $(M \cup M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$. $\square$

Theorem 4. *Let (T, γ) be a σ-topological space with J an ideal on T. Then the operator $Cl^{*\sigma} : \wp(T) \to \wp(T)$ defined by $Cl^{*\sigma}(M) = M \cup M^{*\sigma}$ for all $M \in \wp(T)$, is a Kuratowski closure operator and it generates a σ-topology $\gamma^*(J) = \{M \in \wp(T) : Cl^{*\sigma}(M^c) = M^c\}$ which is finer than γ.*

Proof. (i) Since $\emptyset^{*\sigma} = \emptyset$, then $Cl^{*\sigma}(\emptyset) = \emptyset \cup \emptyset^{*\sigma} = \emptyset \cup \emptyset = \emptyset$.
(ii) $Cl^{*\sigma}(M) = M \cup M^{*\sigma}$. This implies $M \subset Cl^{*\sigma}(M)$.
(iii) $Cl^{*\sigma}(M \cup N) = (M \cup N) \cup (M \cup N)^{*\sigma} = (M \cup N) \cup (M^{*\sigma} \cup N^{*\sigma}) = (M \cup M^{*\sigma}) \cup (N \cup N^{*\sigma}) = Cl^{*\sigma}(M) \cup Cl^{*\sigma}(N)$.
(iv) Let $M, N \subset T$ with $M \subset N$. Then $Cl^{*\sigma}(M) = M \cup M^{*\sigma} \subset N \cup N^{*\sigma} = Cl^{*\sigma}(N)$. This implies $Cl^{*\sigma}(M) \subset Cl^{*\sigma}(N)$. We have $M \subset Cl^{*\sigma}(M)$. This implies $Cl^{*\sigma}(M) \subset Cl^{*\sigma}(Cl^{*\sigma}(M))$. But $Cl^{*\sigma}(Cl^{*\sigma}(M)) = Cl^{*\sigma}(M \cup M^{*\sigma}) = (M \cup M^{*\sigma}) \cup (M \cup M^{*\sigma})^{*\sigma} \subset (M \cup M^{*\sigma}) \cup M^{*\sigma} = M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Hence $Cl^{*\sigma}(Cl^{*\sigma}(M)) = Cl^{*\sigma}(M)$. Consequently $Cl^{*\sigma}(M)$ is a Kuratowski closure operator.
Now we have to show that $\gamma^*(J) = \{M \in \wp(T) : Cl^{*\sigma}(M^c) = M^c\}$ is a σ-topology on T.
Since $Cl^{*\sigma}(\emptyset) = \emptyset$, then $\emptyset^c \in \gamma^*(J)$. This implies $T \in \gamma^*(J)$. Also since $T \subset Cl^{*\sigma}(T) \subset T$, then $Cl^{*\sigma}(T) = T$. This implies $T^c \in \gamma^*(J)$. Hence $\emptyset \in \gamma^*(J)$
Let $M_1, M_2, ..., M_n, ... \in \gamma^*(J)$. Then $Cl^{*\sigma}(M_i^c) = M_i^c$ for all $i \in \mathbb{N}$. Now $\bigcap\limits_{i \in \mathbb{N}} M_i^c \subset M_i^c$ for all $i \in \mathbb{N}$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset Cl^{*\sigma}(M_i^c) = M_i^c$ for all $i \in \mathbb{N}$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset (\bigcap\limits_{i \in \mathbb{N}} M_i^c) \subset Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c)$. This implies $Cl^{*\sigma}(\bigcap\limits_{i \in \mathbb{N}} M_i^c) = (\bigcap\limits_{i \in \mathbb{N}} M_i{}^c)$. Thus $Cl^{*\sigma}(\bigcup\limits_{i \in \mathbb{N}} M_i)^c = (\bigcup\limits_{i \in \mathbb{N}} M_i)^c$. Hence $\bigcup\limits_{i \in \mathbb{N}} M_i \in \gamma^*(J)$. Therefore $\gamma^*(J)$ is closed under countable union.
Again let $M_j \in \gamma^*(J), j = 1, 2, 3, ...n$. Then $Cl^{*\sigma}(M_j^c) = M_j^c$ for all $j = 1, 2, 3, ...n$. Therefore $Cl^{*\sigma}\{(\bigcap\limits_{j=1}^{n} M_j)^c\} = Cl^{*\sigma}(\bigcup\limits_{j=1}^{n} M_j^c) = \bigcup\limits_{j=1}^{n} Cl^{*\sigma}(M_j^c) = \bigcup\limits_{j=1}^{n} (M_j^c) = (\bigcap\limits_{j=1}^{n} M_j)^c$. This implies $\bigcap\limits_{j=1}^{n} M_j \in \gamma^*(J)$. Therefore $\gamma^*(J)$ is closed under finite intersection. Thus $\gamma^*(J)$ is a σ-topology on T.

Next from Theorem 3 (xiv), we have $M^{*\sigma} \subset Cl^{\sigma}(M)$ implies $M \cup M^{*\sigma} \subset M \cup Cl^{\sigma}(M) = Cl^{\sigma}(M)$ implies $Cl^{*\sigma}(M) \subset Cl^{\sigma}(M)$. Hence $\gamma \subset \gamma^*(J)$. □

The member of $\gamma^*(J)$ is called $\sigma^*(J)$-open set or simply σ^*-open set and the complement of $\sigma^*(J)$-open set is called $\sigma^*(J)$-closed set or simply σ^*-closed set.

Result 3. *Let (T, γ) be a σ-topological space. If $J = \{\emptyset\}$, then $\gamma = \gamma^*(J)$.*

Proof. If $p \in Cl^{\sigma}(M)$, then (by Theorem 1 (i)), $V_p \cap M \neq \emptyset$, for all $V_p \in \gamma(p)$. This implies $V_p \cap M \notin \{\emptyset\} = J$, for all $V_p \in \gamma(p)$ implies $p \in M^{*\sigma}$ implies $p \in M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Since p is an arbitrary member of $Cl^{\sigma}(M)$, then $Cl^{\sigma}(M) \subset Cl^{*\sigma}(M)$. Also by Theorem 3 (xiv), $M^{*\sigma} \subset Cl^{\sigma}(M)$. This implies $M \cup M^{*\sigma} \subset M \cup Cl^{\sigma}(M)$ implies $Cl^{*\sigma}(M) \subset Cl^{\sigma}(M)$. Hence $Cl^{*\sigma}(M) = Cl^{\sigma}(M)$, for all $M \in \wp(T)$. Consequently $\gamma^*(J) = \gamma$ implies $\gamma = \gamma^*(\{\emptyset\})$. □

Theorem 5. *Let (T, γ) be a σ-topological space and J_1, J_2 be two ideals on T. If $J_1 \subset J_2$, then $\gamma^*(J_1) \subset \gamma^*(J_2)$.*

Proof. Let $O \in \gamma^*(J_1)$. Then $Cl^{*\sigma}_{J_1}(O^c) = O^c \Rightarrow O^c \cup O^{c^{*\sigma}}(J_1) = O^c$. This implies $O^{c^{*\sigma}}(J_1) \subset O^c$ implies $O^{c^{*\sigma}}(J_2) \subset O^{c^{*\sigma}}(J_1) \subset O^c$ (by Theorem 3 (v)). This implies $O^{c^{*\sigma}}(J_2) \cup O^c = O^c$ implies $Cl^{*\sigma}_{J_2}(O^c) = O^c$ implies $O \in \gamma^*(J_2)$. Since $O \in \gamma^*(J_1)$ is arbitrary, then $\gamma^*(J_1) \subset \gamma^*(J_2)$. □

Theorem 6. *Let (T, γ) be a σ-topological space with J an ideal on T. Then*
(i) $I \in J$ implies $I^c \in \gamma^*(J)$.
(ii) $M^{*\sigma} = Cl^{*\sigma}(M^{*\sigma})$, for all $M \in \wp(T)$.

Proof. : (i) We have for all $I \in J$, $(M \cup I)^{*\sigma} = M^{*\sigma}$. If we take $M = \emptyset$, then $I^{*\sigma} = \emptyset^{*\sigma} = \emptyset$, for all $I \in J$. Hence $Cl^{*\sigma}(I) = I \cup I^{*\sigma} = I \cup \emptyset = I$. Therefore $I^c \in \sigma^*(J)$. This implies I is $\gamma^*(J)$-closed, for all $I \in J$.
(ii) We have $(M^{*\sigma})^{*\sigma} \subset M^{*\sigma}$. This implies $M^{*\sigma} = M^{*\sigma} \cup (M^{*\sigma})^{*\sigma} = Cl^{*\sigma}(M^{*\sigma})$. Hence $M^{*\sigma}$ is a $\sigma^*(J)$-closed. □

Theorem 7. *Let (T, γ) be a σ-topological space and $M \subset T$. Then M is σ^*-closed if and only if $M^{*\sigma} \subset M$.*

Proof. If M is σ^*-closed, then $M = Cl^{*\sigma}(M) = M \cup M^{*\sigma}$. This implies $M^{*\sigma} \subset M$. Conversely let $M^{*\sigma} \subset M$. This implies $M = M \cup M^{*\sigma} = Cl^{*\sigma}(M)$. Hence M is σ^*-closed. □

Theorem 8. *Let (T, γ_1) and (T, γ_2) be two σ-topological spaces and J be an ideal on T. Then $\gamma_1 \subset \gamma_2$ implies $M^{*\sigma}(J, \gamma_2) \subset M^{*\sigma}(J, \gamma_1)$.*

Proof. Let $p \in M^{*\sigma}(J, \gamma_2)$. This implies $V_p \cap M \notin J$ for all $V_p \in \gamma_2(p)$ implies $V_p \cap M \notin J$ for all $V_p \in \gamma_1(p)$. This implies $p \in M^{*\sigma}(J, \gamma_1)$. Since p is an arbitrary element of $M^{*\sigma}(J, \gamma_2)$, then $M^{*\sigma}(J, \gamma_2) \subset M^{*\sigma}(J, \gamma_1)$. □

Theorem 9. *Let (T, γ) be a σ-topological space and J be an ideal on T. Then the collection $\beta(J, \gamma) = \{M \setminus I : M \in \gamma, I \in J\}$ is a basis for the σ-topology $\gamma^*(J)$.*

Proof. Let $M \in \gamma^*(J)$ and $p \in M$. Then M^c is σ^*-closed, i.e, $Cl^{*\sigma}(M^c) = M^c$ and hence $M^c \cup (M^c)^{*\sigma} = M^c$ implies $(M^c)^{*\sigma} \subset M^c$. This implies $p \notin (M^c)^{*\sigma}$ and there exists $V_p \in \gamma(p)$ such that $V_p \cap M^c \in J$. Take $K = V_p \cap M^c$, then $p \notin K$ and $K \in J$. Thus $p \in V_p \setminus K = V_p \cap K^c = V_p \cap (V_p \cap M^c)^c = V_p \cap (V_p^c \cup M) = (V_p \cap V_p^c) \cup (V_p \cup M) = V_p \cap M \subset M$. Hence $p \in V_p \setminus K \subset M$, where $V_p \setminus K \in \beta(J, \gamma)$. Thus $\beta(J, \gamma)$ is an open base of $\gamma^*(J)$. $\square$

The example given below proves that $M^{*\sigma} \cap N^{*\sigma} = (M \cap N)^{*\sigma}$ is not satisfied in general.

Example 2. *Let T=$\{p,q,r,s\}$, γ=$\{\emptyset, T, \{p\}, \{s\}, \{p,s\}, \{q,s\}, \{r,s\}, \{p,r,s\}, \{p,q,s\}, \{q,r,s\}\}$, $J = \{\emptyset, \{p\}\}$. Then σ-open sets containing p are: T, $\{p\}$, $\{p,s\}$, $\{p,r,s\}$, $\{p,q,s\}$; σ-open sets containing q are: T, $\{q,s\}$, $\{p,q,s\}$, $\{q,r,s\}$; σ-open sets containing r are: T, $\{r,s\}$, $\{p,r,s\}$, $\{q,r,s\}$; σ-open sets containing s are: T, $\{s\}$, $\{p,s\}$, $\{q,s\}$, $\{r,s\}$, $\{p,q,s\}$, $\{p,r,s\}$, $\{q,r,s\}$. Take M=$\{p,q\}$ and N=$\{p,s\}$. Then $M^{*\sigma} = \{q\}$ and $N^{*\sigma} = \{q, r, s\}$ and hence $M^{*\sigma} \cap N^{*\sigma} = \{q\}$. Now $(M \cap N)^{*\sigma} = \{p\}^{*\sigma} = \emptyset$ and so $M^{*\sigma} \cap N^{*\sigma} \neq (M \cap N)^{*\sigma}$.*

3. ψ_σ-Operator

In this part, we have introduced another set operator ψ_σ in (T, γ, J). This operator is as like similar of ψ-operator [5, 10], in ideal topological space. Definition of ψ_σ-operator is given below:

Definition 7. *Let (T, γ, J) be an ideal σ-topological space. An operator $\psi_\sigma : \wp(T) \to \gamma$ is defined as follows:*

for every $M \in \wp(T)$, $\psi_\sigma(M)$=$\{p \in T$: there exists a $V \in \gamma(p)$ such that $V \setminus M \in J\}$.

Observe that $(T \setminus M)^{*\sigma} = \{p \in T : V \cap (T \setminus M) \notin J \text{ } for \text{ } every \text{ } V \in \gamma(p)\}$.
This implies

$$\begin{aligned} T \setminus (T \setminus M)^{*\sigma} &= T \setminus \{p \in T : V \cap (T \setminus M) \notin J \text{ } for \text{ } every \text{ } V \in \gamma(p)\} \\ &= \{p \in T : \exists V \in \gamma(p) \text{ } such \text{ } that \text{ } V \cap (T \setminus M) \in J\} \\ &= \{p \in T : \exists V \in \gamma(p) \text{ } such \text{ } that \text{ } V \setminus M \in J\} \\ &= \psi_\sigma(M) \end{aligned}$$

Hence $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma}$.

Here we have to find out the value of $\psi_\sigma(M)$ of a set in σ-topological space.

Example 3. *Let T=$\{p,q,r,\}$, γ=$\{\emptyset, T, \{r\}, \{p,r\}, \{q,r\}\}$, $J = \{\emptyset, \{r\}\}$. Then for M=$\{p,q\}$, $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma} = T \setminus \{r\}^{*\sigma} = T \setminus \emptyset = T$.*

The characteristics of the operator ψ_σ has been studied in the following results:

Theorem 10. *Let (T, γ, J) be an ideal σ-topological space. Then the following properties hold:*
(i) If $M \subset N$, then $\psi_\sigma(M) \subset \psi_\sigma(N)$.
(ii) If $M, N \in \wp(T)$, then $\psi_\sigma(M) \cup \psi_\sigma(N) \subset \psi_\sigma(M \cup N)$.
(iii) If $M, N \in \wp(T)$, then $\psi_\sigma(M) \cap \psi_\sigma(N) = \psi_\sigma(M \cap N)$.
(iv) If $M \subset T$, then $\psi_\sigma(M) = \psi_\sigma(\psi_\sigma(M))$ if and only if $(T \setminus M)^{*\sigma} \subset ((T \setminus M)^{*\sigma})^{*\sigma}$.
(v) If $M \in J$, then $\psi_\sigma(M) = T \setminus T^{*\sigma}$.
(vi) If $M \subset T, J_1 \in J$, then $\psi_\sigma(M \setminus J_1) = \psi_\sigma(M)$.
(vii) If $M \subset T, J_1 \in J$, then $\psi_\sigma(M \cup J_1) = \psi_\sigma(M)$.
(viii) If $V \in \gamma$, then $V \subset \psi_\sigma(V)$.
(ix) If $(M \setminus N) \cup (N \setminus M) \in J$, then $\psi_\sigma(M) = \psi_\sigma(N)$.
(x) $Int^{\sigma^*}(M) = M \cap \psi_\sigma(M)$.

Proof. (i) $M \subset N$ implies $(T \setminus M) \supset (T \setminus N)$. This implies $(T \setminus M)^{*\sigma} \supset (T \setminus N)^{*\sigma}$ (by Theorem 3 (iii)). This implies $T \setminus (T \setminus M)^{*\sigma} \subset T \setminus (T \setminus N)^{*\sigma}$. Hence $\psi_\sigma(M) \subset \psi_\sigma(N)$.
(ii) We know $M \subset M \cup N$ and $N \subset M \cup N$. This implies $\psi_\sigma(M) \subset \psi_\sigma(M \cup N)$ and $\psi_\sigma(N) \subset \psi_\sigma(M \cup N)$ (by Theorem 10 (i)). Hence $\psi_\sigma(M) \cup \psi_\sigma(N) \subset \psi_\sigma(M \cup N)$.
(iii) Since $M \cap N \subset M$ and $M \cap N \subset N$. This implies $\psi_\sigma(M \cap N) \subset \psi_\sigma(M)$ and $\psi_\sigma(M \cap N) \subset \psi_\sigma(N)$ (by Theorem 10 (i)). Hence $\psi_\sigma(M \cap N) \subset \psi_\sigma(M) \cap \psi_\sigma(N)$.

For reverse inclusion let $p \in \psi_\sigma(M) \cap \psi_\sigma(N)$. Then $p \in \psi_\sigma(M)$ and $p \in \psi_\sigma(N)$. Then there exist $V, O \in \gamma(p)$ such that $V \setminus M \in J$ and $O \setminus N \in J$. This implies $(V \setminus M) \cup (O \setminus N) \in J$, since J is finite additive. Now

$$\begin{aligned}
(V \setminus M) \cup (O \setminus N) &= [(V \cap M^c) \cup O] \cap [(V \cap M^c) \cup N^c] \\
&= (V \cup O) \cap (M^c \cup O) \cap (V \cup N^c) \cap (M^c \cup N^c) \\
&\supset (V \cap O) \cap (M^c \cup N^c) \\
&= (V \cap O) \setminus (M \cap N)
\end{aligned}$$

This implies $(V \cap O) \setminus (M \cap N) \in J$, since J is heredity. Since $V \cap O \in \gamma(p)$ then $p \in \psi_\sigma(M \cap N)$. Thus $\psi_\sigma(M) \cap \psi_\sigma(N) \subset \psi_\sigma(M \cap N)$. Hence we obtain $\psi_\sigma(M) \cap \psi_\sigma(N) = \psi_\sigma(M \cap N)$.
(iv) Let $\psi_\sigma(M) = \psi_\sigma(\psi_\sigma(M))$. Then $T \setminus (T \setminus M)^{*\sigma} = T \setminus [T \setminus \psi_\sigma(M)]^{*\sigma} = T \setminus [T \setminus \{T \setminus (T \setminus \psi_\sigma(M))\}]^{*\sigma}$. This implies $(T \setminus M)^{*\sigma} = ((T \setminus M)^{*\sigma})^{*\sigma}$.

Conversely, suppose that $(T \setminus M)^{*\sigma} = ((T \setminus M)^{*\sigma})^{*\sigma}$ holds. Then $T \setminus (T \setminus M)^{*\sigma} = T \setminus ((T \setminus M)^{*\sigma})^{*\sigma} = T \setminus [T \setminus \{T \setminus (T \setminus \psi_\sigma(M))\}]^{*\sigma}$. This implies $\psi_\sigma(M) = T \setminus (T \setminus \psi_\sigma(M))^{*\sigma} = \psi_\sigma(\psi_\sigma(M))$.

(v) We have $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma} = T \setminus T^{*\sigma}$ (by Theorem 3 (xi)).

(vi) We have $T \setminus [T \setminus (M \setminus J_1)]^{*\sigma} = T \setminus [T \setminus (M \cap J_1^c)]^{*\sigma} = T \setminus [T \cap (M^c \cup J_1)]^{*\sigma} = T \setminus [(T \cap M^c) \cup (T \cap J_1)]^{*\sigma} = T \setminus [(T \setminus M) \cup J_1]^{*\sigma} = T \setminus (T \setminus M)^{*\sigma}$ (by Theorem 3 (xi)). So $\psi_\sigma(M \setminus J_1) = \psi_\sigma(M)$.

(vii) We have $T\setminus[T\setminus(M\cup J_1)]^{*\sigma} = T\setminus[T\cap(M^c\cap J_1^c)]^{*\sigma} = T\setminus[(T\setminus M)\setminus J_1]^{*\sigma} = T\setminus(T\setminus M)^{*\sigma}$ (by Theorem 3 (xi)). So $\psi_\sigma(M\cup J_1) = \psi_\sigma(M)$.

(viii) Let $p\in V$. Then $p\notin T\setminus V$ and hence $V\cap(T\setminus V)=\emptyset\in J$. Thus $p\notin(T\setminus V)^{*\sigma}$. This implies $p\in T\setminus(T\setminus V)^{*\sigma}$ and hence $p\in\psi_\sigma(V)$. So $V\subset\psi_\sigma(V)$.

(ix) Let $J_1 = M\setminus N$ and $J_2 = N\setminus M$. Since $J_1\cup J_2\in J$, then by heredity $J_1, J_2\in J$. Also $N = (M\setminus J_1)\cup J_2$. This implies $\psi_\sigma(N) = \psi_\sigma((M\setminus J_1)\cup J_2)$. So $\psi_\sigma(N) = \psi_\sigma((M\setminus J_1)$ and hence $\psi_\sigma(N) = \psi_\sigma(M)$, (by Theorem 10 (vi) and (vii)).

(x) Let $p\in M\cap\psi_\sigma(M)$. Then $p\in M$ and $p\in\psi_\sigma(M)$. Thus $p\in M$ and there exists a $V_p\in\gamma(p)$ such that $V_p\setminus M\in J$ implies $V_p\setminus(V_p\setminus M)$ is a σ^*-open neighborhood of p and hence $p\in Int^{\sigma^*}(M)$. Hence $M\cap\psi_\sigma(M)\subset Int^{\sigma^*}(M)$. Again, if $p\in Int^{\sigma^*}(M)$, then there exists a σ^*-open neighborhood $V_p\setminus I$ of p where $V_p\in\gamma$ and $I\in J$ such that $p\in V_p\setminus I\subset M$ which implies $V_p\setminus M\subset I$ and $V_p\setminus M\in J$. Hence $p\in M\cap\psi_\sigma(M)$. Hence $Int^{\sigma^*}(M) = M\cap\psi_\sigma(M)$. □

Note 2. *We have $V\subset\psi_\sigma(V)$, for every $V\in\gamma$. But there exists a set M which is not σ-open set but satisfies $M\subset\psi_\sigma(M)$.*

Example 4. *Let T=$\{p,q,r,\}$, γ=$\{\emptyset,T,\{r\},\{p,r\},\{q,r\}\}$, $J=\{\emptyset,\{r\}\}$. Then for M=$\{p,q\}$, $\psi_\sigma(M) = T\setminus(T\setminus M)^{*\sigma} = T\setminus\{r\}^{*\sigma} = T\setminus\emptyset = T$. Thus $M\subset\psi_\sigma(M)$ but M is not a σ-open set.*

The example given below shows that $\psi_\sigma(M)\cup\psi_\sigma(N) = \psi_\sigma(M\cup N)$ does not hold in general.

Example 5. *In Example 2 we consider $M=\{r,s\}$ and $N=\{q,r\}$. Then $\psi_\sigma(M) = T\setminus\{p,q\}^{*\sigma} = T\setminus\{q\} = \{p,r,s\}$ and $\psi_\sigma(N) = T\setminus\{p,s\}^{*\sigma} = T\setminus\{q,r,s\} = \{p\}$. Therefore $\psi_\sigma(M)\cup\psi_\sigma(N) = \{p,r,s\}$ and $\psi_\sigma(M\cup N) = T\setminus\{p\}^{*\sigma} = T\setminus\emptyset = T$. Hence $\psi_\sigma(M)\cup\psi_\sigma(N)\neq\psi_\sigma(M\cup N)$.*

Definition 8. *Let γ be a σ-topological space on a non empty set T. If an ideal J satisfies the property $\gamma\cap J=\{\emptyset\}$ then the ideal J is called σ-codense ideal.*

Theorem 11. *Let (T,γ,J) be an ideal σ-topological space. Then the properties given below are equivalent.*
(i) $\gamma\cap J=\{\emptyset\}$.
(ii) $\psi_\sigma(\emptyset)=\emptyset$.
(iii) If $J_1\in J$ then $\psi_\sigma(J_1)=\emptyset$.

Proof. $(i)\Rightarrow(ii)$: Let $\gamma\cap J=\{\emptyset\}$. Then $T = T^{*\sigma}$. Then $\psi_\sigma(\emptyset) = T\setminus(T\setminus\emptyset)^{*\sigma} = T\setminus T^{*\sigma} = \emptyset$.

$(ii)\Rightarrow(iii)$: Let $\psi_\sigma(\emptyset)=\emptyset$ holds. Then $\psi_\sigma(J_1) = T\setminus(T\setminus J_1)^{*\sigma} = T\setminus T^{*\sigma}$ (by Theorem 3 (xi)) $= T\setminus(T\setminus\emptyset)^{*\sigma} = \psi_\sigma(\emptyset) = \emptyset$.

$(iii)\Rightarrow(i)$: Let $J_1\in J$ be such that $\psi_\sigma(J_1)=\emptyset$. Now $\psi_\sigma(J_1)=\emptyset$ implies $T\setminus(T\setminus J_1)^{*\sigma}=\emptyset$. This implies $T\setminus T^{*\sigma}=\emptyset$, since $J_1\in J$ (by Theorem 3 (xi)). Thus $T=T^{*\sigma}$. Hence $\gamma\cap J=\{\emptyset\}$. □

4. σ-Compatible Ideal

In this section, we have studied a particular type of ideal and its several features. This ideal is as like similar of μ-compatible ideal [9] on supra topological space. This particular type of ideal is:

Definition 9. *Let (T, γ, J) be an ideal σ-topological space. We say the σ-structure is σ-compatible with the ideal J denoted $\gamma \sim J$, if the condition holds: for every $M \subset T$, if for all $p \in M$, there exists $V \in \gamma(p)$ such that $V \cap M \in J$, then $M \in J$.*

Theorem 12. *Let (T, γ, J) be an ideal σ-topological space. Then $\gamma \sim J$ if and only if $\psi_\sigma(M) \setminus M \in J$ for every $M \subset T$.*

Proof. Suppose $\gamma \sim J$. Observe that $p \in \psi_\sigma(M) \setminus M$ if and only if $p \notin M$ and there exists $V_p \in \gamma(p)$ such that $V_p \setminus M \in J$. Now for each $p \in \psi_\sigma(M) \setminus M$ and $V_p \in \gamma(p)$, $V_p \cap (\psi_\sigma(M) \setminus M) \in J$ (by heredity) and hence $(\psi_\sigma(M) \setminus M) \in J$, since $\gamma \sim J$.
Conversely, suppose the given condition holds and $M \subset T$ and assume that for each $p \in M$, there exists $V_p \in \gamma(p)$ such that $V_p \cap M \in J$. Observe that $\psi_\sigma(T \setminus M) \setminus (T \setminus M) = M \setminus M^{*\sigma}$=$\{p \in T$: there exists $V_p \in \gamma(p)$ such that $p \in V_p \cap M \in J\}$. Thus we have $M \subset \psi_\sigma(T \setminus M) \setminus (T \setminus M) \in J$ and hence $M \in J$, by heredity of J. $\square$

Example 6. *Let T=$\{p,q\}$, γ=$\{\emptyset,T,\{p\},\{q\}\}$, $J = \{\emptyset, \{p\}\}$. Then $\emptyset^{*\sigma} = \emptyset$, $\{p\}^{*\sigma} = \emptyset$, $\{q\}^{*\sigma} = \{q\}$ and $\{T\}^{*\sigma} = \{q\}$. Then $\psi_\sigma(\emptyset) = T \setminus T^{*\sigma} = \{p, q\} \setminus \{q\} = \{p\}$, $\psi_\sigma(\{p\}) = T \setminus (T \setminus \{p\})^{*\sigma} = T \setminus \{q\}^{*\sigma} = T \setminus \{q\} = \{p\}$, $\psi_\sigma(\{q\}) = T \setminus (T \setminus \{q\})^{*\sigma} = T \setminus \{p\}^{*\sigma} = T \setminus \emptyset = T$, $\psi_\sigma(T) = T \setminus \emptyset^{*\sigma} = T \setminus \emptyset = T$. Then we see that $\psi_\sigma(\emptyset) \setminus \emptyset = \{p\} \in J$, $\psi_\sigma(\{q\}) \setminus \{q\} = T \setminus \{q\} = \{p\} \in J$, $\psi_\sigma(\{p\}) \setminus \{p\} = \{p\} \setminus \{p\} = \emptyset \in J$ and $\psi_\sigma(T) \setminus T = T \setminus T = \emptyset \in J$. So $\gamma \sim J$.*

Corollary 1. *Let (T, γ, J) be an ideal σ-topological space with $\gamma \sim J$. Then $\psi_\sigma(\psi_\sigma(M)) = \psi_\sigma(M)$ for every $M \subset T$.*

Proof. We know $\psi_\sigma(M) \subset \psi_\sigma(\psi_\sigma(M))$. Also since $\gamma \sim J$, then for every $M \subset T$, $\psi_\sigma(M) \setminus M \in J$. This implies $\psi_\sigma(M) \setminus M = J_1$ for some $J_1 \in J$. This implies $\psi_\sigma(M) \subset M \cup J_1$. Then $\psi_\sigma(\psi_\sigma(M)) \subset \psi_\sigma(M \cup J_1) = \psi_\sigma(M)$. Thus $\psi_\sigma(\psi_\sigma(M)) = \psi_\sigma(M)$. $\square$

Example 7. *Consider T=$\{p,q\}$, γ=$\{\emptyset,T,\{p\},\{q\}\}$ and $J = \{\emptyset, \{p\}\}$. Then by Example 6, $\gamma \sim J$ and $\psi_\sigma(\psi_\sigma(\phi)) = \psi_\sigma(\emptyset)$, $\psi_\sigma(\psi_\sigma(\{p\})) = \psi_\sigma(\{p\})$, $\psi_\sigma(\psi_\sigma(\{q\})) = \psi_\sigma(T) = T = \psi_\sigma(\{q\})$ and $\psi_\sigma(\psi_\sigma(T)) = \psi_\sigma(T)$*

Newcomb in [11] has defined $M = N$ $(mod\ J)$, if $(M \setminus N) \cup (N \setminus M) \in J$. Further, he studied several characteristics of $M = N$ $(mod\ J)$. Here we shall observe that if $M = N$ $(mod\ J)$ then $\psi_\sigma(M) = \psi_\sigma(N)$.
Now we define Baire set in (T, γ, J).

Definition 10. *Let (T, γ, J) be an ideal σ-topological space. A subset M of T is called a Baire set with respect to γ and J denoted by $M \in \boldsymbol{B}_r(T, \gamma, J)$, if there exists a σ-open set $V \in \gamma$ such that $M = V \ (mod\ J)$.*

Theorem 13. *Let (T, γ, J) be an ideal σ-topological space with $\gamma \sim J$. If $V \cup O \in \gamma$ and $\psi_\sigma(V) = \psi_\sigma(O)$, then $V = O \ (mod\ J)$.*

Proof. $V \in \gamma$ implies $V \subset \psi_\sigma(V)$ and hence $V \setminus O \subset \psi_\sigma(V) \setminus O = \psi_\sigma(O) \setminus O \in J$. By heredity of J, $V \setminus O \in J$. Similarly, $O \setminus V \in J$. Then $(V \setminus O) \cup (O \setminus V) \in J$, by finite additivity of J. So $V = O \ (mod\ J)$. □

Clearly, $M = N \ (mod\ J)$ is an equivalence relation. In this favour, following theorem is observable:

Theorem 14. *Let (T, γ, J) be an ideal σ-topological space with $\gamma \sim J$. If $M, N \in \boldsymbol{B}_r(T, \gamma, J)$ and $\psi_\sigma(M) = \psi_\sigma(N)$. Then $M = N \ (mod\ J)$.*

Proof. Let $V, O \in \gamma$ such that $M = V \ (mod\ J)$ and $N = O \ (mod\ J)$. Now $\psi_\sigma(M) = \psi_\sigma(N)$ and $\psi_\sigma(N) = \psi_\sigma(O)$ (by Theorem 10 (ix)). Since $\psi_\sigma(M) = \psi_\sigma(V)$ implies that $\psi_\sigma(V) = \psi_\sigma(O)$, hence $V = O \ (mod\ J)$ (by Theorem 13). Hence $M = N \ (mod\ J)$, by transitivity. □

Theorem 15. *Let (T, γ, J) be an ideal σ-topological space.*
(i) If $N \in \boldsymbol{B}_r(T, \gamma, J) \setminus J$, then there exists $M \in \gamma \setminus \{\emptyset\}$ such that $N = M \ (mod\ J)$.
(ii) Let $\gamma \cap J = \{\emptyset\}$, then $N \in \boldsymbol{B}_r(T, \gamma, J) \setminus J$ if and only if there exists $M \in \gamma \setminus \{\emptyset\}$ such that $N = M \ (mod\ J)$.

Proof. (i) Let $N \in \mathbf{B}_r(T, \gamma, J) \setminus J$, then $N \in \mathbf{B}_r(T, \gamma, J)$. Now if there does not exist $M \in \gamma \setminus \{\emptyset\}$ such that $N = M \ (mod\ J)$, we have $N = \emptyset \ (mod\ J)$. This implies $N \in J$, which is a contradiction.
(ii) Here we shall prove converse part only. Let $M \in \gamma \setminus \{\emptyset\}$ such that $N = M \ (mod\ J)$. Then $M = (N \setminus J_2) \cup J_1$, where $J_2 = N \setminus M$, $J_1 = M \setminus N$ both belong to J. If $N \in J$, then $M \in J$, by heredity and additivity, which contradicts the fact $\gamma \cap J = \{\emptyset\}$. □

5. $\psi_\sigma - C$ SETS

Modak and Bandyopadhyay in [8] have introduced a generalized set with the help of ψ-operator in ideal topological space. In this part, we have studied a set with the help of ψ_σ-operator in (T, γ, J) space. Further, we have studied the properties of this type of sets.

Definition 11. *Let (T, γ, J) be an ideal σ-topological space. A subset M of T is called a $\psi_\sigma - C$ sets, if $M \subset Cl^\sigma(\psi_\sigma(M))$.*
The family of all ψ_σ-C sets in (T, γ, J) is denoted by $\psi_\sigma(T, \gamma)$.

Theorem 16. *Let (T, γ, J) be an ideal σ-topological space. If $M \in \gamma$ then $M \in \psi_\sigma(T, \gamma)$.*

Proof. By Theorem 10 (viii), it follows that $\gamma \subset \psi_\sigma(T, \gamma)$. □

Now by the given example we are to show that the reverse inclusion is not true:

Example 8. *From Example 4 we get $M \in \psi_\sigma(T, \gamma)$ but $M \notin \gamma$.*

By the following example, we are to show that any σ-closed in (T, γ, J) may not be a $\psi_\sigma - C$ set.

In the following example, by $C^\sigma(\gamma)$ we denote the family of all σ-closed sets in (T, γ).

Example 9. *We consider Example 2. Here $M = \{q\} \in C^\sigma(\gamma)$. Then $\psi_\sigma(M) = T \setminus (T \setminus M)^{*\sigma} = T \setminus \{p, r, s\}^{*\sigma} = T \setminus \{q, r, s\} = \{p\}$. Hence $Cl^\sigma(\psi_\sigma(M)) = \cap\{C : \psi_\sigma(M) \subset C, T \setminus C \in \gamma\} = \{p\}$. Therefore $M \in C^\sigma(\gamma)$ but $M \notin \psi_\sigma(T, \gamma)$.*

Theorem 17. *Let $\{M_\alpha : \alpha \in \Delta\}$ be a family of non-empty $\psi_\sigma - C$ sets in an ideal σ-topological space (T, γ, J), then $\bigcup\limits_{\alpha\in\Delta} \in \psi_\sigma(T, \gamma)$.*

Proof. For each $\alpha \in \Delta$, $M_\alpha \subset Cl^\sigma(\psi_\sigma(M_\alpha)) \subset Cl^\sigma(\psi_\sigma(\bigcup\limits_{\alpha\in\Delta} M_\alpha))$. This implies that $\bigcup\limits_{\alpha\in\Delta} M_\alpha \subset Cl^\sigma(\psi_\sigma(\bigcup\limits_{\alpha\in\Delta} M_\alpha))$. Thus $\bigcup\limits_{\alpha\in\Delta} M_\alpha \in \psi_\sigma(T, \gamma)$. □

6. Conclusion

In this writeup, we have introduced a new topology called σ-topology and defined ideals on that spaces. Using this idea, we have discussed relationship of various operators namely $()^{*\sigma}$ operator, ψ_σ-operator. The result of this writeup can be extended to σ-connected sets, σ-compact sets, σ-paracompact sets. The separation axioms can also be introduced in this space. The other properties of ψ_σ-sets can be found and one can introduce some operators on this type of sets to the development of mathematical knowledge.

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