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Solving the Generalized Rosenau-KdV Equation by the Meshless Kernel-Based Method of Lines

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ABSTRACT

This current investigation consists of the numerical solutions of the Generalized Rosenau-KdV equation by using the meshless kernel-based method of lines, which is a truly meshless method. The governing equation is a nonlinear partial differential equation but the use of the method of lines leads to an ordinary differential equation. Thus, the partial differential equation is replaced by the ordinary differential equation. The numerical efficiency of the used technique is tested by different numerical examples. Numerical values of error norms and physical invariants are compared with known values in the literature. Moreover, Multiquadric, Gaussian, and Wendland's compactly supported functions are used in computations. It is seen that the used truly meshless method in computations is very effective with high accuracy and reliability.

Keywords: Generalized Rosenau-Kdv equation, Meshless Kernel-based method of lines, Radial basis function.

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Introduction

The nonlinear evolution equation is one of the most considerable scientific research areas. Many scientists improved various mathematical models to designate wave behavior during the past several decades. One of these models is called the KdV equation. In order to describe wave propagation and spread interaction, this equation can be used in $[1-3]$. The equation indicates the long-time evolution of wave phenomena, in which the effect of the nonlinear terms UU_x is counterbalanced by the dispersion U_{xxx}. There are a lot of works on this equation in the literature, see $[4-9]$ and references therein.

In this paper, we consider the Generalized Rosenau-KdV equation which is a nonlinear partial differential equation. It is defined by the following form:

$$u_t+u_x+u_{xxx}+u_{xxxxt}+\beta(u^p)_x=0 \tag{1}$$

where $\beta>0$, $p\geq 2$ is an integer. When p = 2, the Rosenau-KdV equation is obtained. Rosenau equation was proposed to describe the dynamics of dense discrete systems in $[10, 11]$. In the search, numerical outcomes will be conducted for different values of p. In calculations following initial and boundary conditions will be used:

$$u(a,t)=u(b,t)=0,\quad 0\leq t\leq T, \tag{2}$$

$$u(x,0)=f(x) \tag{3}$$

The solitary solution and invariants for the generalized Rosenau-KdV equation are given in $[12, 13]$. There are a large number of theoretical and numerical studies for equation (1) which are seen in references $[12-21]$.

The sech-ansätze method was used for the solitary solutions of the equation by Esfahani in [12]. Razborova et. al. [13] studied dynamics of dispersive shallow water wave of the Rosenau–KdV equation with power law nonlinearity. Solitary and periodic solutions were derived by Zuo in [14]. The solitary wave ansatz method is used to obtain topological 1-soliton solution of the generalized Rosenau-KdV equation in [15]. Conservative linear difference scheme was used to obtain numerical solutions of Rosenau-KdV and generalized Rosenau-KdV in [16]. Also, Zheng and Zhou [17] presented an average linear finite difference scheme for the numerical solution of the initial-boundary value problem of the generalized Rosenau–KdV equation. In the study [18], authors used a conservative Crank–Nicolson finite difference scheme for the initial-boundary value problem of the generalized Rosenau–KdV equation. It can be seen that the difference scheme shows a discrete analog of the main conservation laws associated to the equation in this paper. Karakoç et. al. [19] proposed the finite element method based on collocation. In the studies [20, 21], the authors solved the equation by using meshless method based on radial basis functions.

One of the important issues is a computation with high-dimensional data in many areas of science and engineering. As known, many traditional methods such as finite elements, finite differences, finite volumes, and boundary elements method require a regular domain mesh generation to solve problems. However, the meshless methods require neither domain nor surface discretization because they are independent of a mesh. So, instead of generating the mesh, they use scattered nodes, which can be randomly distributed, through the computational domain. This is a great advantage since mesh generation is one of the most time-consuming parts of any mesh-based numerical simulation. Thus, meshfree methods provide an attractive alternative for solving certain problems. Up to now, the generalized Rosenau KdV equation has not been solved by using the meshless kernel-based method of lines. This method depends upon the meshless solution technique and so there is no need to an extra discretization. It is a way of approximating partial differential equations by ordinary differential equations. Therefore, the problem of correct time-stepping will be automatically solved by the ODE solver. Our main intention in this present paper is to indicate that the method is appropriate and reliable to obtain a numerical solution to partial differential equations. That's why, in this paper, we construct the proposed method to obtain the numerical results for the generalized Rosenau KdV equation.

The design of this paper is as follows: In Sec. 2, we construct the implementation of the Meshless kernel-based method of lines. In Sec. 3, numerical outcomes are illustrated. End the study with a short conclusion given in Sec. 4.

Governing of the Proposed Method to the Generalized Rosenau-KdV Equation

Our main intention of this investigation is to solve the mentioned equation by applying the meshless kernel-based method of lines. This method leads to a system of ordinary differential equations. The advantages of the present method are that there will no time discretization at all, and there will be no unnatural linearization of the differential equation as in diversified other articles. The problem of correct time-stepping will be automatically solved by the ODE solver we call.

Here, the approximate solution u is considered by a linear combination as follows [22] :

$$u(x,t)=\sum_{j=1}^{N}\alpha_j(t)\,v_j(x) \tag{4}$$

where $\alpha_j(t)$ is an unknown term and $v_j(x)$ is spatial term obtained by using different radial basis functions. The most commonly used RBFs are Gaussian (G), Multiquadric (MQ) and Wendland's compactly supported functions which are listed in the following, respectively:

1. $\phi(r)=\exp(-r^2/\varepsilon^2)$,
2. $\phi(r)=\sqrt{(\varepsilon r)^2+1}$, where ε is a shape parameter (see the details in [23]).
3. $\phi_{l,k}(r)=(1-r)_+^k h(r)$

Wendland's compactly supported functions (W) (see the details in [24]) which are defined as follows:

$$W_{4,2}(r)=(1-r)_+^6(3+18r+35r^2),$$
$$W_{5,3}(r)=(1-r)_+^8(1+8r+25r^2+32r^3),$$
$$W_{6,4}(r)=(1-r)_+^{10}(5+50r+210r^2+450r^3+429r^4),$$
$$W_{7,5}(r)=(1-r)_+^{12}(9+108r+566r^2+1644r^3+2697r^4+2048r^5)$$

where r denotes the Euclidean distance between two collocation points. It is seen that these base functions depend on space variables. For ease notation in the rest of the paper, $\phi_{l,k}$ will be used as $W_{l,k}$. Partial derivatives of $u(x,t)$ easily evaluated as follows:

$$u_t(x,t)=\sum_{j=1}^{N}\alpha_j'(t)\,v_j(x) \tag{5}$$

$$u_x(x,t)=\sum_{j=1}^{N}\alpha_j(t)\,v_j'(x)\,) \tag{6}$$

By writing necessary derivative terms in the equation (1) we get

$$\sum_{j=1}^{N}\alpha_j'(t)\,v_j(x)+\sum_{j=1}^{N}\alpha_j(t)\,v_j'(x)+\sum_{j=1}^{N}\alpha_j(t)\,v_j'''(x)+\sum_{j=1}^{N}\alpha_j'(t)\,v_j^{iv}(x)+\beta\left(\left(\sum_{j=1}^{N}\alpha_j(t)\,v_j(x)\right)^p\right)'=0 \tag{7}$$

where the last term is a nonlinear term. After taking a derivative of nonlinear term the equation (7) can be written as follows:

$$\sum_{j=1}^{N}\left(v_j(x)+v_j^{iv}(x)\right)*\alpha_j'(t)=-\sum_{j=1}^{N}\alpha_j(t)\,v_j'(x)-\sum_{j=1}^{N}\alpha_j(t)\,v_j'''(x)-$$
$$\beta p\left(\sum_{j=1}^{N}\alpha_j(t)\,v_j(x)\right)^{p-1}\sum_{j=1}^{N}\alpha_j(t)\,v_j'(x) \tag{8}$$

This equality is written in the following symbolic form

$$(V + V^{iv}) * \alpha'(t) = -(V' * \alpha(t)) - (V''' * \alpha(t)) - \beta p(V * \alpha(t))^{p-1} * (V' * \alpha(t)) \quad (9)$$

where V is an invertible matrice and its entries are base functions and $\alpha(t)$ is a vector. Therefore, the equation (9) can be written as;

$$\alpha'(t) = -(V + V^{iv})^{-1} * [V' * \alpha(t) + V''' * \alpha(t) + \beta p(V * \alpha(t))^{p-1} * (V' * \alpha(t))] \quad (10)$$

Finally, the governing equation is converted to an ordinary differential equation. In our computations, the equation (10) is solved by using ode113 in MATLAB. The solver ode113 uses the Adams-Bashforth-Moulton predictor-corrector method.

Numerical Results

This section illustrates some numerical results of the governing equation by using the method described above. Numerical values of error norms and invariants are prominent to test the accuracy of the method. Error norms are defined as follows:

$$L_2 = \sqrt{h \sum_{j=1}^{N} \left| u_j^{exact} - u_j^{num.} \right|^2} \quad (11)$$

$$L_\infty = \max_{1 \le j \le N} \left| u_j^{exact} - u_j^{num.} \right| \quad (12)$$

For a numerical comparison of invariants following mass and energy conservations are used [12]:

$$Q(t) = \int_a^b u(x,t)\, dx \text{and} \quad (13)$$

$$E(t) = \|u\|^2 + \|u_{xx}\|^2 \quad (14)$$

In numerical treatments degree of a nonlinear term is taken as $p = 2$, $p = 3$, and $p = 5$.
Case 1: When p = 2 and $\beta = 0.5$, the solitary wave solution of the Rosenau-KdV equation is defined as follows [15]:

$$u(x,t) = \left(-\frac{35}{24} + \frac{35}{312}\sqrt{313}\right) \times \operatorname{sech}^4 \frac{1}{24}\sqrt{-26 + 2\sqrt{313}} \left[x - \left(\frac{1}{2} + \frac{\sqrt{313}}{26}\right) t\right] \quad (15)$$

Solution domain is taken as $-70 \le x \le 100$ with $h = 1$ up to time $T = 60$ with $\Delta t = 0.1$. Values of invariants are evaluated as $Q = 5.498173$ and $E = 1.9897829$ at time $t = 0$. Computed values of invariants are tabulated in Tables 1 and 2. As seen in the tables, the values of invariants are preserved. At the end of running time analytical value of amplitude is evaluated as 0.5258 at the location $x = 71$. For all numerical approximations same amplitude value and location data were evaluated. It is observed that solitary wave property is preserved by using different radial basis functions. Solitary wave simulations are plotted in Figure 1.

Table 1. Error norms and invariants for $p = 2$ and $T = 40$

Method	L_2	L_∞	Q	E
$W_{7,5}$	5.195308e-6	1.024193e-6	5.4981736	1.9897829
$W_{6,4}$	5.049681e-6	1.736524e-6	5.4981736	1.9897828
G	8.541866e-4	3.372133e-4	5.4981736	1.9897971
MQ	2.025134e-4	4.563390e-5	5.4972047	1.9897816
[20]	1.152193e-3	4.02987e-4	5.49816	1.98978
[15]	5.297873e-3	1.878952e-3	5.49773	1.98470

Table 1. Error norms and invariants for $p = 2$ and $T = 40$

Method	L_2	L_∞	Q	E
$W_{7,5}$	5.656446e-6	9.697467e-7	5.4981690	1.9897829
$W_{6,4}$	8.161006e-6	2.783771e-6	5.4981692	1.9897828
G	1.086041e-3	4.022695e-4	5.4981704	1.9897996
MQ	3.503223e-4	7.847234e-5	5.4960065	1.9897815
[19]	1.519562e-3	5.146861e-4	5.49815	1.98978

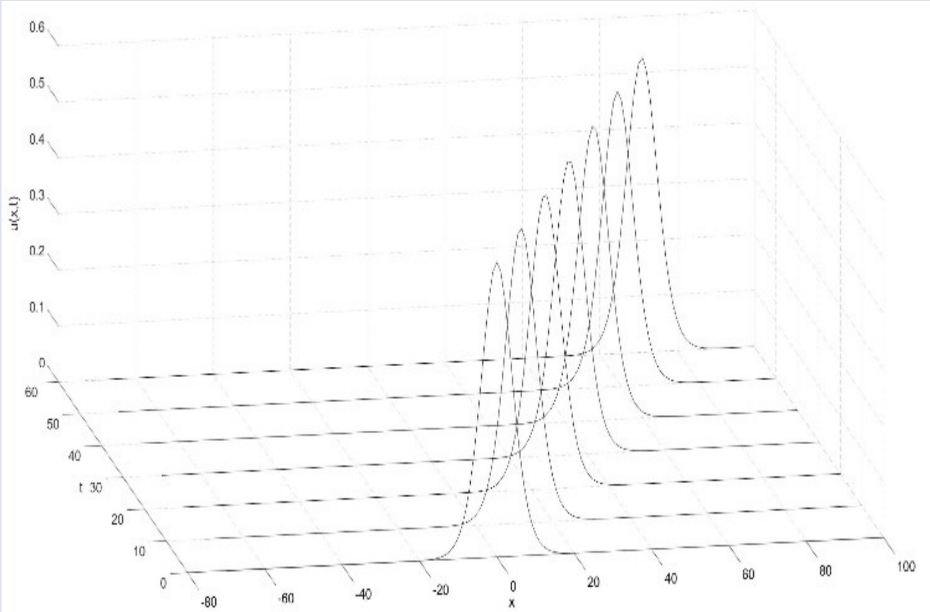

Figure 1. Solitary wave motion for $p = 2$

Case 2: For $p = 3$ and $\beta = 1$, the soliton solution is given as follows [12,13]:

$$u(x,t) = \frac{1}{4}\sqrt{-15+3\sqrt{41}} \times \text{sech}^2 \frac{1}{4}\sqrt{\frac{-5+\sqrt{41}}{2}}\left[-\frac{1}{10}\left(+\sqrt{41}\right)t\right] \quad (16)$$

Calculations are done in the domain $-60 \leq x \leq 100$ with $h = 1$ up to time $T = 40$ for $\Delta t = 0.1$ to make detailed comparisons with references. Comparison of evaluated numerical values is given in Table 3. It has been seen that very sensitive numerical values are computed. At the initial time, invariants are found as $Q = 4.8989794$ and $E = 1.6825477$. Numerical values of invariants are almost the same as the initial values. Therefore, it is seen that the performance of the method is very high and reliable. The simulation of progressive waves keeping original forms are seen in Figure 2. In computations, it is seen that amplitude values are equal to the exact value 0.5096 at the position $x = 46$ for all radial basis functions.

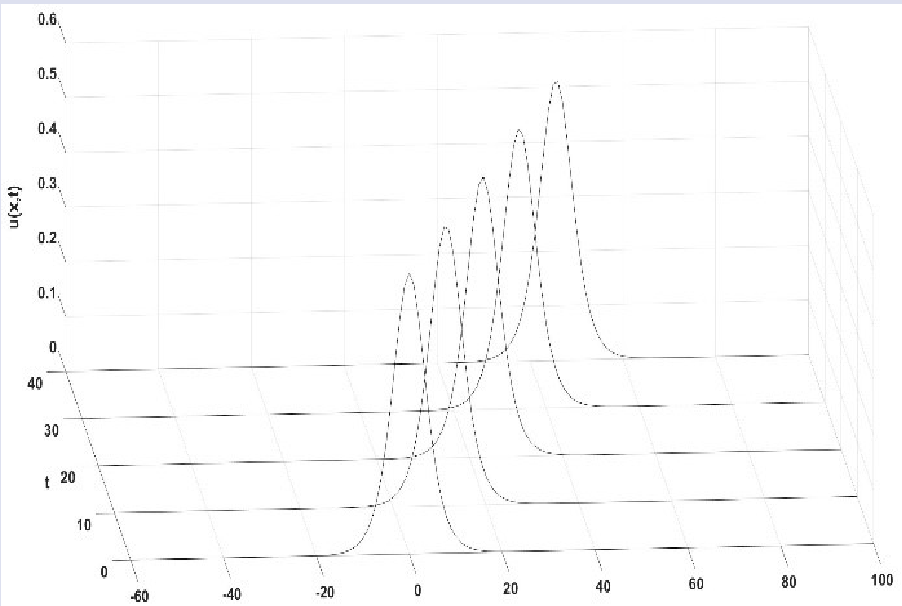

Figure 2. Solitary wave motion for $p = 3$

Table 3. Error norms and invariants for $p = 3$.

Method	L_2	L_∞	Q	E
$W_{7,5}$	1.0290e-06	3.0720e-07	4.8989794	1.6825477
$W_{6,4}$	6.1124e-06	2.3119e-06	4.8989795	1.6825477
G	1.8629e-06	5.7893e-07	4.8989797	1.6825477
MQ	3.0026e-04	6.4671e-05	4.8975866	1.6825450
[20]	1.7880e-03	6.3620e-04	4.8989794	1.682539
[15]	1.3498e-02			
[16]		7.5394e-03		1.6825466

Case 3: When $p = 5$ and $\beta = 1$ the soliton solution is defined as [16]:

$$u(x,t) = \sqrt[4]{\frac{4}{15}(-5+\sqrt{34})} \times sech\frac{1}{3}\sqrt{-5+\sqrt{34}}\left[x - \frac{1}{10}(5+\sqrt{34})t\right] \tag{17}$$

where the solution domain $-60 \leq x \leq 100$ and time $T = 40$. In computations mesh step and the time step is taken as $h = 1$ and $\Delta t = 0.1$. Comparison of numerical results with the results of some other papers is presented in Table 4. Numerical values of invariants at the beginning of the solitary wave are calculated as $Q = 7.0936431$ and $E = 3.1107123$. The single solitary wave profile is illustrated in Figure 3. It can be observed that solitary wave keeps its original form during computing time. This situation implies that the energy is conservative. Solitary wave has amplitude$= 0.6828$ at $x = 43$. As seen is computed results, the present numerical method by using different radial basis functions is slightly better than other referenced works.

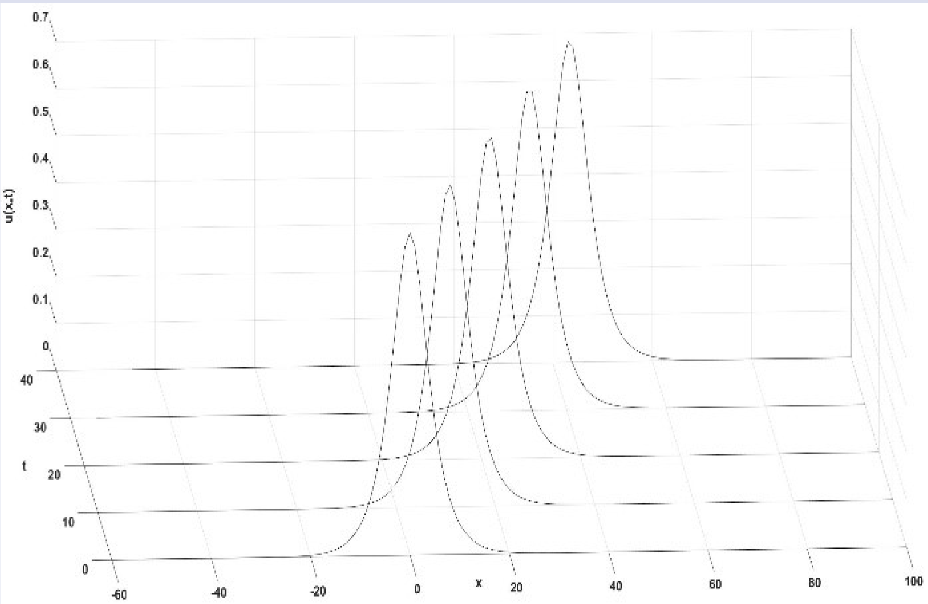

Figure 2. Solitary wave motion for $p = 5$

Table 4. Error norms and invariants for $p = 5$.

Method	L_2	L_∞	Q	E
$W_{7,5}$	2.2926e-06	8.1216e-07	7.0936430	3.1107122
$W_{6,4}$	1.6895e-05	6.0986e-06	7.0936431	3.1107119
G	3.8193e-05	2.2361e-05	7.0936631	3.1107123
MQ	4.8448e-04	1.2184e-04	7.0909712	3.1107111
[20]	3.3217e-03	1.1897e-03	7.0936431	3.205919
[16]	1.7998e-02			
[17]		1.2020e-02		3.1107099

Conclusions

In this paper, the meshless kernel-based method of lines is applied successfully to get the numerical solution of the Generalized-Rosenau–KdV equation for different nonlinear cases. In computations degree of the nonlinear term is used as 2,3 and 5. It is seen that the used proposed method is a very suitable technique solving for the given nonlinear partial differential equation and similar nonlinear equations. The method can be applied easily because there is no need an extra linearization. Therefore, the governing equation is replaced by an ordinary differential equation and it is solved easily by using MATLAB ode-solver code. When we compare the studied in the literature with our numerical results, it can be seen that the results are obtained with high accuracy. It is said that the used meshless technique is a powerful solution method, and we believe that the current method can be applied to construct new solutions for these types of equations in future studies.

Conflicts of interest

The authors state that there is no conflict of interest regarding the publication of this study.

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