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Algebra of frontier points via semi-kernels

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Abstract

In topological spaces, the study of interior and closure of a set are renowned concepts where the interior is defined as the union of open sets and the closure is defined as the intersection of closed sets. In literature, it is also a significant study while a set is defined as the intersection of open sets, and the union of closed sets. These respective ideas are known as the kernel of a set and its complementary function. Utilizing these ideas, some authors have introduced various kinds of results in topological spaces. Some mathematicians have extended these concepts via Levine's semi-open sets to semi-kernel and its complementary function. The study of these notions is also a remarkable part of the field of topological spaces as the collection of semi-open sets does not form a topology again. In this paper, we have taken the semi-kernel and its complementary function into account to introduce new types of frontier points. After that we have studied and presented several characterizations of these new types of frontiers and established relationships among them. Finally, we have shown that semi-homeomorphic images of these new types of frontiers are invariant.

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1. Introduction

The semi-open set in a topological space was first introduced by Levine [1] and its notion has opened a new branch in the research field by the name of generalized open set. A good number of mathematicians have studied this set extensively [2-11]. The study of $\wedge$-sets and $\vee$-sets in topological spaces was introduced by Maki [12]. These two types of sets are not defined in the traditional way using interior (denoted as *'Int'*) and closure (denoted as ***'Cl'***) operators in topological spaces. Though the set $\wedge$ is not an open set and the set $\vee$ is not a closed set, they are dual to each other. These two sets have been utilized in the study of separation axioms (see [7,10]). Dontchev and Maki [13], and Maheshwari and Prasad [14] imposed these notions in the field of semi-open sets by the name of $\wedge_s$-set and $\vee_s$-set. These sets help us to study generalized separation axioms in detail (see [3,4,8,9,11,13,15]).

In this paper, our main work is to study some new types of boundaries or frontiers with the help of $\wedge_s$-sets and $\vee_s$-sets that are not conventional type frontiers (see [16-22]) because they are not defined in terms of *'Int'* and *'Cl'* operators. Throughout this paper, the notations $\mathcal{T}$ and $\mathcal{S}$ are used to denote the topological spaces (X,τ) and (Y,σ) respectively. In a topological space $\mathcal{T}$, the family of all closed (resp. semi-open) sets is identified by the notations $C(\mathcal{T})$ (resp. $SO(\mathcal{T})$).

In a topological space $\mathcal{T}$, a subset P of X is a $\wedge_s$ (resp. $\vee_s$)-set [23] if $P = P^{\wedge_s}$ (resp. $P^{\vee_s}$), where $P^{\wedge_s} = \cap\{N : N \supseteq P, N \in SO(\mathcal{T})\}$ and $P^{\vee_s} = \cup\{M : M \subseteq P, X \setminus M \in SO(\mathcal{T})\}$. In [13,14], $P^{\wedge_s}$ is called the semi-kernel of P.

The aim of this paper is to solve the question: What happens if the frontier points are defined by semi-open sets related to the operators $()^{\wedge_s}$ and $()^{\vee_s}$?

2. Frontier Points Via Semi-kernels

We begin this section with the following example:

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Example 2.1. Let $X = \{p,q,r\}$ and $\tau = \{\varnothing,\{p\},\{q,r\},X\}$. Then $C(\mathcal{T}) = \{\varnothing,\{p\},\{q,r\},X\}$. For $\{p\}$, $\{p\} \subseteq Cl(Int(\{p\}))$; for $\{q\}$, $\{q\} \nsubseteq Cl(Int(\{q\}))$; for $\{r\}$, $\{r\} \nsubseteq Cl(Int(\{r\}))$; for $\{p,q\}$, $\{p,q\} \nsubseteq Cl(Int(\{p,q\}))$; for $\{p,r\}$, $\{p,r\} \nsubseteq Cl(Int(\{p,r\}))$; for $\{q,r\}$, $\{q,r\} \subseteq Cl(Int(\{q,r\}))$ and for X, $X \subseteq Cl(Int(X))$. Thus $SO(\mathcal{T}) = \{\varnothing,\{p\},\{q,r\},X\}$. Consider $P = \{q\}$. Then $P^{\wedge_s} = \{q,r\}$ and $(X \setminus P)^{\wedge_s} = X$. Therefore $P^{\wedge_s} \cap (X \setminus P)^{\wedge_s} \neq \varnothing$.

In this regards, we define:

Definition 2.2. Let $\mathcal{T}$ be a topological space. Define the frontier operator $Bd^{\wedge_s} : \wp(X) \to \wp(X)$ by $Bd^{\wedge_s}(P) = P^{\wedge_s} \cap (X \setminus P)^{\wedge_s}$, $P \in \wp(X)$, where $\wp(X)$ is the power set of X.

Theorem 2.3. Let P and Q be two subsets of a topological space $\mathcal{T}$. Then

1. $Bd^{\wedge_s}(\varnothing) = \varnothing = Bd^{\wedge_s}(X)$;
2. $Bd^{\wedge_s}(P) = Bd^{\wedge_s}(X \setminus P)$;
3. $Bd^{\wedge_s}(P) = P^{\wedge_s} \setminus P^{\vee_s}$;
4. for $P \in SO(\mathcal{T})$, $Bd^{\wedge_s}(P) = P \setminus P^{\vee_s}$;
5. for semi-closed and semi-open set P, $Bd^{\wedge_s}(P) = \varnothing$;
6. for $P \in \tau$, $Bd^{\wedge_s}(P) = P \setminus P^{\vee_s}$;
7. for $\wedge_s$-set P, $Bd^{\wedge_s}(P) = P \setminus P^{\vee_s}$;
8. for $\vee_s$-set P, $Bd^{\wedge_s}(P) = P^{\wedge_s} \setminus P$;
9. $Bd^{\wedge_s}(Bd^{\wedge_s}(P)) \neq Bd^{\wedge_s}(P)$;
10. $Bd^{\wedge_s}(P \cup Q) \subseteq Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q)$.

Proof. 1. Follows from the fact that $\varnothing^{\wedge_s} = \varnothing$.

3. $Bd^{\wedge_s}(P) = P^{\wedge_s} \cap (X \setminus P)^{\wedge_s} = P^{\wedge_s} \cap (X \setminus P^{\vee_s})$ ([23]) $= P^{\wedge_s} \setminus P^{\vee_s}$.

4. $Bd^{\wedge_s}(P) = P^{\wedge_s} \cap (X \setminus P)^{\wedge_s} = P \cap (X \setminus P^{\vee_s}) = P \setminus P^{\vee_s}$.

5. Follows from the fact that for semi-open and semi-closed set P, $P^{\vee_s} = P^{\wedge_s} = P$.

6. Follows from $\tau \subseteq SO(\mathcal{T})$.

7, 8. Follows from Definitions of $\wedge_s$-set and $\vee_s$-set.

9. It will be followed by the Example 2.4.

10. $Bd^{\wedge_s}(P \cup Q) = (P \cup Q)^{\wedge_s} \cap [X \setminus (P \cup Q)]^{\wedge_s} = (P \cup Q)^{\wedge_s} \cap [(X \setminus P) \cap (X \setminus Q)]^{\wedge_s} \subseteq (P \cup Q)^{\wedge_s} \cap [(X \setminus P)^{\wedge_s} \cap (X \setminus Q)^{\wedge_s}] = (P^{\wedge_s} \cup Q^{\wedge_s}) \cap [(X \setminus P)^{\wedge_s} \cap (X \setminus Q)^{\wedge_s}] = [((X \setminus P)^{\wedge_s} \cap (X \setminus Q)^{\wedge_s}) \cap P^{\wedge_s}] \cup [((X \setminus P)^{\wedge_s} \cap (X \setminus Q)^{\wedge_s}) \cap Q^{\wedge_s}] \subseteq [(X \setminus P)^{\wedge_s} \cap P^{\wedge_s}] \cup [(X \setminus Q)^{\wedge_s} \cap Q^{\wedge_s}] = Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q)$.

Example 2.4. Let $X = \{p,q,r\}$ and $\tau = \{\varnothing,\{p\},\{q,r\},X\}$. Then $SO(\mathcal{T}) = \{\varnothing,\{p\},\{q,r\},X\}$. Consider $P = \{q\}$. Then $Bd^{\wedge_s}(P) = \{q,r\}$ and $Bd^{\wedge_s}(Bd^{\wedge_s}(P)) = \varnothing$. Thus $Bd^{\wedge_s}(Bd^{\wedge_s}(P)) \neq Bd^{\wedge_s}(P)$.

Note that $Bd^{\wedge_s}(P)$ is not a semi-open set, in general.

For the converse of the relation (5) of Theorem 2.3, we have the following result:

Theorem 2.5. Let P be a subset of a topological space $\mathcal{T}$. Then for $Bd^{\wedge_s}(P) = \varnothing$, $P^{\wedge_s} = P = P^{\vee_s}$.

Proof. Given that $P^{\wedge_s} \cap (X \setminus P)^{\wedge_s} = \varnothing$. Then $P^{\wedge_s} \subseteq X \setminus (X \setminus P)^{\wedge_s}$. This implies that $P^{\wedge_s} \subseteq P^{\vee_s}$ and hence $P \subseteq P^{\wedge_s} \subseteq P^{\vee_s} \subseteq P$ [23].

Lemma 2.6. Let P and Q be two subsets of a topological space $\mathcal{T}$. Then

$Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \cap Q)$.

Proof. For the proof of this theorem, we consider following relations:

- $Bd^{\wedge_s}(P \cap Q) = Bd^{\wedge_s}[X \setminus (P \cap Q)] = Bd^{\wedge_s}[(X \setminus P) \cup (X \setminus Q)] \subseteq Bd^{\wedge_s}(X \setminus P) \cup Bd^{\wedge_s}(X \setminus Q) = Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q)$.
- $Bd^{\wedge_s}(P \setminus Q) = Bd^{\wedge_s}[P \cap (X \setminus Q)] \subseteq Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(X \setminus Q) = Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q)$.
- $Bd^{\wedge_s}(Q \setminus P) = Bd^{\wedge_s}[Q \cap (X \setminus P)] \subseteq Bd^{\wedge_s}(Q) \cup Bd^{\wedge_s}(X \setminus P) = Bd^{\wedge_s}(Q) \cup Bd^{\wedge_s}(P)$.

Therefore from the above three relations, we have

$Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(Q \setminus P) \subseteq Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q)$.

On the other hand, $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}[(P \setminus Q) \cup (P \cap Q)] \cup Bd^{\wedge_s}[(Q \setminus P) \cup (P \cap Q)] \subseteq Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(Q \setminus P)$.

Hence $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \cap Q)$.

Theorem 2.7. Let P and Q be two subsets of a topological space $\mathcal{T}$. Then following properties hold:

1. $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(P \cup Q)$;
2. $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \cup Q) \cup Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \cap Q)$;
3. $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \cup Q)$;
4. $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(P \Delta Q) = Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(Q \setminus P)$;
5. $Bd^{\wedge_s}(Q) \cup Bd^{\wedge_s}(P \Delta Q) = Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(Q \setminus P)$,

where Δ stands for symmetric difference.

Proof. 1. Putting $X \setminus Q$ in place of Q in Lemma 2.6, we have $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(X \setminus Q) = Bd^{\wedge_s}[P \setminus (X \setminus Q)] \cup Bd^{\wedge_s}[(X \setminus Q) \setminus P] \cup Bd^{\wedge_s}[P \cap (X \setminus Q)]$. This implies that $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}[X \setminus (P \cup Q)] \cup Bd^{\wedge_s}(P \setminus Q) = Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(P \cup Q) \cup Bd^{\wedge_s}(P \setminus Q)$.

2. Putting $X \setminus P$ in place of P in Lemma 2.6, we have $Bd^{\wedge_s}(X \setminus P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}[(X \setminus P) \setminus Q] \cup Bd^{\wedge_s}[Q \setminus (X \setminus P)] \cup Bd^{\wedge_s}[(X \setminus P) \cap Q]$. This implies that $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}[X \setminus (P \cup Q)] \cup Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(Q \setminus P) = Bd^{\wedge_s}(P \cup Q) \cup Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(Q \setminus P)$.

3. Putting $X \setminus P$ in place of P and $X \setminus Q$ in place of Q in Lemma 2.6, we have $Bd^{\wedge_s}(X \setminus P) \cup Bd^{\wedge_s}(X \setminus Q) = Bd^{\wedge_s}[(X \setminus P) \setminus (X \setminus Q)] \cup Bd^{\wedge_s}[(X \setminus Q) \setminus (X \setminus P)] \cup Bd^{\wedge_s}[(X \setminus P) \cap (X \setminus Q)]$. This implies that $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(Q) = Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}[X \setminus (P \cup Q)] = Bd^{\wedge_s}(Q \setminus P) \cup Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(P \cup Q)$.

4. From Lemma 2.6, we have $Bd^{\wedge_s}(P) \cup Bd^{\wedge_s}(P \Delta Q) = Bd^{\wedge_s}[P \setminus (P \Delta Q)] \cup Bd^{\wedge_s}[P \cap (P \Delta Q)] \cup Bd^{\wedge_s}[(P \Delta Q) \setminus P] = Bd^{\wedge_s}(P \cap Q) \cup Bd^{\wedge_s}(P \setminus Q) \cup Bd^{\wedge_s}(Q \setminus P)$.

5. Similar to (4).

It is noticeable that for a subset P of a topological space $\mathcal{T}$, $P^{\vee_s} \cap (X \setminus P)^{\vee_s} = \varnothing$. But their union is neither empty nor the whole space, in general.

Example 2.8. Let $X = \{p,q,r\}$ and $\tau = \{\varnothing,\{p\},\{p,q\},X\}$. Then closed sets are $\varnothing,\{r\},\{q,r\}$ and X. Now for $\{p\}$, $\{p\} \subseteq Cl(Int(\{p\}))$; for $\{q\}$, $\{q\} \nsubseteq Cl(Int(\{q\}))$; for $\{r\}$, $\{r\} \nsubseteq Cl(Int(\{r\}))$; for $\{p,q\}$, $\{p,q\} \subseteq Cl(Int(\{p,q\}))$; for $\{p,r\}$, $\{p,r\} \subseteq Cl(Int(\{p,r\}))$; for $\{q,r\}$, $\{q,r\} \nsubseteq Cl(Int(\{q,r\}))$. Therefore $SO(\mathcal{T}) = \{\varnothing,\{p\},\{p,q\},\{p,r\},X\}$ and the semi-closed sets are $\varnothing$, $\{q\}$, $\{r\}$, $\{q,r\}$ and X. Let $P = \{p,q\}$. Then $P^{\vee_s} = \{q\}$ and $(X \setminus P)^{\vee_s} = \{r\}$. Therefore $P^{\vee_s} \cup (X \setminus P)^{\vee_s} \neq X$,$\varnothing$.

In view of the Example 2.8, we define a new frontier operator as:

Definition 2.9. Let $\mathcal{T}$ be a topological space. Define the frontier operator $Bd^{\vee_s} : \wp(X) \to \wp(X)$ by $Bd^{\vee_s}(P) = P^{\vee_s} \cup (X \setminus P)^{\vee_s}$, $P \in \wp(X)$.

Theorem 2.10. Let P be a subset of a topological space $\mathcal{T}$. Then following results hold:

1. $Bd^{\vee_s}(\varnothing) = Bd^{\vee_s}(X) = X$;
2. $Bd^{\vee_s}(P) = Bd^{\vee_s}(X \setminus P)$;
3. $Bd^{\vee_s}(P) = P^{\vee_s} \cup (X \setminus P^{\wedge_s})$;
4. for $P \in SO(\mathcal{T})$, $Bd^{\vee_s}(P) = P^{\vee_s} \cap (X \setminus P)$;
5. for semi-closed set P, $Bd^{\vee_s}(P) = P \cup (X \setminus P^{\wedge_s})$;
6. for $\wedge_s$-set P, $Bd^{\vee_s}(P) = P^{\vee_s} \cup (X \setminus P)$;
7. for $\vee_s$-set P, $Bd^{\vee_s}(P) = P \cup (X \setminus P^{\wedge_s})$;
8. for semi-closed and semi-open set P, $Bd^{\vee_s}(P) = X$;
9. $X \setminus Bd^{\vee_s}(P) = Bd^{\wedge_s}(P)$.

Proof. 1, 2. Obvious from Definition.

3. Obvious from the fact $(X \setminus P)^{\wedge_s} = X \setminus P^{\vee_s}$.

4. $Bd^{\vee_s}(P) = P^{\vee_s} \cup (X \setminus P)^{\vee_s} = P^{\vee_s} \cup (X \setminus P)$, since $X \setminus P$ is semi-closed.

5. $Bd^{\vee_s}(P) = P^{\vee_s} \cup (X \setminus P)^{\vee_s} = P \cup (X \setminus P^{\wedge_s})$, since $P = P^{\vee_s}$ [23].

7. $Bd^{\wedge_s}(P) = P^{\vee_s} \cup (X \setminus P)^{\vee_s} = P \cup (X \setminus P^{\wedge_s})$.

9. $X \setminus Bd^{\vee_s}(P) = X \setminus [P^{\vee_s} \cup (X \setminus P)^{\vee_s}] = (X \setminus P^{\vee_s}) \cap [X \setminus (X \setminus P)^{\vee_s}] = P^{\wedge_s} \setminus P^{\vee_s} = Bd^{\wedge_s}(P)$.

Example 2.11. Let $X = \{p,q,r\}$ and $\tau = \{\varnothing,\{p\},\{q,r\},X\}$. Then the semi-closed sets are $\varnothing$, $\{p\}$, $\{q,r\}$ and X. Consider $P = \{p,q\}$. Then $Bd^{\vee_s}(P) = \{p\}$ and $Bd^{\vee_s}(Bd^{\vee_s}(P)) = X$. Thus $Bd^{\vee_s}(Bd^{\vee_s}(P)) \neq Bd^{\vee_s}(P)$.

Note that $Bd^{\vee_s}(P)$ is not a semi-closed set, in general.

Definition 2.12. Let $\mathcal{T}$ be a topological space. We define the frontier operator $Bd_{\vee_s} : \wp(X) \to \wp(X)$ by $Bd_{\vee_s}(P) = P \setminus P^{\vee_s}$, $P \in \wp(X)$.

Theorem 2.13. Let P be a subset of a topological space $\mathcal{T}$. Then

1. $P = P^{\vee_s} \cup Bd_{\vee_s}(P)$;
2. $P^{\vee_s} \cap Bd_{\vee_s}(P) = \varnothing$;
3. for $Bd_{\vee_s}(P) = \varnothing$, $P = P^{\vee_s}$;

4. for a semi-closed set P, $Bd_{\vee_s}(P) = \varnothing$.

Proof. 1, 2. Obvious from Definition.

3. Follows from 1.

4. For semi-closed set P, $P = P^{\vee_s}$ and hence $Bd_{\vee_s}(P) = P \setminus P^{\vee_s} = \varnothing$.

Theorem 2.14. Let P be a subset of a topological space $\mathcal{T}$. Then $Bd_{\vee_s}(P) = \varnothing$ if and only if P is a $\vee_s$-set.

Proof. Suppose $Bd_{\vee_s}(P) = \varnothing$. Then $P \setminus P^{\vee_s} = \varnothing$ implies $P \subseteq P^{\vee_s}$. Furthermore, $P^{\vee_s} \subseteq P$.

Conversely, suppose that P is a $\vee_s$-set. Then $P = P^{\vee_s}$ and hence $Bd_{\vee_s}(P) = \varnothing$.

Definition 2.15. Let $\mathcal{T}$ be a topological space. We define the frontier operator $Bd_{\wedge_s} : \wp(X) \to \wp(X)$ by $Bd_{\wedge_s}(P) = P^{\wedge_s} \setminus P$, $P \in \wp(X)$.

Theorem 2.16. Let P be a subset of a topological space $\mathcal{T}$. Then

1. $P \cup Bd_{\wedge_s}(P) = P^{\wedge_s}$;
2. $P \cap Bd_{\wedge_s}(P) = \varnothing$;
3. $Bd_{\wedge_s}(P) = \varnothing$ if and only if $P = P^{\wedge_s}$;
4. for semi-open set P, $Bd_{\wedge_s}(P) = \varnothing$.

Proof. 1, 2. Obvious from Definition.

3. Assume $Bd_{\wedge_s}(P) = \varnothing$. Then $P^{\wedge_s} \subseteq P$, and from Definition of $()^{\wedge_s}$, $P \subseteq P^{\wedge_s}$. Thus $P^{\wedge_s} = P$.

Converse is trivial.

4. Since P is semi-open, so $P^{\wedge_s} = P$. Hence the result.

Theorem 2.17. Let P be a subset of a topological space $\mathcal{T}$. Then $Bd_{\wedge_s}(P) = \varnothing$ if and only if P is a $\wedge_s$-set.

Theorem 2.18. Let P be a subset of a topological space $\mathcal{T}$. Then

1. $Bd_{\vee_s}(P) \cap Bd_{\wedge_s}(P) = \varnothing$;
2. $Bd_{\wedge_s}(P) \cup Bd_{\vee_s}(P) = Bd^{\wedge_s}(P)$;
3. $Bd_{\wedge_s}(P) = Bd^{\wedge_s}(P) \setminus Bd_{\vee_s}(P)$;
4. $Bd_{\vee_s}(P) = Bd^{\wedge_s}(P) \setminus Bd_{\wedge_s}(P)$.

Proof. 1. $Bd_{\vee_s}(P) \cap Bd_{\wedge_s}(P) = (P \setminus P^{\vee_s}) \cap (P^{\wedge_s} \setminus P) = \varnothing$.

2. $Bd_{\wedge_s}(P) \cup Bd_{\vee_s}(P) = (P^{\wedge_s} \setminus P) \cup (P \setminus P^{\vee_s}) = P^{\wedge_s} \setminus P^{\vee_s} = Bd^{\wedge_s}(P)$.

3, 4. Follows from (1) and (2) because $\{Bd_{\vee_s}(P), Bd_{\wedge_s}(P)\}$ is a partition of $Bd^{\wedge_s}(P)$.

We conclude this paper with the following definition and result.

Definition 2.19. [24] Let $\mathcal{T}$ and $\mathcal{S}$ be two topological spaces. A mapping $f : \mathcal{T} \to \mathcal{S}$ is called semi-homeomorphism if f is a bijection and images and pre-images of semi-open sets are semi-open.

Theorem 2.20. Let $\mathcal{T}$ and $\mathcal{S}$ be two topological spaces and $f : \mathcal{T} \to \mathcal{S}$ be a semi-homeomorphism. Then for a subset P of X,

1. $f(P^{\wedge_s}) = [f(P)]^{\wedge_s}$;
2. $f(P^{\vee_s}) = [f(P)]^{\vee_s}$.

Proof. We give the proof of (1) only. We shall first show that $f(P^{\wedge_s}) \subseteq [f(P)]^{\wedge_s}$.

If not, there exists $y \in f(P^{\wedge_s})$ but $y \notin [f(P)]^{\wedge_s}$. This implies that $y \notin \cap\{V \in SO(\mathcal{S}) : f(P) \subseteq V\}$. Then there exists $U \in SO(\mathcal{S})$ such that $f(P) \subseteq U$ but $y \notin U$. Therefore $f^{-1}(U) \in SO(\mathcal{T})$, $P = f^{-1}(f(P)) \subseteq f^{-1}(U)$ and $f^{-1}(y) \notin f^{-1}(U)$ (because if $f^{-1}(y) \in f^{-1}(U)$, then $f(f^{-1}(U)) = U$ implies $y \in U$, a contradiction). Thus $f^{-1}(y) \notin P^{\wedge_s}$ and hence $y \notin f(P^{\wedge_s})$, a contradiction. Therefore $f(P^{\wedge_s}) \subseteq [f(P)]^{\wedge_s}$.

We shall now show that $[f(P)]^{\wedge_s} \subseteq f(P^{\wedge_s})$.

If not, there exists $y \in [f(P)]^{\wedge_s}$ but $y \notin f(P^{\wedge_s})$. Then $f^{-1}(y) \notin P^{\wedge_s} = \cap\{V \in SO(\mathcal{T}) : P \subseteq V\}$. This implies that there exists $U \in SO(\mathcal{T})$ such that $P \subseteq U$ but $f^{-1}(y) \notin U$. Thus $f(P) \subseteq f(U)$ but $y \notin f(U)$. Moreover, $f(U) \in SO(\mathcal{S})$. So $y \notin [f(P)]^{\wedge_s}$, a contradiction. Thus $[f(P)]^{\wedge_s} \subseteq f(P^{\wedge_s})$. Hence the result follows.

Conclusions

The mathematical findings are:

- The frontier operator $Bd^{\wedge_s} : \wp(X) \to \wp(X)$ is $\varnothing$-preserving and sub-additive but not X-preserving and not idempotent.
- The frontier operator $Bd^{\vee_s} : \wp(X) \to \wp(X)$ is X-preserving but not $\varnothing$-preserving and not idempotent.
- For any set $P \subseteq X$, its images under the frontier operators $Bd_{\wedge_s}$, $Bd_{\vee_s} : \wp(X) \to \wp(X)$ are disjoint. Moreover, $\{Bd_{\wedge_s}(P), Bd_{\vee_s}(P)\}$ forms a partition of $Bd^{\wedge_s}(P)$.
- Semi-homeomorphic image of $\wedge_s$ (resp. $\vee_s$) -set is again $\wedge_s$ (resp. $\vee_s$) -set.
- For a semi-homeomorphism $f : \mathcal{T} \to \mathcal{S}$, $f[Bd^{\wedge_s}(P)] = Bd^{\wedge_s}[f(P)]$; $f[Bd^{\vee_s}(P)] = Bd^{\vee_s}[f(P)]$; $f[Bd_{\wedge_s}(P)] = Bd_{\wedge_s}[f(P)]$ and $f[Bd_{\vee_s}(P)] = Bd_{\vee_s}[f(P)]$.

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Conflicts of interest

The authors state that they did not have a conflict of interests.

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