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Mathematical Analysis of Discrete Fractional Prey-Predator Model with Fear Effect and Square Root Functional Response

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ABSTRACT

This paper investigates the dynamics of a discrete fractional prey-predator system. The prey-predator interaction is modelled using the square root functional response, which appropriately models systems in which the prey exhibits a strong herd structure, implying that the predator generally interacts with the prey along the herd's outer corridor. Some recent field experiments and studies show that predators affect prey by directly killing and inducing fear in prey, reducing prey species' reproduction rate. Considering these facts, we propose a mathematical model to study herd behaviour and fear effect in the prey-predator system. We show algebraically equilibrium points and their stability condition. Condition for Neimark-Sacker bifurcation, Flip bifurcation and Fold bifurcation are given. Phase portraits and bifurcation diagrams are portraits that depict the model's behaviour based on some hypothetical data. Numerical simulations reveal the model's rich dynamics as a result of fear and fractional order.

Keywords: Prey-predator, Discrete, Fractional, Bifurcation, Fear effect.

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Introduction

Because of their complex behaviour, population models have piqued the interest of researchers. Prey-predator models are an essential component of population models. Based on observed ecological interactions among individuals of the species at various trophic levels, mathematical modelling is a helpful tool for understanding and predicting the long-term survival of various species. There are different types of prey-predator models, such as the continuous model [1-2], discrete model [3-5], fractional model[6-9], etc. Nowadays, the fractional-order system can explain more natural phenomena that were previously ignored by the classical theory of the integer-order dynamical system. Discrete-time population models become more realistic than continuous models when the population sizes are relatively small and in cases where births and deaths occur discrete times or within specific intervals. A discrete form of fractional-order model is now a popular mathematical tool [10]. In this article, we consider the discrete fractional order model.

Din [11] discussed chaos control in a discrete-time prey-predator system. Zhao and Du ([12]) investigated a discrete-time prey-predator model with an Allee effect. Santra and Mahapatra [13] studied the dynamics of a discrete-time prey-predator model under imprecise biological parameters. Santra et al. [14] investigated bifurcation and chaos of a discrete predator-prey model with Crowley-Martin functional response. For more dynamical investigations related to different versions of prey-predator models, we refer to Baydemir et al. [15], Santra et al. [16], Rech [17], Singh and Deolia [18], Khan and Khalique [19,21], Rozikov and Shoyimardonov [20] and references therein.

In reality, a class of prey population exhibits herd behaviour so that the capturing rate of prey by a predator will be different from usual models. Incorporate this herd behaviour of prey; we consider square root functional response [22-23] in our proposed model. The basic assumption in developing such functional interactions is that the predator hunts on the outskirts of a moving herd. If we assume that the herd has a square shape for simplicity, then the interaction between the predator and the prey will occur on the group's borders, which means that it will hunt one (or more) from four times the square root of the prey density. A similar assumption can be made for the prey population's circle herd shape.

Furthermore, predation fear directly impacts prey reproduction; that's why we modify the prey reproduction term incorporating fear factor [24-25] in our proposed model. Two significant factors are limiting wild animal activity: energy and time constraints. To avoid predation, prey may shorten their activity periods and devote some of their foraging time to vigilance; however, prey must balance defence time and foraging intake. A high level of anti-predator behaviour over a long period causes ageing to be sped up and leads to starvation, impacting growth. As a result, there are costs and benefits for prey in prey defence. In this case, we include the cost as a type of prey growth reduction caused by predation fear. In this paper, we study the prey-predator with fear and herd behaviour using the discrete fractional-order model. The study's intention is: what is the effect of fractional order and fear on the proposed model system?

Mathematical Model

Consider the following fractional-order prey-predator model with square root functional response and fear effect :

$$D^{\alpha}x = rx\left(1-\frac{x}{k}\right)\frac{1}{1+\varphi y} - by\sqrt{x}$$
$$D^{\alpha}y = cy\sqrt{x} - dy$$

with initial condition

$$x(t)\geq 0,\ y(t)\geq 0$$

Where x and y denote the density of prey and predator populations respectively at any time t. The biological meaning of the system parameters are as follows: r is the intrinsic per capita growth rate of prey population, k is the environmental carrying capacity of prey population, φ is the fear effect due to predation, b is the maximal per capita consumption rate of predators, c is the efficiency with which predators convert consumed prey into new predators, d is the per capita death rate of predators, and α is the fractional-order satisfying $\alpha\in(0,1]$ and $D^{\alpha}\equiv\frac{d^{\alpha}}{dt^{\alpha}}$ is in the sense of Caputo derivative.

Using the fractional-order discretization process, we get the following discrete fractional-order system

$$x\to x+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[rx\left(1-\frac{x}{k}\right)\frac{1}{1+\varphi y}-by\sqrt{x}\right]$$
$$y\to y+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[cy\sqrt{x}-dy\right]$$

where h is the step size and r,k,φ,b,c,d are all positive constants. By the biological meaning of the model variables, we only consider the system in the region $\Omega=\{(x,y):\ x\geq 0, y\geq 0\}$ in the $(x,y)-$ plane.

General Stability Analysis

Equilibria

Fixed points of the system (1) are determined by solving the following non-linear system of equations:

$$x = x+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[rx\left(1-\frac{x}{k}\right)\frac{1}{1+\varphi y}-by\sqrt{x}\right]$$
$$y = y+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[cy\sqrt{x}-dy\right]$$

We get three non-negative fixed points by solving the above equations:

(i) $P_0=(0,0)$

(ii) $P_1=(k,0)$

(iii) $P_2=(x_2,y_2)$

here $x_2=\left(\frac{d}{c}\right)^2$ and y_2 is a positive solution of

$$y^2+\frac{y}{\varphi}-\frac{rd}{\varphi bc}\left(1-\frac{d^2}{kc^2}\right)=0$$

Local Stability Analysis

The discussion about the dynamical behavior of model (1) is carried out in this sub-section. The Jacobian Matrix J for the system (1) is

$$J=\begin{bmatrix}1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x}{k}\right)\frac{1}{1+\varphi y}-\frac{by}{2\sqrt{x}}\right] & \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx\left(1-\frac{x}{k}\right)\frac{\varphi}{(1+\varphi y)^2}-b\sqrt{x}\right]\\ \frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy}{2\sqrt{x}}\right] & 1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x}-d\right]\end{bmatrix}$$

The characteristic equation of the matrix J is $\lambda^2-T\lambda+D=0$, where

$$T=2+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x}{k}\right)\frac{1}{1+\varphi y}-\frac{by}{2\sqrt{x}}+c\sqrt{x}-d\right]$$

$$\begin{aligned}D=&\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x}{k}\right)\frac{1}{1+\varphi y}-\frac{by}{2\sqrt{x}}\right]\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x}-d\right]\right]\\&-\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx\left(1-\frac{x}{k}\right)\frac{\varphi}{(1+\varphi y)^2}-b\sqrt{x}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy}{2\sqrt{x}}\right]\right]\\=&1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x}-d\right]+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x}{k}\right)\frac{1}{1+\varphi y}-\frac{by}{2\sqrt{x}}\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x}-d\right]\right]\\&-\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx\left(1-\frac{x}{k}\right)\frac{\varphi}{(1+\varphi y)^2}-b\sqrt{x}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy}{2\sqrt{x}}\right]\right]\end{aligned}$$

Hence the discrete-time system (1) is said to be:

(i) a dissipative dynamical system if $|D|<1$,

(ii) a conservative dynamical system if and only if $|D|=1$,

(iii) an undissipated dynamical system otherwise.

Stability and dynamic behavior at P_1

The Jacobian matrix at the fixed point $P_1=(k,0)$ is

$$J=\begin{bmatrix}1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)} & -\frac{b\sqrt{k}h^{\alpha}}{\Gamma(1+\alpha)}\\ 0 & 1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\end{bmatrix}$$

The equilibrium point P_1 is said to be:

(i) Sink if $\left|1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)}\right|<1$, and $\left|1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\right|<1$,

(ii) Source if $\left|1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)}\right|>1$, and $\left|1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\right|>1$,

(iii) Saddle if $\left|1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)}\right|>1,$ and $\left|1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\right|<1;$

or $\left|1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)}\right|<1,$ and $\left|1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\right|>1,$

(iv) Non-hyperbolic if $\left|1-\frac{rh^{\alpha}}{\Gamma(1+\alpha)}\right|=1$ or

$\left|1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{k}-d\right]\right|=1.$

Dynamic behavior around the interior fixed point

From the Jacobian matrix at the interior fixed point $P_2\left(x_2,y_2\right),$ we get

$$1-T+D=-1-\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}+c\sqrt{x_2}-d\right]$$
$$+\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}\right]\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]\right]$$
$$-\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx_2\left(1-\frac{x_2}{k}\right)\frac{\varphi}{(1+\varphi y_2)^2}-b\sqrt{x_2}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy_2}{2\sqrt{x_2}}\right]\right]$$

$$1+T+D=3+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}+c\sqrt{x_2}-d\right]$$
$$+\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}\right]\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]\right]$$
$$-\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx_2\left(1-\frac{x_2}{k}\right)\frac{\varphi}{(1+\varphi y)^2}-b\sqrt{x_2}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy_2}{2\sqrt{x_2}}\right]\right]$$

If $1-T+D>0,$ then interior equilibrium point $P_2\left(x_2,y_2\right)$ is:

(i) Sink if $1+T+D>0$ and $D<1$,
(ii) Source if $1+T+D>0$ and $D>1$,
(iii) Saddle if $1+T+D<0$,
(iv) Non-hyperbolic if $1+T+D=0$ and $T\neq 0,2,$

or $T^2-4D<0$ and $D=1$.

Bifurcation Analysis

This section obtains the conditions for Neimark-Sacker bifurcation, flip bifurcation, and fold bifurcation for model (1). Neimark-Sacker bifurcation causes closed invariant curves into the system, which shows more complex behaviour. Another bifurcation, flip bifurcation, occurs when the system switches to a new limit cycle twice the period of the existing one. Fold bifurcation, in which two fixed points collide and disappear into the system.

Neimark-Sacker Bifurcation

Condition for the occurrence of Neimark-Sacker bifurcation (Elaydi [26]) at an interior fixed point $P_2\left(x_2,y_2\right)$ is $D=1$.

i.e.

$$\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]+h\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]\right]$$
$$=\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx_2\left(1-\frac{x_2}{k}\right)\frac{\varphi}{(1+\varphi y_2)^2}-b\sqrt{x_2}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy_2}{2\sqrt{x_2}}\right]\right]$$

Flip Bifurcation

Condition for the occurrence of Flip bifurcation (Elaydi [26]) at an interior fixed point $P_2\left(x_2,y_2\right)$ is $1+T+D=0$.

i.e.

$$3+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}+c\sqrt{x_2}-d\right]$$
$$+\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}\right]\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]\right]$$
$$=\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx_2\left(1-\frac{x_2}{k}\right)\frac{\varphi}{(1+\varphi y_2)^2}-b\sqrt{x_2}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy_2}{2\sqrt{x_2}}\right]\right]$$

Fold bifurcation

Condition for the occurrence of Fold bifurcation (Elaydi [26]) at an interior fixed point $P_2\left(x_2,y_2\right)$ is $1-T+D=0$

i.e.

$$-1-\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}+c\sqrt{x_2}-d\right]$$
$$+\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[r\left(1-\frac{2x_2}{k}\right)\frac{1}{1+\varphi y_2}-\frac{by_2}{2\sqrt{x_2}}\right]\right]\left[1+\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[c\sqrt{x_2}-d\right]\right]$$
$$=\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[-rx_2\left(1-\frac{x_2}{k}\right)\frac{\varphi}{(1+\varphi y_2)^2}-b\sqrt{x_2}\right]\right]\left[\frac{h^{\alpha}}{\Gamma(1+\alpha)}\left[\frac{cy_2}{2\sqrt{x_2}}\right]\right]$$

Numerical Simulations

In this section, numerical simulations were run using a hypothetical set of parameter values as shown in table 1. These parameter values have biological and mathematical significance. This section numerically analyses the model to investigate more results on fractional-order and the fear effect on prey due to predation. In a specific range, bifurcation diagrams for prey and predator are created for the model. To better understand the system, phase portraits are drawn in a specific section of the bifurcation diagram. This section of the study's parameters are step size, fear effect, and fractional order.

Table 1: Parameter values

r	k	φ	b	c	d	h	α
0.5	1.0	0.1	0.7	0.5	0.3	0.3	0.9

To know the effect of step size in the system dynamics, we draw the bifurcation diagram in figure 1 for $h\in[0.1,1.0]$ and phase portraits in figure 2 for (A) $h=0.3$, (B) $h=0.36$ rest of the parameters from table 1. Neimark-Sacker bifurcation occurs w.r.t. this parameter and the system enter into an unstable zone when step size h crosses the threshold value 0.3445 .

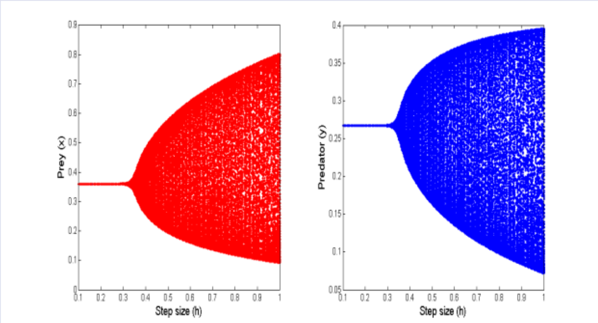

Figure 1. The bifurcation diagram of the system concerning the step size h in the range [0.1,1] and remaining parameters are from Table 1.

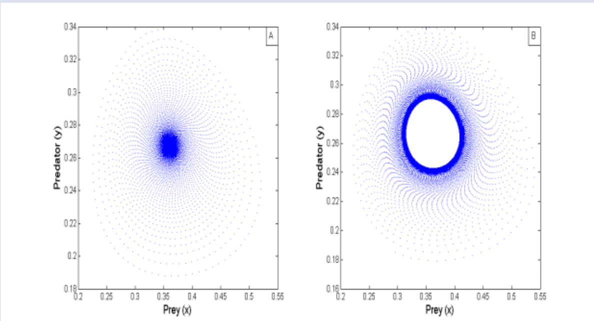

Figure 2. Phase portraits of the system for step size (A) $h = 0.3$, (B) $h = 0.36$, and remaining parameters are from Table 1.

Now, we are interested to know the effect of fear in the system dynamics, and we draw the bifurcation diagram in figure 3 for $\varphi \in [0.1,1.0]$ and phase portraits in figure 4 for (A) $\varphi = 0.6$, (B) $\varphi = 0.9$ rest of the parameters from table 1. Neimark-Sacker bifurcation occurs w.r.t. this parameter and the system enter into an unstable zone when fear effect φ crosses the threshold value 0.7855 .

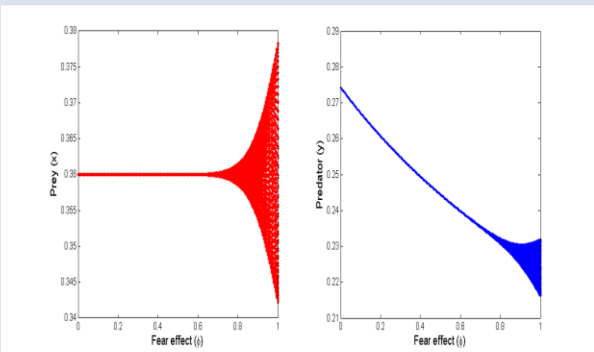

Figure 3. The bifurcation diagram of the system concerning the fear effect φ in the range [0,1] and remaining parameters are from Table 1.

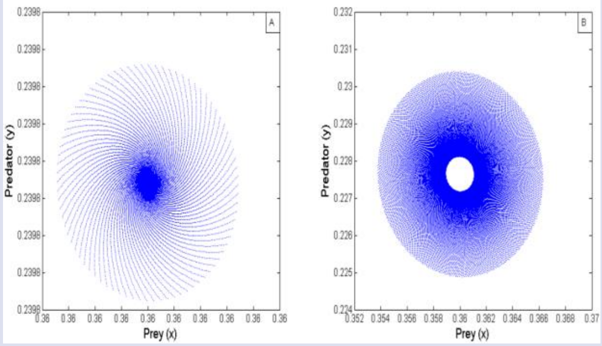

Figure 4: Phase portraits of the system for fear effect (A) $\varphi = 0.6$, (B) $\varphi = 0.9$ and remaining parameters are from Table 1.

Lastly, we are interested to know the effect of fractional-order in the system dynamics. We draw the bifurcation diagram in figure 5 for $\alpha \in [0.1,1.0]$ and phase portraits in figure 6 for (A) $\alpha = 0.8$, (B) $\alpha = 0.9$ the rest of the parameters from table 1. Neimark-Sacker bifurcation occurs w.r.t. this parameter and the system enter into a stable zone when fractional-order α crosses the threshold value 0.8556 . The low weight of α mean strong memory and high value of α mean weak memory. So faint memory can stabilize the model.

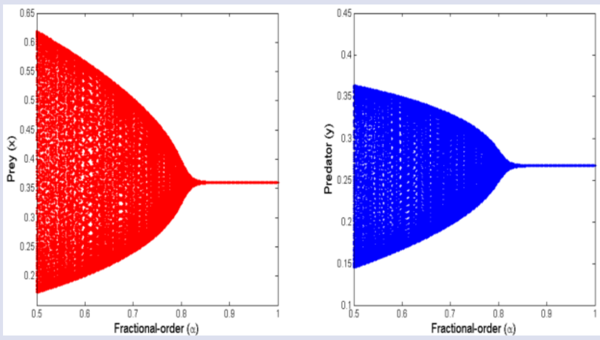

Figure 5. The bifurcation diagram of the system for the fractional-order α in the range [0,1] and remaining parameters are from Table 1.

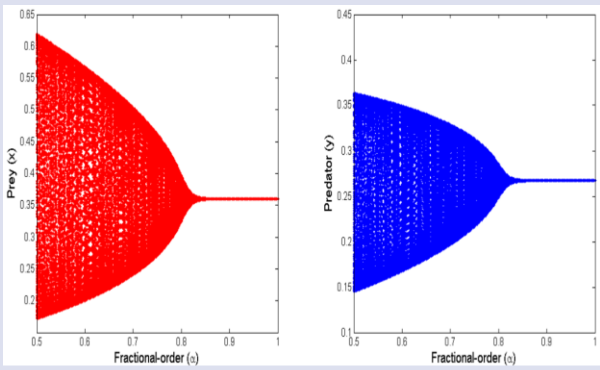

Figure 6. Phase portraits of the system for fractional-order (A) $\alpha = 0.8$, (B) $\alpha = 0.9$ and remaining parameters are from Table 1.

Conclusion

In this work, we investigated the dynamic behaviours of the discrete fractional-order predator-prey system. Fear effect on a prey-predator interaction is studied using the discrete fractional-order model. We established the conditions for a flip bifurcation, fold bifurcation, and a Neimark-Sacker bifurcation of the map at a unique positive fixed point. We have discussed the fear effects concerning the local stability of the interior equilibrium point. Finally, we conclude that fear destabilizes the model, and fear hurts the predator population. Fractional order has to stabilize effect on the model system.

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Conflicts of interest

There are no conflicts of interest in this work.

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