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Research Article

Ostrowski's Type Inequalities for the Complex Integral on Paths

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ABSTRACT. In this paper, we extend the Ostrowski inequality to the integral with respect to arc-length by providing upper bounds for the quantity

$$\left|f(v)\,\ell(\gamma)-\int_{\gamma}f(z)\,|dz|\right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t\in[a,b]$ with the length $\ell(\gamma)$, $u=z(a)$, $v=z(x)$ with $x\in(a,b)$ and $w=z(b)$ while f is holomorphic in G, an open domain and $\gamma\subset G$. An application for circular paths is also given.

Keywords: Complex integral, continuous functions, holomorphic functions, Ostrowski inequality.

2020 Mathematics Subject Classification: 26D15, 26D10, 30A10, 30A86.

1. INTRODUCTION

In 1938, A. Ostrowski [8], proved the following inequality concerning the distance between the integral mean $\frac{1}{b-a}\int_a^b f(t)\,dt$ and the value $f(x)$, $x\in[a,b]$.

Theorem 1.1 (Ostrowski, 1938 [8]). *Let $f:[a,b]\to\mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b) such that $f':(a,b)\to\mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty:=\sup_{t\in(a,b)}|f'(t)|<\infty$. Then,*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\|f'\|_\infty(b-a) \tag{1.1}$$

for all $x\in[a,b]$ and the constant $\frac{1}{4}$ is the best possible.

In [6], S. S. Dragomir and S. Wang, by the use of the *Montgomery integral identity* [7, p. 565],

$$f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt=\frac{1}{b-a}\int_a^b p(x,t)\,f'(t)\,dt\,,\quad x\in[a,b]\,, \tag{1.2}$$

where $p:[a,b]^2\to\mathbb{R}$ is given by

$$p(x,t):=\begin{cases}t-a & \text{if}\quad t\in[a,x]\,,\\[2ex] t-b & \text{if}\quad t\in(x,b],\end{cases}$$

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gave a simple proof of Ostrowski's inequality and applied it for special means (identric mean, logarithmic mean, etc.) and to the problem of estimating the error bound in approximating the Riemann integral $\int_a^b f(t)\,dt$ by one arbitrary Riemann sum (see [6, Section 3]). For extensions of Ostrowski's inequality in terms of the p-norms of the derivative, see [1], [2] and [3]. For a recent survey on Ostrowski's inequality, see [4].

In order to extend this result for the complex integral, we need some preparations as follows. Suppose γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$ and f is a complex function which is continuous on γ. Put $z(a) = u$ and $z(b) = w$ with $u, w \in \mathbb{C}$. We define the integral of f on $\gamma_{u,w} = \gamma$ as

$$\int_\gamma f(z)\,dz = \int_{\gamma_{u,w}} f(z)\,dz := \int_a^b f(z(t))\,z'(t)\,dt.$$

We observe that the actual choice of parametrization of γ does not matter. This definition immediately extends to paths that are piecewise smooth. Suppose γ is parametrized by $z(t)$, $t \in [a,b]$, which is differentiable on the intervals $[a,c]$ and $[c,b]$, then assuming that f is continuous on γ we define

$$\int_{\gamma_{u,w}} f(z)\,dz := \int_{\gamma_{u,v}} f(z)\,dz + \int_{\gamma_{v,w}} f(z)\,dz,$$

where $v := z(c)$. This can be extended for a finite number of intervals. We also define the integral with respect to arc-length

$$\int_{\gamma_{u,w}} f(z)\,|dz| := \int_a^b f(z(t))\,|z'(t)|\,dt$$

and the length of the curve γ is then

$$\ell(\gamma) = \int_{\gamma_{u,w}} |dz| = \int_a^b |z'(t)|\,dt.$$

Let f and g be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a piecewise smooth path from $z(a) = u$ to $z(b) = w$. Then, we have the *integration by parts formula*

$$\int_{\gamma_{u,w}} f(z)\,g'(z)\,dz = f(w)\,g(w) - f(u)\,g(u) - \int_{\gamma_{u,w}} f'(z)\,g(z)\,dz. \tag{1.3}$$

We recall also the *triangle inequality* for the complex integral, namely

$$\left|\int_\gamma f(z)\,dz\right| \le \int_\gamma |f(z)|\,|dz| \le \|f\|_{\gamma,\infty}\,\ell(\gamma), \tag{1.4}$$

where $\|f\|_{\gamma,\infty} := \sup_{z\in\gamma} |f(z)|$.

We also define the p-norm with $p \ge 1$ by

$$\|f\|_{\gamma,p} := \left(\int_\gamma |f(z)|^p\,|dz|\right)^{1/p}.$$

For $p = 1$, we have

$$\|f\|_{\gamma,1} := \int_\gamma |f(z)|\,|dz|.$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder's inequality, we have

$$\|f\|_{\gamma,1} \le [\ell(\gamma)]^{1/q}\,\|f\|_{\gamma,p}.$$

In the recent paper [5], we obtained the following result for functions of complex variable:

Theorem 1.2. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a,b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$,*

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \|f'\|_{\gamma_{u,v};\infty} \int_{\gamma_{u,v}} |z-u|\,|dz| + \|f'\|_{\gamma_{v,w};\infty} \int_{\gamma_{v,w}} |z-w|\,|dz| \\ &\le \left[\int_{\gamma_{u,v}} |z-u|\,|dz| + \int_{\gamma_{v,w}} |z-w|\,|dz| \right] \|f'\|_{\gamma_{u,w};\infty} \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \max_{z\in\gamma_{u,v}} |z-u|\, \|f'\|_{\gamma_{u,v};1} + \max_{z\in\gamma_{v,w}} |z-w|\, \|f'\|_{\gamma_{v,w};1} \\ &\le \max\left\{ \max_{z\in\gamma_{u,v}} |z-u|, \max_{z\in\gamma_{v,w}} |z-w| \right\} \|f'\|_{\gamma_{u,w};1}. \end{aligned} \tag{1.6}$$

If $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\begin{aligned} &\left| f(v)(w-u) - \int_\gamma f(z)\,dz \right| \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,v};p} + \left(\int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{v,w};p} \\ &\le \left(\int_{\gamma_{u,v}} |z-u|^q\,|dz| + \int_{\gamma_{v,w}} |z-w|^q\,|dz| \right)^{1/q} \|f'\|_{\gamma_{u,w};p}. \end{aligned} \tag{1.7}$$

Motivated by the above results, in this paper, we extend the Ostrowski inequality to the complex integral, by providing upper bounds for the quantity

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right|$$

under the assumptions that γ is a smooth path parametrized by $z(t)$, $t \in [a,b]$, with the length $\ell(\gamma)$, $u = z(a)$, $v = z(x)$ with $x \in (a,b)$ and $w = z(b)$ while f is holomorphic in G, an open domain and $\gamma \subset G$. An application for circular paths is also given.

2. Ostrowski Type Results

We have the following result for functions of complex variable:

Theorem 2.3. *Let f be holomorphic in G, an open domain and suppose $\gamma \subset G$ is a smooth path from $z(a) = u$ to $z(b) = w$. If $v = z(x)$ with $x \in (a, b)$, then $\gamma_{u,w} = \gamma_{u,v} \cup \gamma_{v,w}$ and*

$$\left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \leq \ell(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};1} + \ell(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};1} \tag{2.8}$$
$$\leq \begin{cases} \frac{1}{2}\left[\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|\right]\|f'\|_{\gamma_{u,w};1}, \\[2ex] \left[\ell^p(\gamma_{u,v}) + \ell^p(\gamma_{v,w})\right]^{1/p}\left(\|f'\|^q_{\gamma_{u,v};1} + \|f'\|^q_{\gamma_{v,w};1}\right)^{1/q} \\ p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\[2ex] \frac{1}{2}\ell(\gamma_{u,w})\left[\|f'\|_{\gamma_{u,w};1} + \left|\|f'\|_{\gamma_{u,v};1} - \|f'\|_{\gamma_{v,w};1}\right|\right]. \end{cases}$$

Proof. Using the integration by parts formula, we have

$$\begin{aligned}
\int_{\gamma_{u,v}} f(z)\,|dz| &= \int_a^x f(z(t))\,|z'(t)|\,dt \\
&= \int_a^x f(z(t))\,d\left(\int_a^t |z'(s)|\,ds\right) \\
&= f(z(t))\int_a^t |z'(s)|\,ds\Big|_a^x - \int_a^x \frac{df(z(t))}{dt}\left(\int_a^t |z'(s)|\,ds\right)dt \\
&= f(z(x))\int_a^x |z'(s)|\,ds - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right)z'(t)\,dt \\
&= f(v)\,\ell(\gamma_{u,v}) - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right)z'(t)\,dt
\end{aligned}$$

and

$$\begin{aligned}
\int_{\gamma_{v,w}} f(z)\,|dz| &= \int_x^b f(z(t))\,|z'(t)|\,dt \\
&= -\int_x^b f(z(t))\,d\left(\int_t^b |z'(s)|\,ds\right) \\
&= -f(z(t))\int_t^b |z'(s)|\,ds\Big|_x^b + \int_x^b \frac{df(z(t))}{dt}\left(\int_t^b |z'(s)|\,ds\right)dt \\
&= f(z(x))\int_x^b |z'(s)|\,ds + \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right)z'(t)\,dt \\
&= f(v)\,\ell(\gamma_{v,w}) + \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right)z'(t)\,dt\,.
\end{aligned}$$

If we add these two equalities, we get

$$
\begin{aligned}
&\int_{\gamma_{u,v}} f(z)\,|dz| + \int_{\gamma_{v,w}} f(z)\,|dz| \\
&= f(v)\,\ell(\gamma_{u,v}) + f(v)\,\ell(\gamma_{v,w}) - \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt \\
&+ \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt,
\end{aligned}
$$

which gives the following equality of interest

$$
\begin{aligned}
&f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \\
&= \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt - \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt.
\end{aligned} \tag{2.9}
$$

By taking the modulus in (2.9) and using the properties of modulus, we get

$$
\begin{aligned}
&\left| f(v)\,\ell(\gamma_{u,w}) - \int_{\gamma_{u,w}} f(z)\,|dz| \right| \\
&\le \left| \int_a^x f'(z(t))\left(\int_a^t |z'(s)|\,ds\right) z'(t)\,dt \right| \\
&+ \left| \int_x^b f'(z(t))\left(\int_t^b |z'(s)|\,ds\right) z'(t)\,dt \right| \\
&\le \int_a^x |f'(z(t))|\,|z'(t)| \left(\int_a^t |z'(s)|\,ds\right) dt \\
&+ \int_x^b |f'(z(t))|\,|z'(t)| \left(\int_t^b |z'(s)|\,ds\right) dt =: B(x)
\end{aligned} \tag{2.10}
$$

for $x \in [a,b]$. We have

$$
\int_a^t |z'(s)|\,ds \le \int_a^x |z'(s)|\,ds \text{ for } t \in [a,x]
$$

and

$$
\int_t^b |z'(s)|\,ds \le \int_x^b |z'(s)|\,ds \text{ for } t \in [x,b],
$$

then

$$
\begin{aligned}
B(x) &\le \int_a^x |z'(s)|\,ds \int_a^x |f'(z(t))|\,|z'(t)|\,dt \\
&+ \int_x^b |z'(s)|\,ds \int_x^b |f'(z(t))|\,|z'(t)|\,dt \\
&= \ell(\gamma_{u,v}) \int_{\gamma_{u,v}} |f'(z)|\,|dz| + \ell(\gamma_{v,w}) \int_{\gamma_{v,w}} |f'(z)|\,|dz|
\end{aligned}
$$

and by (2.10), we get the first inequality in (2.8). The second part follows by Hölder's inequalities

$$mn+cd\leq\begin{cases}\max\{m,c\}(n+d)\\[2ex](m^p+c^p)^{1/p}(n^q+d^q)^{1/q}, \text{ for } p,\ q>1,\ \frac{1}{p}+\frac{1}{q}=1,\end{cases}$$

where $m,n,c,d\geq 0$. □

Corollary 2.1. *With the assumption of Theorem 2.3 and if there exists $m\in\gamma$ such that $\ell(\gamma_{u,m})=\ell(\gamma_{m,w})$, then*

$$\left|f(m)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2}\ell(\gamma)\|f'\|_{\gamma_{u,w};1} \tag{2.11}$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}|f'(z)||dz|=\int_{\gamma_{s,w}}|f'(z)||dz|$, then

$$\left|f(s)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2}\ell(\gamma)\|f'\|_{\gamma_{u,w};1}. \tag{2.12}$$

We have also the following result for p-norms:

Theorem 2.4. *With the assumption of Theorem 2.3, we have for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$ that*

$$\begin{aligned}&\left|f(v)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\\&\leq\frac{1}{q+1}\left[\ell^{1+1/q}(\gamma_{u,v})\|f'\|_{\gamma_{u,v};p}+\ell^{1+1/q}(\gamma_{v,w})\|f'\|_{\gamma_{v,w};p}\right]\\&\leq\frac{1}{q+1}\begin{cases}\frac{1}{2^{1+1/q}}\left[\ell(\gamma_{u,w})+|\ell(\gamma_{u,v})-\ell(\gamma_{v,w})|\right]^{1+1/q}\\\times\left[\|f'\|_{\gamma_{u,v};p}+\|f'\|_{\gamma_{v,w};p}\right],\\[2ex]\left[\ell^{q+1}(\gamma_{u,v})+\ell^{q+1}(\gamma_{v,w})\right]^{1/q}\|f'\|_{\gamma_{u,w};p},\\[2ex]\frac{1}{2^{1/p}}\left[\|f'\|^p_{\gamma_{u,w};p}+\left|\|f'\|^p_{\gamma_{u,v};p}-\|f'\|^p_{\gamma_{v,w};p}\right|\right]^{1/p}\\\times\left[\ell^{1+1/q}(\gamma_{u,v})+\ell^{1+1/q}(\gamma_{v,w})\right].\end{cases}\end{aligned} \tag{2.13}$$

Proof. Using the weighted Hölder integral inequality for $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\begin{aligned}&\int_a^x|f'(z(t))||z'(t)|\left(\int_a^t|z'(s)|ds\right)dt\\&\leq\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\left(\int_a^x\left(\int_a^t|z'(s)|ds\right)^q|z'(t)|dt\right)^{1/q}\\&=\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\left(\frac{\left(\int_a^x|z'(s)|ds\right)^{q+1}}{q+1}\right)^{1/q}\\&=\frac{\left(\int_a^x|z'(s)|ds\right)^{1+1/q}}{(q+1)^{1/q}}\left(\int_a^x|f'(z(t))|^p|z'(t)|dt\right)^{1/p}\end{aligned}$$

and

$$\begin{aligned}
&\int_x^b |f'(z(t))| |z'(t)| \left(\int_t^b |z'(s)| ds\right) dt \\
&\leq \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\int_x^b \left(\int_t^b |z'(s)| ds\right)^q |z'(t)| dt\right)^{1/q} \\
&= \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \left(\frac{\left(\int_x^b |z'(s)| ds\right)^{q+1}}{q+1}\right)^{1/q} \\
&= \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p}
\end{aligned}$$

for $x \in (a, b)$. If we add these two inequalities, we get

$$\begin{aligned}
B(x) &\leq \frac{\left(\int_a^x |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_a^x |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&\quad + \frac{\left(\int_x^b |z'(s)| ds\right)^{1+1/q}}{(q+1)^{1/q}} \left(\int_x^b |f'(z(t))|^p |z'(t)| dt\right)^{1/p} \\
&= \frac{1}{q+1}\left[\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right],
\end{aligned}$$

which proves the first inequality in (2.13). We also have

$$\begin{aligned}
&\ell^{1+1/q}(\gamma_{u,v}) \left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \ell^{1+1/q}(\gamma_{v,w}) \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p} \\
&\leq \max\left\{\ell^{1+1/q}(\gamma_{u,v}), \ell^{1+1/q}(\gamma_{v,w})\right\} \\
&\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&= [\max\{\ell(\gamma_{u,v}), \ell(\gamma_{v,w})\}]^{1+1/q} \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right] \\
&= \frac{1}{2^{1+1/q}} [\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|]^{1+1/q} \\
&\times \left[\left(\int_{\gamma_{u,v}} |f'(z)|^p |dz|\right)^{1/p} + \left(\int_{\gamma_{v,w}} |f'(z)|^p |dz|\right)^{1/p}\right]
\end{aligned}$$

and

$$\ell^{1+1/q}(\gamma_{u,v})\left(\int_{\gamma_{u,v}}|f'(z)|^p|dz|\right)^{1/p}+\ell^{1+1/q}(\gamma_{v,w})\left(\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right)^{1/p}$$
$$\leq\max\left\{\left(\int_{\gamma_{u,v}}|f'(z)|^p|dz|\right)^{1/p},\left(\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right)^{1/p}\right\}$$
$$\times\left[\ell^{1+1/q}(\gamma_{u,v})+\ell^{1+1/q}(\gamma_{v,w})\right]$$
$$=\left[\max\left\{\int_{\gamma_{u,v}}|f'(z)|^p|dz|,\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right\}\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}(\gamma_{u,v})+\ell^{1+1/q}(\gamma_{v,w})\right]$$
$$=\frac{1}{2^{1/p}}\left[\int_{\gamma_{u,w}}|f'(z)|^p|dz|+\left|\int_{\gamma_{u,v}}|f'(z)|^p|dz|-\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right|\right]^{1/p}$$
$$\times\left[\ell^{1+1/q}(\gamma_{u,v})+\ell^{1+1/q}(\gamma_{v,w})\right].$$

By Hölder's discrete inequality, we have

$$\ell^{1+1/q}(\gamma_{u,v})\left(\int_{\gamma_{u,v}}|f'(z)|^p|dz|\right)^{1/p}+\ell^{1+1/q}(\gamma_{v,w})\left(\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right)^{1/p}$$
$$\leq\left[\ell^{q+1}(\gamma_{u,v})+\ell^{q+1}(\gamma_{v,w})\right]^{1/q}\left[\int_{\gamma_{u,v}}|f'(z)|^p|dz|+\int_{\gamma_{v,w}}|f'(z)|^p|dz|\right]^{1/p}$$
$$=\left[\ell^{q+1}(\gamma_{u,v})+\ell^{q+1}(\gamma_{v,w})\right]^{1/q}\left(\int_{\gamma_{u,w}}|f'(z)|^p|dz|\right)^{1/p},$$

which prove the last part of (2.13). □

We have:

Corollary 2.2. *With the assumption of Theorem 2.4 and if there exists $m\in\gamma$ such that $\ell(\gamma_{u,m})=\ell(\gamma_{m,w})$, then*

$$\left|f(m)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2^{1+1/q}(q+1)}\ell^{1+1/q}(\gamma_{u,w})\left[\|f'\|_{\gamma_{u,m};p}+\|f'\|_{\gamma_{m,w};p}\right]\tag{2.14}$$

and if $s\in\gamma$ such that $\int_{\gamma_{u,s}}|f'(z)|^p|dz|=\int_{\gamma_{s,w}}|f'(z)|^p|dz|$, then

$$\left|f(s)\ell(\gamma)-\int_\gamma f(z)|dz|\right|\leq\frac{1}{2^{1/p}(q+1)}\left[\ell^{1+1/q}(\gamma_{u,s})+\ell^{1+1/q}(\gamma_{s,w})\right]\|f'\|_{\gamma_{u,w};p}.\tag{2.15}$$

Finally we have:

Theorem 2.5. *With the assumption of Theorem 2.3, we have*

$$\begin{aligned}
(2.16)\qquad & \left| f(v)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| \\
& \leq \frac{1}{2}\left[\ell^2(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};\infty} + \ell^2(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};\infty}\right] \\
& \leq \frac{1}{2}\begin{cases} \frac{1}{4}\left[\ell(\gamma_{u,w}) + |\ell(\gamma_{u,v}) - \ell(\gamma_{v,w})|\right]^2 \\ \times\left[\|f'\|_{\gamma_{u,v};\infty} + \|f'\|_{\gamma_{v,w};\infty}\right], \\ \\ \left[\ell^{2q}(\gamma_{u,v}) + \ell^{2q}(\gamma_{v,w})\right]^{1/q}\left(\|f'\|^p_{\gamma_{u,v};\infty} + \|f'\|^p_{\gamma_{v,w};\infty}\right)^{1/p}, \\ p,\ q > 1 \text{ and } \frac{1}{p} + \frac{1}{q} = 1, \\ \\ \left[\ell^2(\gamma_{u,v}) + \ell^2(\gamma_{v,w})\right]\|f'\|_{\gamma_{u,w};\infty}. \end{cases}
\end{aligned}$$

Proof. We have

$$\begin{aligned}
& \int_a^x |f'(z(t))|\,|z'(t)| \left(\int_a^t |z'(s)|\,ds\right) dt \\
& \leq \max_{t\in[a,x]} |f'(z(t))| \int_a^x \left(\int_a^t |z'(s)|\,ds\right)|z'(t)|\,dt \\
& = \frac{1}{2}\max_{t\in[a,x]} |f'(z(t))| \left(\int_a^x |z'(s)|\,ds\right)^2 = \frac{1}{2}\ell^2(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};\infty}
\end{aligned}$$

and

$$\begin{aligned}
& \int_x^b |f'(z(t))|\,|z'(t)| \left(\int_t^b |z'(s)|\,ds\right) dt \\
& \leq \max_{t\in[x,b]} |f'(z(t))| \int_x^b \left(\int_t^b |z'(s)|\,ds\right)|z'(t)|\,dt \\
& = \frac{1}{2}\max_{t\in[x,b]} |f'(z(t))| \left(\int_x^b |z'(s)|\,ds\right)^2 = \frac{1}{2}\ell^2(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};\infty},
\end{aligned}$$

which by addition produce

$$B(x) \leq \frac{1}{2}\left[\ell^2(\gamma_{u,v})\,\|f'\|_{\gamma_{u,v};\infty} + \ell^2(\gamma_{v,w})\,\|f'\|_{\gamma_{v,w};\infty}\right]$$

and by utilising the inequality (2.10), we get the first part of (2.16).

The second part is obvious and we omit the details. □

Corollary 2.3. *With the assumption of Theorem 2.3 and if there exists $m \in \gamma$ such that $\ell(\gamma_{u,m}) = \ell(\gamma_{m,w})$, then*

$$\begin{aligned}
\left| f(m)\,\ell(\gamma) - \int_\gamma f(z)\,|dz| \right| & \leq \frac{1}{8}\left[\|f'\|_{\gamma_{u,m};\infty} + \|f'\|_{\gamma_{m,w};\infty}\right]\ell^2(\gamma_{u,w}) \\
& \leq \frac{1}{4}\|f'\|_{\gamma_{u,w};\infty}\,\ell^2(\gamma_{u,w}).
\end{aligned}$$

3. Examples for Circular Paths

Let $[a,b] \subseteq [0,2\pi]$ and the circular path $\gamma_{[a,b],R}$ centered in 0 and with radius $R>0$

$$z(t) = R\exp(it) = R(\cos t + i\sin t), \ t\in[a,b].$$

If $[a,b]=[0,\pi]$, then we get a half circle, while for $[a,b]=[0,2\pi]$, we get the full circle. We have

$$z'(t) = Ri\exp(it), \ t\in[a,b]$$

and $|z'(t)| = R$ for $t\in[a,b]$ giving that

$$\ell\left(\gamma_{[a,b],R}\right) = \int_a^b |z'(t)|\,dt = R(b-a).$$

If $x\in[a,b]$ and $v = R\exp(ix)$, then by (2.8), we have

$$\left|R(b-a)f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right|$$
$$\leq R(x-a)R\int_a^x |f'(R\exp(it))|\,dt + R(b-x)R\int_x^b |f'(R\exp(it))|\,dt,$$

that is equivalent to

$$\left|(b-a)f(R\exp(ix)) - \int_a^b f(R\exp(it))\,dt\right| \tag{3.17}$$
$$\leq R(x-a)\int_a^x |f'(R\exp(it))|\,dt + R(b-x)\int_x^b |f'(R\exp(it))|\,dt$$

for $x\in[a,b]$. In particular, if we take $x=\frac{a+b}{2}$ in (3.17), then we get

$$\left|(b-a)f\left(R\exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{2}R(b-a)\int_a^b |f'(R\exp(it))|\,dt. \tag{3.18}$$

If $m\in[a,b]$ is such that

$$\int_a^m |f'(R\exp(it))|\,dt = \int_m^b |f'(R\exp(it))|\,dt,$$

then from (3.17), we get

$$\left|(b-a)f(R\exp(mi)) - \int_a^b f(R\exp(it))\,dt\right| \tag{3.19}$$
$$\leq \frac{1}{2}R(b-a)\int_a^b |f'(R\exp(it))|\,dt.$$

By making use of (2.13), we get

$$\left|R(b-a)f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right|$$
$$\leq \frac{1}{q+1}\left[R^{1+1/q}(x-a)^{1+1/q}R^{1/p}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p}\right.$$
$$\left. + R^{1+1/q}(b-x)^{1+1/q}R^{1/p}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right],$$

that is equivalent to

$$\left|(b-a) f(R\exp(ix)) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{q+1} R \left[(x-a)^{1+1/q}\left(\int_a^x |f'(R\exp(it))|^p\,dt\right)^{1/p} + (b-x)^{1+1/q}\left(\int_x^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right] \tag{3.20}$$

for $x \in [a,b]$. If we take $x = \frac{a+b}{2}$ in (3.20), then we get

$$\left|(b-a) f\left(R\exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{(q+1)\,2^{1+1/q}} R (b-a)^{1+1/q} \left[\left(\int_a^{\frac{a+b}{2}} |f'(R\exp(it))|^p\,dt\right)^{1/p} + \left(\int_{\frac{a+b}{2}}^b |f'(R\exp(it))|^p\,dt\right)^{1/p}\right]. \tag{3.21}$$

If $c \in [a,b]$ is such that

$$\int_a^c |f'(R\exp(it))|^p\,dt = \int_c^b |f'(R\exp(it))|^p\,dt,$$

then by (3.20), we get

$$\left|(b-a) f(R\exp(ic)) - \int_a^b f(R\exp(it))\,dt\right| \leq \frac{1}{(q+1)\,2^{1/p}} R \left[(c-a)^{1+1/q} + (b-c)^{1+1/q}\right] \left(\int_a^b |f'(R\exp(it))|^p\,dt\right)^{1/p}. \tag{3.22}$$

Further, if we use (2.16), then we have

$$\begin{aligned}&\left|R(b-a) f(R\exp(ix)) - R\int_a^b f(R\exp(it))\,dt\right| \\ &\leq \frac{1}{2} R^2 \left[(x-a)^2 \sup_{t\in[a,x]} |f'(R\exp(it))| + (b-x)^2 \sup_{t\in[x,b]} |f'(R\exp(it))|\right] \\ &\leq \frac{1}{2} R^2 \left[(x-a)^2 + (b-x)^2\right] \sup_{t\in[a,b]} |f'(R\exp(it))|,\end{aligned}$$

that is equivalent to

$$\begin{aligned}
(3.23)\qquad &\left|(b-a) f(R\exp(xi)) - \int_a^b f(R\exp(it))\,dt\right| \\
&\leq \frac{1}{2}R\left[(x-a)^2 \sup_{t\in[a,x]}|f'(R\exp(it))| + (b-x)^2 \sup_{t\in[x,b]}|f'(R\exp(it))|\right] \\
&\leq R\left[\frac{1}{4}(b-a)^2 + \left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}|f'(R\exp(it))|
\end{aligned}$$

for $x\in[a,b]$. In particular, we have

$$\begin{aligned}
(3.24)\qquad &\left|(b-a) f\left(R\exp\left(\frac{a+b}{2}i\right)\right) - \int_a^b f(R\exp(it))\,dt\right| \\
&\leq \frac{1}{8}R(b-a)^2\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}|f'(R\exp(it))| + \sup_{t\in\left[\frac{a+b}{2},b\right]}|f'(R\exp(it))|\right] \\
&\leq \frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}|f'(R\exp(it))|.
\end{aligned}$$

We give now examples for some fundamental complex functions. Consider the function $f(z)=z^n$, $z\in\mathbb{C}$ with $n\geq 1$. Then $f'(z)=nz^{n-1}$,

$$f(R\exp(ix)) = R^n\exp(nxi),$$

$$|f'(R\exp(it))| = nR^{n-1}|\exp((n-1)it)| = nR^{n-1},\ t\in[a,b]$$

and

$$\int_a^b f(R\exp(it))\,dt = R^n\int_a^b \exp(nti)\,dt = R^n\frac{\exp(nbi)-\exp(nai)}{ni}.$$

Making use of the inequality (3.23), we get

$$\left|(b-a)R^n\exp(nxi) - R^n\frac{\exp(nbi)-\exp(nai)}{ni}\right| \leq R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]nR^{n-1},$$

which is equivalent to

$$(3.25)\qquad \left|(b-a)\exp(nxi) - \frac{\exp(nbi)-\exp(nai)}{ni}\right| \leq n\left[\frac{1}{2}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]$$

for $x\in[a,b]$. If we take in (3.25) $x=\frac{a+b}{2}$, then we get

$$(3.26)\qquad \left|(b-a)\exp\left(n\left(\frac{a+b}{2}\right)i\right) - \frac{\exp(nbi)-\exp(nai)}{ni}\right| \leq \frac{1}{4}n(b-a)^2$$

for $n\geq 1$. Consider the exponential function $f(z)=\exp(z)$, $z\in\mathbb{C}$. Then $f'(z)=\exp(z)$,

$$|f'(R\exp(it))| = |\exp(R(\cos t + i\sin t))| = \exp(R\cos t),\ t\in[a,b]$$

and by the inequality (3.17), we get

$$\left|(b-a)\exp(R\exp(ix))-\int_a^b \exp(R\exp(it))\,dt\right| \le R\left[(x-a)\int_a^x \exp(R\cos t)\,dt+(b-x)\int_x^b \exp(R\cos t)\,dt\right], \tag{3.27}$$

while from the inequality (3.23), we get

$$\begin{aligned}\left|(b-a)\exp(R\exp(ix))-\int_a^b \exp(R\exp(it))\,dt\right| &\le \frac{1}{2}R\left[(x-a)^2\sup_{t\in[a,x]}\exp(R\cos t)+(b-x)^2\sup_{t\in[x,b]}\exp(R\cos t)\right]\\ &\le R\left[\frac{1}{4}(b-a)^2+\left(x-\frac{a+b}{2}\right)^2\right]\sup_{t\in[a,b]}\exp(R\cos t)\end{aligned} \tag{3.28}$$

for $x\in[a,b]$. From the inequality (3.20), we get

$$\begin{aligned}\left|(b-a)\exp(R\exp(ix))-\int_a^b \exp(R\exp(it))\,dt\right| &\le \frac{1}{q+1}R\left[(x-a)^{1+1/q}\left(\int_a^x \exp(pR\cos t)\,dt\right)^{1/p}\right.\\ &\left.+(b-x)^{1+1/q}\left(\int_x^b \exp(pR\cos t)\,dt\right)^{1/p}\right]\end{aligned} \tag{3.29}$$

for $x\in[a,b]$, where $p,q>1$ with $\frac{1}{p}+\frac{1}{q}=1$. If in the inequality (3.27) we take $x=\frac{a+b}{2}$, then we get

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp(R\exp(it))\,dt\right| \le \frac{1}{2}R(b-a)\int_a^b \exp(R\cos t)\,dt, \tag{3.30}$$

while from the inequality (2.15), we get

$$\begin{aligned}\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b \exp(R\exp(it))\,dt\right| &\le \frac{1}{8}(b-a)^2R\left[\sup_{t\in\left[a,\frac{a+b}{2}\right]}\exp(R\cos t)+\sup_{t\in\left[\frac{a+b}{2},b\right]}\exp(R\cos t)\right]\\ &\le \frac{1}{4}R(b-a)^2\sup_{t\in[a,b]}\exp(R\cos t).\end{aligned} \tag{3.31}$$

From (3.29), we have

$$\left|(b-a)\exp\left(R\exp\left(\frac{a+b}{2}i\right)\right)-\int_a^b\exp\left(R\exp\left(it\right)\right)dt\right| \tag{3.32}$$
$$\le\frac{1}{(q+1)\,2^{1+1/q}}R\,(b-a)^{1+1/q}$$
$$\times\left[\left(\int_a^{\frac{a+b}{2}}\exp\left(pR\cos t\right)dt\right)^{1/p}+\left(\int_{\frac{a+b}{2}}^b\exp\left(pR\cos t\right)dt\right)^{1/p}\right].$$

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