On Fixed Point Results for Generalized Contractions in Non-Newtonian Metric Spaces

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ABSTRACT

The work of non-Newtonian calculus was begun in 1972. This calculus provides a different area to the classical one. Non-Newtonian metric concept was defined in 2002 by Basar and Cakmak. Then Binbaşıoğlu et al. had given the metric spaces of non-Newtonian in 2016. Also, they started to study on the fixed-point theory by defining some topological properties in non-Newtonian metric spaces. In this work, we give some fixed-point theorems and results for self-mappings satisfying certain conditions in the non-Newtonian metric spaces.

Keywords: Fixed point, Non-Newtonian metric space, Contraction mapping, Generalized contraction mapping.

Introduction

There exist too many studies on fixed-point theory in different spaces [1-12]. Also, there are many applications of the theory and mappings that meet certain conditions of contraction and have been a crucial area of different research works.

The non-Newtonian calculus is alternative to what is customary. The non-Newtonian calculus in various fields including information technology, fractal geometry, economic growth, finance, wave theory, quantum physics, in medicine for examples tumor therapy, cancer-chemotherapy, in mathematics for examples functional analysis, differential equations, approximation theory, problems of decision making, and chaos theory has many applications. The non-Newtonian metric concept was defined in 2002 by Basar et al. and then Binbaşıoğlu et al. gave the metric spaces of non-Newtonian in 2016. Also, they started to study on the fixed-point theory in non-Newtonian metric spaces.

In this work, we present fixed-point theorems and results for self-mappings satisfying certain conditions in the non-Newtonian metric spaces.

Preliminaries

We mention that some basic knowledge related to structure of non-Newtonian calculus.

Definition

A generator is called as an injective function from ℝ to a subset of ℝ [6].

Remark

Every generator generates an arithmetic. An arithmetic is generated by a generator [6].
\[ |a|_N \text{ denotes the } \beta - \text{absolute value for a number } a \in A \subset \mathbb{R}(N) \text{ defined by } \beta(|\beta^{-1}(a)|) \text{ and so } \]

\[ \sqrt{a^{2N}} = |a|_N = \beta(|\beta^{-1}(a)|). \]

Thus,

\[ |a|_N = \beta(|\beta^{-1}(a)|) = \begin{cases} a, & \beta(0) \prec a, \\ \beta(0), & \beta(0) = a, \\ \beta(0) \succ a. & \end{cases} \]

Let us take any \( c \in \mathbb{R}(N) \). If \( c \prec \beta(0) \), then \( c \) is called a positive non-Newtonian real number. If \( c \succ \beta(0) \), then \( c \) is called a non-Newtonian negative real number. If \( c = \beta(0) \), then \( c \) is called an unsigned non-Newtonian real number. Non-Newtonian positive and negative real numbers are denoted by \( \mathbb{R}^+(N) \) and \( \mathbb{R}^-(N) \) respectively [6].

**Definition**

Let us take \( X \neq \emptyset \) and suppose that \( d_N: X \times X \to \mathbb{R}^+(N) \) satisfies the following conditions for \( a, b, c \in X \):

\( (\text{NM1}) d_N(a, b) = \beta(0) \) if \( a = b \),

\( (\text{NM2}) d_N(a, b) = d_N(b, a) \),

\( (\text{NM3}) d_N(a, b) \preceq d_N(a, c) \preceq d_N(b, c) \).

Then \( d_N \) is a non-Newtonian metric on \( X \). Also \( (X, d_N) \) is a non-Newtonian metric space [6].

**Example**

Assume that \( d_N \) is defined as \( d_N(a, b) = |a \prec b|_N \) for all \( a, b \in \mathbb{R}(N) \), then \( (\mathbb{R}(N), d_N) \) is a non-Newtonian metric space [6].

**Main Results**

**Theorem**

Let \( d_N \) be a non-Newtonian complete metric on \( X \) and \( c, d \) be positive integers. If a mapping \( K: X \to X \) satisfies

\[ d_N(K^c a, K^d b) \preceq k \times d_N(a, b) \times d_N(a, K^c a) \times d_N(b, K^d b) \times d_N(a, K^c a) \times d_N(b, K^d b) \]

for all \( a, b \in X \), where \( k, l, m, n, p \) are non-Newtonian positive real numbers with \( k \preceq l \preceq m \preceq n \preceq p \preceq \beta(0) \), then \( K \) has a unique fixed-point.

**Proof**

Take \( a_0 \in X, t \geq \beta(0) \), we construct

\[ a_{2t+1} = K^c a_{2t}, \]

\[ a_{2t+2} = K^d a_{2t+1}. \]

Then

\[ d_N(a_{2t+1}, a_{2t+2}) = d_N(K^c a_{2t}, K^d a_{2t+1}) \]

\[ \preceq k \times d_N(a_{2t}, a_{2t+1}) \times d_N(a, K^c a) \times d_N(b, K^d b) \times d_N(a, K^c a) \times d_N(b, K^d b). \]
It implies that
\[
\varepsilon(\beta(1) \varepsilon n) \varepsilon d_N(a_{2t+1}, a_{2t+2}) \varepsilon (k \varepsilon l \varepsilon n) \varepsilon d_N(a_{2t}, a_{2t+1}).
\]
So
\[
d_N(a_{2t+1}, a_{2t+2}) \varepsilon \varepsilon r \varepsilon d_N(a_{2t}, a_{2t+1}), \text{ where } r = \frac{(k \varepsilon l \varepsilon n)}{\beta(1) \varepsilon l \varepsilon n}
\]
implies that
\[
d_N(a_{2t+2}, a_{2t+3}) \varepsilon s \varepsilon d_N(a_{2t+1}, a_{2t+2}),
\]
where \( s = \frac{(k \varepsilon m \varepsilon p)}{\beta(1) \varepsilon m \varepsilon n} \).

Therefore, we get for each \( t = 0, 1, 2, \ldots \)
\[
d_N(a_{2t+1}, a_{2t+2}) \varepsilon r \varepsilon d_N(a_{2t}, a_{2t+1})
\]
\[
\varepsilon r \varepsilon \varepsilon \varepsilon d_N(a_{2t-1}, a_{2t})
\]
\[
\varepsilon r \varepsilon \varepsilon \varepsilon d_N(a_{2t-2}, a_{2t-1})
\]
\[
\varepsilon \ldots \varepsilon \varepsilon \varepsilon d_N(a_0, a_1),
\]
\[
d_N(a_{2t+2}, a_{2t+3}) \varepsilon s \varepsilon d_N(a_{2t+1}, a_{2t+2})
\]
\[
\varepsilon \ldots \varepsilon \varepsilon \varepsilon d_N(a_0, a_1).
\]

So, for \( y < z \) we have
\[
d_N(a_{2y+1}, a_{2y+2}) \varepsilon d_N(a_{2y+1}, a_{2y+2})
\]
\[
\varepsilon d_N(a_{2y+2}, a_{2y+3}) \varepsilon \ldots \varepsilon d_N(a_{2z}, a_{2z+1})
\]
\[
\varepsilon [r \varepsilon \sum_{i=y+1}^{z} (r \varepsilon s)^{\nu N} \varepsilon d_N(a_0, a_1)]
\]
\[
\varepsilon [r \varepsilon \varepsilon \varepsilon \varepsilon \varepsilon d_N(a_0, a_1)]
\]
\[
\varepsilon \beta(1) \varepsilon r \varepsilon \varepsilon \varepsilon d_N(a_0, a_1).
\]

Then we deduced
\[
d_N(a_{2y}, a_{2y+1}) \varepsilon (\beta(1) \varepsilon r) \varepsilon \varepsilon \varepsilon (r \varepsilon s)^{\nu N} \varepsilon d_N(a_0, a_1),
\]
\[
d_N(a_{2y}, a_{2z}) \varepsilon (\beta(1) \varepsilon r) \varepsilon \varepsilon \varepsilon (r \varepsilon s)^{\nu N} \varepsilon d_N(a_0, a_1),
\]
\[
d_N(a_{2y+1}, a_{2z+1}) \varepsilon (\beta(1) \varepsilon r) \varepsilon \varepsilon \varepsilon (r \varepsilon s)^{\nu N} \varepsilon d_N(a_0, a_1).
\]

For \( 0 < w < v \), \( d_N(a_w, a_v) \varepsilon q_w \), where
\[
q_w = (\beta(1) \varepsilon r) \varepsilon \varepsilon \varepsilon (r \varepsilon s)^{\nu N} \varepsilon d_N(a_0, a_1) \text{ with an integer part of } \frac{w}{z}.
\]

So \( \{a_w\} \) is non-Newtonian Cauchy. Since \( (X, d_N) \) is non-Newtonian complete, there exists \( x \in X \) such that
\[
a_w \rightarrow x.
\]

For a non-Newtonian real number \( 0 \varepsilon \beta(e) \), choose \( d_0 \in \mathbb{N} \) such that
Theorem

We suppose that for some $a_{2t}$, now we show that $d_N(a_{2t-1}, a_{2t}) < \frac{\beta(e)}{\beta(3)} A d_N(x, a_{2t-1}) < \frac{\beta(e)}{\beta(3)} A d_N(x, a_{2t})$ for all $t \geq d_q$, where

$$
A = \max\{\frac{\beta(1) + n}{\beta(1)} l, \frac{k + p}{\beta(1)} l, \frac{m}{\beta(1)} l, \frac{m}{\beta(1)} l, \frac{m}{\beta(1)} l\}.
$$

Now,

$$
\begin{align*}
& d_N(x, K^w a) < d_N(x, a_{2t}) + d_N(a_{2t}, K^w x) < \\
& < d_N(x, a_{2t}) + k \cdot d_N(x, a_{2t-1}) + l \cdot d_N(a_{2t-1}, K^w a_{2t-1}) + m \cdot d_N(a_{2t-1}, K^w a_{2t-1}) \\
& < d_N(x, a_{2t}) + k \cdot d_N(x, K^w a_{2t-1}) + l \cdot d_N(a_{2t-1}, x) + m \cdot d_N(a_{2t-1}, a_{2t}) \\
& < d_N(x, K^w x) + k \cdot d_N(x, a_{2t}) + l \cdot d_N(a_{2t-1}, x) + m \cdot d_N(a_{2t-1}, a_{2t}) \\
& \leq d_N(x, K^w x) + A \cdot d_N(x, a_{2t}) + A \cdot d_N(a_{2t-1}, a_{2t}) \\
& \leq \frac{\beta(e)}{\beta(3)} + \frac{\beta(e)}{\beta(3)} + \frac{\beta(e)}{\beta(3)} = \beta(e).
\end{align*}
$$

Therefore

$$
d_N(x, K^w x) < \frac{\beta(e)}{\beta(y)} d_N(x, K^w x),
$$

for every $y \in \mathbb{N}$. From $\frac{\beta(e)}{\beta(y)} d_N(x, K^w x) < \beta(0)$ we have $d_N(x, K^w x) = \beta(0)$. This implies that $x = K^w x$.

By using the inequality,

$$
d_N(x, K^w x) < d_N(x, a_{2t+1}) + d_N(a_{2t+1}, K^w x),
$$

now we show that $x = K^w x$.

So $x$ is a fixed-point of $K$.

We suppose that for some $x^*$, there exists another point $x^* \in X$ such that $x^* = Kx^*$. Thus, we have

$$
\begin{align*}
& d_N(x, x^*) = d_N(K^w x, K^w x) = d_N(K^w x, K^w x) \\
& < k \cdot d_N(K^w x, x^*) + l \cdot d_N(K^w x, x^*) \\
& < k \cdot d_N(K^w x, x^*) + l \cdot d_N(K^w x, x^*) + m \cdot d_N(k^w x, x^*) + n \cdot d_N(k^w x, x^*) \\
& \leq (k + n + p) \cdot d_N(K^w x, x^*).
\end{align*}
$$

Consequently, $x^*$ is equal to $x$.

**Theorem**

Let $d_N$ be non-Newtonian complete metric on $X$. If $K : X \to X$ satisfies

$$
\begin{align*}
d_N(Ka, Kb) & < k \cdot d_N(a, b) + l \cdot d_N(a, Ka) \\
& < k \cdot d_N(a, b) + l \cdot d_N(a, Ka) \\
& < k \cdot d_N(a, b) + l \cdot d_N(a, Kb) + m \cdot d_N(a, Kb) + p \cdot d_N(b, Ka)
\end{align*}
$$

for all $a, b \in X$, where $k, l, m, n, p$ are non-Newtonian positive real numbers with $k \cdot l \cdot m \cdot n \cdot p \cdot \beta(1)$, then $K$ has a unique fixed-point.
Proof
Since $d_N$ is a non-Newtonian metric, the above inequality implies that

$$d_N(Ka,Kb) \leq k d_N(a,b) + \frac{l + m}{\beta(2)} d_N(a,Ka) + \frac{n + p}{\beta(2)} d_N(b,Kb).$$

If we substitute $K^v = K^w = K$ in the above theorem, we get the required result.

Corollary
Let $(X, d_N)$ be a non-Newtonian complete metric space and $v, w$ be positive integers. If a self-mapping $K$ on $X$ satisfies

$$d_N(K^v a, K^w b) \leq k d_N(a,b) + l d_N(a,K^v a) + m d_N(b,K^w b) + n d_N(a,K^w b) + p d_N(b,K^v a)$$

for all $a, b \in X$, where $k, l, m, n, p$ be non-Newtonian positive real numbers with $k \leq l \leq m \leq n \leq p \beta(1)$, $l = m = n = p$, then $K$ has a unique fixed-point.

Conclusion

In this paper, we use the concept of non-Newtonian metric space and present some new fixed-point theorems. We expect that our research results can offer a mathematical basis. In the future research, we will explore so concrete applications of the obtained results, here.

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Conflicts of Interest

The author states that did not has conflict of interests.

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