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On Hyperideals of Multiplicative Hyperrings

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ABSTRACT

Let R be a commutative multiplicative hyperring. In this paper, we introduce and study the concepts of n-hyperideal and δ-n-hyperideal of R which are generalization of n-ideals and δ-n-ideals of the in a commutative ring. An element a is called a nilpotent element of R if there exists a positive integer n such that $0 \in a^n$. A hyperideal I ($I \neq R$) of R is called an n- hyperideal of R if for all $a, b \in R$, $a * b \subseteq I$ and a is non-nilpotent element implies that $b \in I$ [15]. Also, I is called a δ-n-hyperideal if for all a, $b \in R$, a$* b \subseteq I$ then either a is nilpotent or b$\in \delta(I)$, where δ is an expansion function over the set of all hyperideals of a multiplicative hyperring. In addition, we give the definition of z_d-hyperideal. Some properties of n-hyperideals, δ-n-hyperideals and z_d-hyperideals of the hyperring R are presented. Finally, the relations between these notions are investigated.

Keywords: Multiplicative hyperring, n-hyperideal, δ-n-hyperideal.

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Introduction

The first publications on algebraic hyperstructures, as a natural generalization of classical algebra, are first encountered in 1934. The group concept, the fundamental definition of algebraic structures, was first generalized to hypergroup theory by Marty [1]. After Marty's definition, many concepts of algebra, especially hypergroups, were generalized to hyperstructures. Subsequently, applications of hyperstructures theory to other branches of science are studied by many researchers. A detailed examination of this theory can be found at [2-4]. The concept of hyperring has been studied in different ways. The definition of hyperring, given by taking " + " hyperoperation and multiplication, was made by Krasner and is known by his name. A class of hyperrings is multiplicative hyperring which satisfies the axioms similar a ring, but product replaced by hyperproduct. The multiplicative hyperring defined by Rota in 1982 and its properties have been studied by many mathematicians [5-9].

In this paper, we consider the notions of n-ideal and δ-n-ideal in commutative rings and extend these notions n-hyperideals and δ-n-hyperideals to commutative multiplicative hyperrings. Furthermore, we characterize for the δ-n hyperideals of commutative multiplicative hyperring.

First of all, let us to introduce some notions and results of algebraic hyperstructures theory, which we will need to development our paper. Let H be a nonempty set and we mean the set of all nonempty subsets of H by $P^*(H)$. A map $\circ$: $H \times H \rightarrow P^*(H)$ is called a hyperoperation on H. Naturally, we can extend the hyperoperation $\circ$ to subsets of H, as follows:

$$X \circ Y = \bigcup_{x\in X, y\in Y} x \circ y, \qquad X \circ h = \bigcup_{x\in X} x \circ h, \qquad h \circ Y = \bigcup_{y\in Y} h \circ y$$

where $\emptyset \neq X, Y \subseteq H$ and $h \in H$.

R is called a multiplicative hyperring with operation + and hyperoperation $\circ$ if

$(R, +)$ is an abelian group,

$(R, \circ)$ is a semihypergroup, i.e, $(x \circ y) \circ z = x \circ (y \circ z)$, for all $x, y, z \in R$,

For all $x, y, z \in R$, we have $x \circ (y + z) \subseteq (x \circ y) + (x \circ z)$ and $(y + z) \circ x \subseteq (y \circ x) + (z \circ x)$,

For all $x, y \in R$, $x \circ (-y) = (-x) \circ y = -(x \circ y)$.

If in (iii) the equality holds, then R has a strongly distributive property. Also R is called commutative if $x \circ y = y \circ x$ for all $x, y \in R$ and an element $e \in R$ is said to be a left (resp. right) scalar identity if $e \circ x = x$, $(resp.\ x \circ e = x)$, $for\ all\ x \in R$. An element e is called scalar identity element if it is both left and right scalar identity element [10]. If $0 \in x \circ y$ and $x \neq 0$, where $\forall x, y \in R$, then $y = 0$, then a commutative multiplicative hyperring R is called a strong hyperdomain [11].

A nonempty subset I of a multiplicative hyperring R is a hyperideal if

$$I - I \subseteq I$$

$$x \circ r \cup r \circ x \subseteq I \text{ for all } x \in I, for\ all\ r \in R$$

The set of all hyperideals of R is denoted by $I(R)$. A hyperideal $I (\neq R)$ of a multiplicative hyperring R is called prime hyperideal if for all $a, b \in R$, $a \circ b \subseteq I$ implies that $a \in I$ or $b \in I$ [12]. An element a is called nilpotent element of R if there exists a positive integer n such that $0 \in a^n$ where for any positive integer $n > 1, a^n = \underbrace{a \circ a \circ \dots \circ a}_{n-times}$ and $a^1 = \{a\}$ and we denoted the set of all nilpotent elements of R by nil(R) (for more details see[8]).

Let $(R, +, \circ)$ and $(S, +', *)$ be two commutative multiplicative hyperrings and $g: R \to S$ be a map. Then g is called a homomorphism (resp. good homomorphism) if g satisfies the following conditions for all $a, b \in R$,

$g(a + b) = g(a) +' g(b)$,

$g(a \circ b) \subseteq g(a) * g(b)$ (resp. $g(a \circ b) = g(a) * g(b)$)

In [10], an expansion function over the set of all hyperideals of a multiplicative hyperring is defined as following:

A function $\delta: I(R) \to I(R)$ that satisfies the following two conditions is called an expansion function of I(R)

(1) $I \subseteq \delta(I)$,

(2) $If\ I \subseteq J, then\ \delta(I) \subseteq \delta(J),\ for\ all\ I, J \in I(R)$

In [13], G. Ulucak defined δ-primary hyperideal as follows; A proper hyperideal I of R is called δ-primary hyperideal of a multiplicative hyperring $(R, +, \circ)$ if for all $a, b \in R$, $a \circ b \subseteq I\ implies\ that\ either\ a \in I\ or\ b \in \delta(I)$, where δ is an expansion function of I(R).

n-Hyperideals of the Multiplicative Hyperring

In this section, the definitions of n-ideal and δ-n-ideal given in [14] and [13], respectively, will be generalized to a commutative multiplicative hyperrings. Now we give a definition of n- hyperideal and some properties of this concept which is in [15]. For the safe of completeness, we will give the proofs of Theorem 2.1. and Theorem 2.2. Throughout this paper, unless otherwise stated, $(R, +, *)$ will be taken as a commutative multiplicative hyperring.

Definition 2.1. Let I be a hyperideal of $(R, +, *)$ and $I \neq R$. For all $a, b \in R$, if $a * b \subseteq I$ and a is non-nilpotent element implies that $b \in I$ then I is called a n-hyperideal of R.

Example 2.1. Let $(\mathbb{Z}_4$, +, .) be a ring and $I = \{\bar{0}, \bar{2}\}$ be an ideal of $\mathbb{Z}_4$. We define hyperoperation ∘ in $\mathbb{Z}_4$. For all $\bar{m}, \bar{n} \in \mathbb{Z}_4$, $\bar{m} \circ \bar{n} = \overline{mn} + I$

o	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{0}$	I	I	I	I
$\bar{1}$	I	$\bar{1} + I$	I	$\bar{1} + I$
$\bar{2}$	I	I	I	I
$\bar{3}$	I	$\bar{1} + I$	I	$\bar{1} + I$

Then $(\mathbb{Z}_4, +, \circ)$ is a multiplicative hyperring and $H = \{\bar{0}, \bar{2}\}$ is a hyperideal of $\mathbb{Z}_4$. Since $\bar{m} \circ \bar{n} \subseteq H$ and $\bar{m}$ is non-nilpotent implies that $\bar{n} \in H$, for all $\bar{m}, \bar{n} \in \mathbb{Z}_4$. Then H is a n-hyperideal of $\mathbb{Z}_4$.

Example 2.2. Let $(\mathbb{Z}, +, *)$ be a multiplicative hyperring with respect to hyperopretion ∗ defined by

$a * b = \{a.b, 2a.b, 3a.b, \dots\}$, for all $a, b \in \mathbb{Z}$ in [4]. $2\mathbb{Z}$ is a hyperideal of $\mathbb{Z}$ but it is not n-hyperideal. Because $4 * 3 \subseteq 2\mathbb{Z}$ and 4 is a non-nilpotent but $3 \notin 2\mathbb{Z}$.

Theorem 2.1. Let $K = \{I_k : k \in \Omega\}$ be a nonempty family of n-hyperideals of a multiplicative hyperring (R, +, ∗). Then $\bigcap_{k\in\Omega} I_k$ is a n-hyperideal of R and if K is a chain, then $\bigcup_{k\in\Omega} I_k$ is a n-hyperideal of R.

Proof. $\bigcap_{k\in\Omega} I_k$ is a n-hyperideal, it is clear from Definition 2.3. We will show that $\bigcup_{k\in\Omega} I_k$ is a n-hyperideal. Let K be a chain. Since $I_k \subseteq R$ and $I_k \neq \emptyset$ for $k \in \Omega$. $\bigcup_{k\in\Omega} I_k \subseteq R$ and $\bigcup_{k\in\Omega} I_k \neq \emptyset$. For all $x, y \in \bigcup_{k\in\Omega} I_k$ and $r \in R$, then there exist $i, j \in \Omega$ such that $x \in I_i$, $y \in I_j$. Suppose $I_i \subseteq I_j$ then $x \in I_j$ since I_j is a hyperideal, $x - y \in I_j$ and $x * r \cup r * x \subseteq I_j$. Hence $x - y \in \bigcup_{k\in\Omega} I_k$ $and\ x * r \cup r * x \subseteq \bigcup_{k\in\Omega} I_k$. Let $x * y \subseteq \bigcup_{k\in\Omega} I_k$ and x is non-nilpotent element for $x, y \in R$. $x * y \subseteq \bigcup_{k\in\Omega} I_k \Rightarrow \exists i \in \Omega, x * y \subseteq I_i$. Since I_i is a n-hyperideal and x is non-nilpotent, $y \in I_i$. Hence $y \in \bigcup_{k\in\Omega} I_k$ and $\bigcup_{k\in\Omega} I_k$ is a n-hyperideal of R.

Theorem 2.2. Let f: (R, +, o)→(S, +', ∗) be a good homomorphism. Then

If J is a n-hyperideal of S, then $f^{-1}(J)$ is a n-hyperideal of R.

If f is an isomorphism and I is a n-hyperideal of R, then $f(I)$ is a n-hyperideal of S.

Proof. i. Since J is a hyperideal and f homomorphism, $f^{-1}(J) = \{r \in R: f(r) \in J\} \neq \emptyset$ is a hyperideal of R. Let us show that $f^{-1}(J)$ is a n-hyperideal. Let $r_1 o r_2 \subseteq f^{-1}(J)$ and r_1 is a non-nilpotent element. Then for all $n \in \mathbb{N}$, $0 \notin (r_1)^n$ so $0_S = f(0) \notin f(r_1)^n$, thus $f(r_1)$ is a non-nilpotent element in S. Since $r_1\ o\ r_2 \subseteq f^{-1}(J)$ and f is a homomorphism, $f(r_1\ o\ r_2) = f(r_1) * f(r_2) \subseteq f(f^{-1}(J)) \subseteq J$. Therefore $f(r_2) \in J$ because J is a n-hyperideal and $f(r_1)$ is a non-nilpotent element. Hence $r_2 \in f^{-1}(J)$ and so $f^{-1}(J)$ is a n-hyperideal of R.

ii. It is clear that $f(I) = \{f(r): r \in I\} \subseteq S$ is a hyperideal of S. Now, we will show that $f(I)$ is a n-hyperideal. For all $s_1, s_2 \in S$, $s_1 * s_2 \subseteq f(I)$ and s_1 is non-nilpotent. Since f is an isomorphism, $f(r_1) = s_1\ and\ f(r_2) = s_2$, for some $r_1, r_2 \in R$. Since s_1 is non-nilpotent, for all $n \in \mathbb{N}, 0 \notin f(r_1)^n = f((r_1)^n)$ and $0 \notin (r_1)^n$, i.e, r_1 is non-nilpotent. $f(r_1\ o\ r_2) \subseteq f(I) \Rightarrow r_1\ o\ r_2 \subseteq I$. From the definition of n-hyperideal, $r_2 \in I$. Hence $s_2 = f(r_2) \in S$, thus $f(I)$ is a n-hyperideal of S.

The set $ann(x) = \{r \in R: 0 \in r * x\}$ is called the annihilator of x in (R, +, ∗) and x is said to be a zerodivisor element of R if $ann(x) \neq 0$. The set of all zerodivisor elements of R denoted by $z_d(R)$.

Definition 2.2. Let I be a proper hyperideal of (R, +, ∗). We say that I is a z_d-hyperideal, precisely when, whenever $a, b \in R$ with $a * b \subseteq I$ implies that $ann(a) \neq \{0\}\ or\ b \in I$.

Example 2.3. Let $(\mathbb{Z}_6, +, .)$ be a ring. We define the following hyperoperation ∗ on $\mathbb{Z}_6$: For all $\bar{a}, \bar{b} \in \mathbb{Z}_6$,

$\bar{a} * \bar{b} = \{\bar{a}.\bar{b}, 2\bar{a}.\bar{b}, 3\bar{a}.\bar{b}, 4.\bar{a}.\bar{b}, 5\bar{a}.\bar{b}\}$. Then $(\mathbb{Z}_6, +, *)$ is a commutative multiplicative hyperring. $H = \{\bar{0}, \bar{2}, \bar{4}\}$ is a z_d-hyperideal of $\mathbb{Z}_6$.

Example 2.4. Let $(\mathbb{Z}, +, *)$ be a multiplicative hyperring w.r.t hyperoperation in Example 2.2. Then $4\mathbb{Z}$ is a hyperideal of $\mathbb{Z}$but it is not z_d-hyperideal. Because $4*3 \subseteq 4\mathbb{Z}$ but $ann(4) = 0$ and $3 \notin 4\mathbb{Z}$.

Theorem 2.3. Every n-hyperideal of R is a z_d-hyperideal.

Proof. Let I be n-hyperideal of R. Assume that $r * s \subseteq I$ and $ann(r) = 0$ for $r, s \in R$. Then $0 \notin r^n$ for all $n \in \mathbb{N}$. Since I is a n-hyperideal of R and r is non-nilpotent element in R, $s \in I$. Hence, I is a z_d-hyperideal of R.

Example 2.5 shows that the converse of the Theorem 2.3 is not true.

Example 2.5. Consider the commutative multiplicative hyperring $\mathbb{Z}_6$ in Example 2.3. $H = \{\bar{0}\}$ is a z_d-hyperideal of $\mathbb{Z}_6$ but H is not n-hyperideal.

Proposition 2.1. If R is a strong hyperdomain, then {0} is a n-hyperideal of R.

Proof. Let $a * b \subseteq 0$ and a is a non nilpotent element for $a, b \in R$. Then $a * b \subseteq 0$ and $0 \notin a^n$ for all $n \in \mathbb{N}$ and so $a \neq 0$. Since R is a strong hyperdomain and $0 \in a * b$, $b = 0$. Therefore $b \in 0$ and so {0} is a n-hyperideal.

In [12], Dasgupta defined the radical of the arbitrary hyperideal I as the intersection of all prime hyperideals containing I and denoted by $Rad(I) = \sqrt{I}$. He also showed that $D \subseteq \mathrm{Rad}(I)$ by defining a set $D = \{r \in R : r^n \subseteq I \; for\ some\ n \in \mathbb{N}\}$ for an arbitrary hyperideal I. We will denote this set D by $D(I)$ for any hyperideal I.

In [16], Anbarloei showed that for $\mathrm{a} \in \mathrm{R}$, $(\mathrm{I{:}a}) = \{\mathrm{r} \in \mathrm{R}{:}\ \mathrm{r} * \mathrm{a} \subseteq \mathrm{I}\}$ is a hyperideal of R. The following proposition prove that n-hyperideal can be characterization with $(I{:}a)$.

Proposition 2.2. Let I be a proper hyperideal of R. I is a n-hyperideal of R if and only if $I = (I{:}a)$ for every $a \notin D(0)$.

Proof. Suppose I is a n-hyperideal of R. Then $x * a \subseteq I$, for all $x \in I$. Therefore $x \in (I{:}a)$ and so $I \subseteq (I{:}a)$. Let $u \in (I{:}a)$ and $a \notin D(0)$. Then $a * u \subseteq I$. By Definition 2.1, $u \in I$. Thus $(I{:}a) \subseteq I$ and so $I = (I{:}a)$. Conversely, let $I = (I{:}a)$, for every $a \notin D(0)$. Suppose $a * u \subseteq I$ for all $a, u \in R$, a is a non-nilpotent. Then $u \in (I{:}a) = I$ and so I is a n-hyperideal.

Proposition 2.3. Let N be a proper hyperideal of (R, +,∗) with identity 1. Then N is a n-hyperideal if and only if for $U, V \in I(R)$, $U * V \subseteq N$, with $U \cap \big(\mathrm{R} - \mathrm{D}(0)\big) \neq \emptyset$ implies $V \subseteq$ N.

Proof. Suppose that $U * V \subseteq N$ with $U \cap \big(\mathrm{R} - D(0)\big) \neq \emptyset$ for hyperideals U and V of R. Since $U \cap \big(\mathrm{R} - D(0)\big) \neq \emptyset$, there exists $x \in U$ such that $x \notin D(0)$. Then $x * V \subseteq N$ and so $V \subseteq (N{:}x)$. Therefore, $V \subseteq N$ by Proposition 2.2. Conversely, $u * v \subseteq N$ and u is non-nilpotent element for all $u, v \in R$. Then $u \notin D(0)$. Let $U =< u >$ and $V =< v >$. Then $\mathrm{U} * \mathrm{V} =< \mathrm{u} > * < \mathrm{v} > \subseteq < \mathrm{u} * \mathrm{v} > \subseteq \mathrm{N}$ and $\mathrm{U} \cap \big(\mathrm{R} - \mathrm{D}(0)\big) \neq \emptyset$. Therefore $V \subseteq N$ and so $b \in N$. Thus, N is a n-hyperideal of R.

Theorem 2.4. Let K be a hyperideal of (R, +,∗) with $\mathrm{K} \cap \big(\mathrm{R} - \mathrm{D}(0)\big) \neq \emptyset$. The following statements are hold:

If J_1, J_2 are n-hyperideals of R with $\mathrm{J_1} * \mathrm{K} = \mathrm{J_2} * \mathrm{K}$, then $J_1 = J_2$.

If $J * K$ is a n-hyperideal of R, then $J * K = J$.

Proof. i. Since J_1 is a hyperideal, $J_1 * K = J_2 * K \subseteq J_1$. Then $J_2 \subseteq J_1$, by Proposition 2.3. Because J_1 is a n-hyperideal, $J_2 * K \subseteq J_1$ and $\mathrm{K} \cap \big(\mathrm{R} - \mathrm{D}(0)\big) \neq \emptyset$. Similarly, $J_1 \subseteq J_2$. Thus, $J_1 = J_2$.

ii. Let $J * K$ be a n-hyperideal of R. Then $J * K$ is a hyperideal and so $J * (J * K) \subseteq J * K$. Since $J * K$ is a n-hyperideal and $\mathrm{K} \cap \big(\mathrm{R} - D(0)\big) \neq \emptyset$, $J \subseteq J * K$. Therefore $J = J * K$.

In Theorem 2.5, another characterization will be given for prime hyperideals to be n-hyperideal.

Theorem 2.5. Let (R, +, ∗) be a commutative multiplicative hyperring with scalar identity (1) and Q be a prime hyperideal of R with $Q \;\cap D(0) \neq \emptyset$ Then Q is an n-hyperideal if and only if $Q \; = D(0)$.

Proof. $D(0) \subseteq Q$ is trivial. Now, we assume that $Q \not\subseteq D(0)$. Then there exist $a \in Q$ such that $0 \notin a^n$, for all $n \in \mathbb{N}$ and so a is non-nilpotent. Since Q is a n-hyperideal and $\mathrm{a} = \mathrm{a} * 1 \subseteq \mathrm{Q}$, $1 \in \mathrm{Q}$ and $\mathrm{a} * 1 \subseteq \mathrm{Q}$, for all $a \in R$. Hence, $a \in Q$ and R= Q, which is a contradiction. Thus, Q $= D(0)$.

Conversely, suppose that $Q \; = D(0)$. Let for $x, y \in R$, $x * y \subseteq Q$ and x is a non-nilpotent. Then $x \notin Q = D(0)$ and $y \in Q$ because Q is a prime hyperideal. Hence, Q is a n-hyperideal of R.

Example 2.6. Let $(\mathbb{Z}, +, *)$ be a multiplicative hyperring in Example2.2. $2\mathbb{Z}$ is a prime hyperideal of $\mathbb{Z}$ but it is not n-hyperideal of $\mathbb{Z}$.

Example 2.7. Consider the multiplicative hyperring ($\mathbb{Z}$, +,∗) in Example2.2, H= $2\mathbb{Z}$ is a prime hyperideal of $\mathbb{Z}$ and $S = \{2,4,6\} \subseteq \mathbb{Z}$. Then $(H{:}S) = \{x \in Z{:}\ x * S \subseteq H\} = \mathbb{Z}$ is a n-hyperideal but H is not n-hyperideal, because $4 * 3 \subseteq H$ and 4 is a non-nilpotent element but $3 \notin H$.

δ – n- Hyperideal of Multiplicative Hyperring

In this section, we will introduce the definition of δ-n-hyperideal over the multiplicative hyperring with scaler identity and we will give a characterization of δ-n-hyperideal. Throughout this section, all hyperideals will be taken as $\mathcal{C}$-hyperideal. $\mathcal{C}$- hyperideals of a multiplicative hyperring defined by Das Gupta in [12] as follows, let $(R, +, *)$ be a multiplicative hyperring and $\mathrm{J} \in I(R)$. J is said to be $\mathcal{C}$- ideal if for any $\mathrm{A} \in \mathcal{C}, A \cap J \neq \emptyset \Rightarrow A \subseteq J$, where $\mathcal{C} = \{r_1 * r_2 * r_3 * \ldots * r_n : \; r_i \in R,\ \mathrm{n} \in \mathbb{N}\}$.

In the following definition, we are using definition of radical I, to state once again if I is $\mathcal{C}$- ideal, then Rad(I)=D(I) in [12].

Definition 3.1. Let $\mathrm{J} \in I(R)$, $J \neq R$ and $\delta{:}\ I(R) \to I(R)$ be an expansion function. We say that J is a δ-n-hyperideal of R if for all $x, y \in R$, $x * y \subseteq J$ then either $x \in \sqrt{0}$ or $y \in \delta(J)$.

Example 3.1. Let $(\mathbb{Z}_8, +, .)$ be a ring and $I = \{\bar{0}, \bar{4}\}$ be an ideal of $\mathbb{Z}_8$. We define hyperoperation in $\mathbb{Z}_8$: For all $\bar{a}, \bar{b} \in \mathbb{Z}_8$, $\bar{a} * \bar{b} = a.b + I$. Then $(\mathbb{Z}_8, +, *)$ is a multiplicative hyperring. Let $\delta{:}\ I(\mathbb{Z}_8) \to I(\mathbb{Z}_8)$ be a function such that $\delta(H) = H$, for all H hyperideal of $\mathbb{Z}_8$. Therefore, δ is an expansion function. $H = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$ is a δ-n-hyperideal.

Example 3.2. Consider the multiplicative hyperring (ℤ, +, *) in Example2.2. Let δ: I(ℤ)→I(ℤ) be a function such that $\delta(H) = H$ for all H hyperideal. Then δ is an expansion function. H= 2ℤ is hyperideal of multiplicative hyperring ℤ but H is not δ-n-hyperideal. Indeed, $4 * 3 \subseteq H$ but $4 \notin \sqrt{0}$ and $3 \notin \delta(H)$.

Proposition 3.1. Let R be multiplicative hyperring with scaler identity and δ be an expansion of hyperideals of R and J a proper hyperideal of R with $\delta(J) \neq R$. If J is a δ-n-hyperideal of R, then $J \subseteq \sqrt{0}$.

Proof. Suppose that $J \nsubseteq \sqrt{0}$. Then there exists element $a \in (J\text{-}\sqrt{0})$. Since $a \in 1 * a \subseteq J$and $a \notin \sqrt{0}$, $1 \in \delta(J)$. Then for all $r \in R$, $r \in 1 * r \subseteq \delta(J)$ and so $\delta(J) \neq R$, a contradiction. Thus $J \subseteq \sqrt{0}$.

Theorem 3.1. Let J be a hyperideal of $(R, +, *)$. If $J = \sqrt{0}$, then J is a δ-n-hyperideal if and only if J is a δ-primary hyperideal.

Proof. Suppose that J is a δ-n-hyperideal and $x, y \in R, x * y \subseteq J$ and $x \notin J$. Then x is not nilpotent and so $y \in \delta(J)$ because J is a δ-n-hyperideal. Hence, J is a δ-primary hyperideal. Conversely, J is a δ-primary hyperideal, $x, y \in R, x * y \subseteq J$ and $x \notin \sqrt{0}$. Then $x \notin J$ and $y \in \delta(J)$ since J is a δ- primary hyperideal. Hence, J is a δ-n-hyperideal of R.

Proposition 3.2. Let δ be an expansion of hyperideals of R. Then the following are hold

i. Let J be a δ-primary hyperideal of R with $\delta(J) \neq R$. Then J is a δ-n-hyperideal of R if and only if $J \subseteq \sqrt{0}$.

ii. Let J be a prime hyperideal of R with $\delta(J) \neq R$. Then J is a δ-n-hyperideal of R if and only if J= $\sqrt{0}$.

Proof. i. It is clear by Proposition 3.1. and Theorem 3.1.

ii.Since J is prime hyperideal, $\sqrt{0} \subseteq$ J. From Proposition 3.1, $J \subseteq \sqrt{0}$ and so $J = \sqrt{0}$. Conversely, since J is a prime hyperideal, J is a δ- primary hyperideal by [13]. From Theorem 3.1, J is a δ-n-hyperideal of R.

Theorem 3.2. For a proper hyperideal I of R and an expansion of fuction δ, the following statements are equivalent:

I is a δ-n-hyperideal of R.

$(I: a) \subseteq \sqrt{0}$ for all a ∈ R – δ(I).

If $a \circ J \subseteq I$ for some a ∈ R and an hyperideal J of R, then $a \in \sqrt{0}$ or J ⊆ δ(I).

If $J \circ K \subseteq I$ for some hyperideals J and K of R implies $J \cap (R - \sqrt{0}) = \emptyset$ or K ⊆ δ(I).

Proof. $(i) \Rightarrow (ii)$ Assume that any $x \in (I: a)$, then $x \circ a \subseteq I$. Since I is a δ-n-ideal of R and $a \notin \delta(I)$, $x \in \sqrt{0}$. Thus, $(I: a) \subseteq \sqrt{0}$.

$(ii) \Rightarrow (iii)$ Suppose that if $a \circ J \subseteq I$ and $J \nsubseteq \delta(I)$. For any $j \in J, a \circ j \subseteq I$ and so $J \subseteq (a: I) \subseteq \sqrt{0}$. Since $J \nsubseteq \delta(I)$, there exist $j \in J$ but $j \notin \delta(I)$ and so $a \in \sqrt{0}$ by (ii).

$(iii) \Rightarrow (iv)$ Let $J \circ K \subseteq I$ and suppose $J \cap (R - \sqrt{0}) \neq \emptyset$. Then there is an element $j \in J - \sqrt{0}$. For any $k \in K, J \circ k \subseteq I$. Then for $j \in J - \sqrt{0}$, $j \circ k \subseteq I$. From (iii), $k \in \delta(I)$ and so K ⊆ δ(I).

$(iv) \Rightarrow (i)$ Let $x \circ y \subseteq I$ for some $x, y \in R$ and $J = (x), K = (y)$. Then $J \circ K \subseteq I$. If $J \cap (R - \sqrt{0}) \neq \emptyset$, then $x \in \sqrt{0}$. If $K \nsubseteq \delta(I)$, then $y \notin \delta(I)$ and so $K \nsubseteq I$, a contraction. Then, $x \in \sqrt{0}$ by our assumption. Thus, I is a δ-n-hyperideal of R.

Example 3.3. Consider the commutative multiplicative hyperring $(\mathbb{Z}_6, +, *)$ in Example 2.3. $H = \{\bar{0}, \bar{2}, \bar{4}\}$, $K = \{\bar{0}, \bar{3}\}, \bar{0}$ and $\mathbb{Z}_6$ are hyperideals of multiplicative hyperring $\mathbb{Z}_6$. $\delta: I(\mathbb{Z}_6) \to I(\mathbb{Z}_6)$ be a function such that

$$\delta(X) = \begin{cases} \mathbb{Z}_6, & X = H, \mathbb{Z}_6 \\ K, & X = K, \{0\} \end{cases}$$

Therefore, δ is an expansion function. $\{0\}$ is a δ-n-hyperideal but it is not δ-primary hyperideal of $\mathbb{Z}_6$.

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Conflicts of interest

There are no conflicts of interest in this work.

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