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Existence of nonoscillatory solutions of second-order neutral differential equations

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Abstract

In this study we shall obtain some sufficient conditions for the existence of positive solutions of variable coefficient nonlinear second-order neutral differential equation with distributed deviating arguments. For some different cases of the range of p(t) by using Banach contraction principle we will give some sufficient conditions for the nonoscillatory solutions of second-order neutral differential equation. With this purpose we will use fixpoint theorem. At the end of the research, there is an example that meets the conditions we have given. Our results improve and extend some existing results.

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1. Introduction

In this work we consider the second-order neutral nonlinear differential equation with distributed deviating arguments of the form

$$\left(x(t) - \int_a^b P(t,\xi)x(t-\xi)d\xi\right)'' + \int_{a_1}^{b_1} f_1(t,x(\sigma_1(t,\xi)))d\xi - \int_{a_2}^{b_2} f_2(t,x(\sigma_2(t,\xi)))d\xi = g(t), \tag{1}$$

where $g \in C([t_0,\infty),\mathbb{R})$, $P(t,\xi) \in C([t_0,\infty) \times [a,b],\mathbb{R})$ for $0 < a < b$ and $\sigma_i(t,\xi) \in C([t_0,\infty) \times [a_i,b_i],\mathbb{R})$ with $\lim_{t\to\infty} \sigma_i(t,\xi) = \infty$ and $0 \le a_i < b_i$, $i = 1,2$.

In this paper, we assume that $f_i(t,x) \in C([t_0,\infty) \times \mathbb{R},\mathbb{R})$ is a nondecreasing in x for $i = 1,2$, $xf_i(t,x) > 0$ for $x \neq 0, i = 1,2$ and satisfies

$$|f_i(t,x) - f_i(t,y)| \le q_i(t)|x-y| \quad \text{for } t \in [t_0,\infty) \quad \text{and} \quad x,y \in [e,f], \tag{2}$$

where $q_i \in C([t_0,\infty),\mathbb{R}^+)$, $i = 1,2$ and $[e,f]$ $(0 < e < f$ or $e < f < 0)$ is any closed interval. Furthermore, suppose that

$$\int_{t_0}^{\infty} sq_i(s)ds < \infty, \quad i = 1,2, \tag{3}$$

$$\int_{t_0}^{\infty} s|f_i(s,d)|ds < \infty, \quad \text{for some } d \neq 0, \ i = 1, 2, \tag{4}$$

$$\int_{t_0}^{\infty} s|g(s)|ds < \infty. \tag{5}$$

The nonoscillatory behavior of solutions of neutral differential equations has been considered by different authors in the past. This work was motivated by the paper of Yang, Zhang and Ge in [1] which is concerned with the existence of nonoscillatory solutions of second-order differential equation of the form

$$(x(t) - p(t)x(t-\tau))'' + f_1(t,x(\sigma_1(t))) - f_2(t,x(\sigma_2(t))) = 0 \tag{6}$$

and T. Candan and R.S. Dahiya in [2] which is concerned with the existence of first and second-order neutral differential equations of the form

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$$\frac{d^k}{dt^k}[x(t)+P(t)x(t-\tau)] + \int_a^b q_1(t,\xi)x(t-\xi)d\xi - \int_c^d q_2(t,\mu)x(t-\mu)d\mu = 0. \quad (7)$$

In 2016, Candan [3] investigated nonoscillatory solutions of higher-order neutral differential equations of the form

$$\left[r(t)\left[\left[x(t) - \int_a^b p_2(t,\xi)x(t-\xi)d\xi\right]^{(n-1)}\right]^{\gamma}\right]' + (-1)^n \int_c^d Q_2(x,\xi)G(x,\xi))d\xi = 0.$$

Neutral differential equations have numerous applications in natural sciences and engineering. Especially, neutral differential equations arise in a variety of real world problems such as in the study of non-Newtonian fluid theory and porous medium problems. In recent years, there have been many studies concerning the oscillatory and nonoscillatory behavior of neutral differential equations. For example, Li, Pintus, and Viglialoro [4] studied "Properties of solutions to porous medium problems with different sources and boundary conditions" in 2019. Also, Li and Rogovchenko [5] studied "On the asymptotic behavior of solutions to a class of third-order nonlinear neutral differential equations" in 2020. Many authors have investigated existence of oscillation and nonoscillation solutions of neutral differential equations. Please, see [1-16] and references cited therein.

The purpose of this article is to give some sufficient conditions for the existence of nonoscillatory solutions of (1) according to different cases of the range of $p(t)$ by using Banach contraction principle.

Let $T_0 = \min\left\{t_1 - b, \inf_{t\geq t_1}\min_{\xi\in[a_1,b_1]}\sigma_1(t,\xi),\ \inf_{t\geq t_1}\min_{\xi\in[a_2,b_2]}\sigma_2(t,\xi)\right\}$ for $t_1 \geq t_0$. By a solution of equation (1), we mean a function $x \in C([T_1,\infty),\mathbb{R})$ in the sense that $x(t) - \int_{a_3}^{b_3} p(t,\xi)x(t-\xi)d\xi$ is two times continuously differentiable on $[t_1, \infty]$ and such that equation (1) is satisfied for $t \geq t_1$.
As is customary, a solution of (1) is said to be oscillatory if it has arbitrarily large zeros. Otherwise the solution is called nonoscillatory.

2. Main Results

Theorem 2.1. Assume that (3) - (5) hold, $P(t,\xi) \geq 0$ and $\int_a^b P(t,\xi)d\xi \leq p < 1$. Then (1) has a bounded nonoscillatory solution.

Proof. Suppose (4) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_2 = d$.
Set

$A = \{x \in X : N_1 \leq x(t) \leq N_2, \quad t \geq t_0\}$,

where N_1 and N_2 are positive constants such that

$N_1 + pN_2 < N_2$.

It is obvious that A is a closed, bounded and convex subset of X. Because of (3) - (5), we can take a $t_1 > t_0$ sufficiently large such that $t - b \geq t_0$, $\sigma_i(t,\xi) \geq t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \geq t_1$ and

$$p + \int_{t_1}^{\infty} s[(b_1-a_1)q_1(s) + (b_2-a_2)q_2(s)]\,ds \leq \theta_1 < 1, \quad (8)$$

$$\int_{t_1}^{\infty} s[(b_1-a_1)f_1(s,d) + |g(s)|]\,ds \leq \alpha - N_1 - pN_2, \quad (9)$$

and

$$\int_{t_1}^{\infty} s[(b_2-a_2)f_2(s,d) + |g(s)|]\,ds \leq N_2 - \alpha, \quad (10)$$

where $\alpha \in (N_1 + pN_2, N_2)$. Define a mapping $S : A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases} \propto - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^\infty (s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ \qquad \left. - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds, \qquad t \geq t_1 \\ (Sx)(t_1), \qquad\qquad t_0 \leq t \leq t_1 \end{cases}$$

It is easy to see that Sx is continuous. For every $x \in A$ and $t \geq t_1$ dealing with (10) we can get

$$\begin{aligned} (Sx)(t) = {} & \alpha - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^\infty (s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ & \left. - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds \\ & \leq \alpha + \int_{t_1}^\infty s[(b_2-a_2)f_2(s,d) + |g(s)|]\,ds \\ & \leq N_2 \end{aligned}$$

and taking (9) into account, we can get

$$\begin{aligned} (Sx)(t) = {} & \alpha - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^\infty (s-t)\left[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \right. \\ & \left. - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\right] ds \\ & \geq \alpha - pN_2 - \int_{t_1}^\infty s[(b_1-a_1)f_1(s,d) + |g(s)|]\,ds \\ & \geq N_1. \end{aligned}$$

Thus we proved that $SA \subset A$. Now we shall show that S is a contraction mapping on A.

In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (8) we have

$$\begin{aligned} |(Sx)(t) - (Sy)(t)| \leq {} & \int_a^b P(t,\xi)|x(t-\xi) - y(t-\xi)|d\xi \\ & + \int_t^\infty (s-t) \int_{a_2}^{b_2} |f_2(s,x(\sigma_2(s,\xi))) - f_2(s,y(\sigma_2(s,\xi)))|d\xi ds \\ & + \int_t^\infty (s-t) \int_{a_1}^{b_1} |f_1(s,x(\sigma_1(s,\xi))) - f_1(s,y(\sigma_1(s,\xi)))|d\xi ds \\ & \leq \int_a^b P(t,\xi)|x(t-\xi) - y(t-\xi)|d\xi \\ & + \int_{t_1}^\infty s \int_{a_1}^{b_1} q_1(s)|x(\sigma_1(s,\xi)) - y(\sigma_1(s,\xi))|d\xi ds \\ & + \int_{t_1}^\infty s \int_{a_1}^{b_1} q_2(s)|x(\sigma_2(s,\xi)) - y(\sigma_2(s,\xi))|d\xi ds \\ & \leq \|x-y\| \left(p + \int_{t_1}^\infty s[(b_1-a_1)q_1(s) + (b_2-a_2)q_2(s)\,]\,ds\right) \leq \theta_1\|x-y\|, \end{aligned}$$

which implies with the sup norm that

$$\|Sx - Sy\| \leq \theta_1\|x-y\|.$$

Since $\theta_1 < 1$, S is a contraction mapping on A. By Banach Contraction Mapping Principle, there exists a unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof.

Theorem 2.2. Assume that (3) - (5) hold, $P(t,\xi) \leq 0$ and $-1 < p \leq \int_a^b P(t,\xi)d\xi$. Then (1) has a bounded nonoscillatory solution.

Proof. Suppose (4) holds with $d > 0$. A similar argument holds for $d < 0$. Let $N_4 = d$.

Set

$$A = \{x \in X : N_3 \leq x(t) \leq N_4, \quad t \geq t_0\},$$

where N_3 and N_4 are positive constants such that

$N_3 < (1-p)N_4$.

It is obvious that A is a closed, bounded and convex subset of X. Because of (3) - (5), we can take a $t_1 > t_0$ sufficiently large such that $t-b \geq t_0$, $\sigma_i(t,\xi) \geq t_0$, $\xi \in [a_i, b_i]$, $i = 1, 2$ for $t \geq t_1$ and

$$p + \int_{t_1}^{\infty} s[(b_1-a_1)q_1(s) + (b_2-a_2)q_2(s)]\,ds \leq \theta_2 < 1, \tag{11}$$

$$\int_{t_1}^{\infty} s[(b_1-a_1)f_1(s,d) + |g(s)|]\,ds \leq \alpha - N_3, \tag{12}$$

and

$$\int_{t_1}^{\infty} s[(b_2-a_2)f_2(s,d) + |g(s)|]\,ds \leq (1-p)N_4 - \alpha, \tag{13}$$

where $\alpha \in (N_3, (1-p)-N_4)$. Define a mapping $S: A \longrightarrow X$ as follows:

$$(Sx)(t) = \begin{cases} \propto - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\Big[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi \\ \qquad - \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\Big]ds, & t \geq t_1 \\ (Sx)(t_1), & t_0 \leq t \leq t_1 \end{cases}$$

It is easy to see that Sx is continuous. For every $x \in A$ and $t \geq t_1$ dealing with (13) we can get

$$(Sx)(t) = \alpha - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\Big[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi$$

$$- \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\Big]ds$$

$$\leq \alpha + pN_4 + \int_{t_1}^{\infty} s[(b_2-a_2)f_2(s,d) + |g(s)|]\,ds \leq N_4$$

and taking (12) in to account, we can get

$$(Sx)(t) = \alpha - \int_a^b P(t,\xi)x(t-\xi)d\xi - \int_t^{\infty}(s-t)\Big[\int_{a_1}^{b_1} f_1(s,x(\sigma_1(s,\xi)))d\xi$$

$$- \int_{a_2}^{b_2} f_2\,(s,x(\sigma_2(s,\xi)))d\xi - g(s)\Big]ds$$

$$\geq \alpha - \int_{t_1}^{\infty} s[(b_1-a_1)f_1(s,d) + |g(s)|]\,ds \geq N_3.$$

Thus we proved that $SA \subset A$. Now we shall show that S is a contraction mapping on A.

In fact, for $x, y \in A$ and $t \geq t_1$, in view of (2) and (11) we have

$$|(Sx)(t) - (Sy)(t)| \leq \int_a^b (-P(t,\xi))|y(t-\xi) - x(t-\xi)|d\xi$$

$$+ \int_t^{\infty}(s-t)\int_{a_2}^{b_2}|f_2(s,x(\sigma_2(s,\xi))) - f_2(s,y(\sigma_2(s,\xi)))|d\xi ds$$

$$+ \int_t^{\infty}(s-t)\int_{a_1}^{b_1}|f_1(s,x(\sigma_1(s,\xi))) - f_1(s,y(\sigma_1(s,\xi)))|d\xi ds$$

$$\leq \int_a^b (-P(t,\xi))|x(t-\xi) - y(t-\xi)|d\xi$$

$$+ \int_{t_1}^{\infty} s \int_{a_1}^{b_1} q_1(s)|x(\sigma_1(s,\xi)) - y(\sigma_1(s,\xi))|d\xi ds$$

$$+ \int_{t_1}^{\infty} s \int_{a_1}^{b_1} q_2(s)|x(\sigma_2(s,\xi)) - y(\sigma_2(s,\xi))|d\xi ds$$

$\leq \|x-y\| \left(p + \int_{t_1}^{\infty} s[(b_1-a_1)q_1(s) + (b_2-a_2)q_2(s)\,]\,ds\right)$

$\leq \theta_2 \|x-y\|,$

which implies with the sup norm that

$\|Sx - Sy\| \leq \theta_2 \|x-y\|.$

Since $\theta_2 < 1$, S is a contraction mapping on A. By Banach Contraction Mapping Principle, there exists a unique fixed point $x \in A$ such that $Sx = x$, which is obviously a positive solution of (1). This completes the proof.

Example 2.3. For $t > 0$ consider the equation

$$\left(x(t) - \int_0^1 \exp(-t-3\xi)x(t-\xi)d\xi\right)'' + \int_1^3 2\exp(-t)x(t-2\xi)d\xi - \int_2^6 \exp(-t)x(t-\xi)d\xi$$
$$= \frac{1}{3}\exp(-t) - \exp(-t-3) + 9\exp(-3t) + 16\exp(-4t). \quad (14)$$

Note that $P(t,\xi) = \exp(-t-3\xi)$, $\sigma_1(t,\xi) = t-2\xi$, $\sigma_2(t,\xi) = t-\xi$, $f_1(t,u) = 2\exp(-t)u$, $f_2(t,u) = \exp(-t)u$ and $g(t) = \frac{1}{3}\exp(-t) - \exp(-t-3) + 9\exp(-3t) + 16\exp(-4t)$. We can check that the conditions of Theorem 2.1 are all satisfied. We note that $x(t) = \exp(-3t) + 1$ is a nonoscillatory solution of (14).

Conflicts of interest

The authors state that did not have a conflict of interests.

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