HSJG
Hagia Sophia Journal of Geometry
e-ISSN 2687-4261
Vol. 2, No. 2, 12-25, 2020
Research Article

On Fibonacci Vectors

Kübra ÇETİNBERK [1], Salim YÜCE [2]

Abstract
The purpose of this article is to study vector products of Fibonacci 3-vectors, Fibonacci 4-vectors and Fibonacci 7-vectors. To achieve this, we first describe the corresponding anti-symmetric matrix for the Fibonacci 3-vector and reconsider the vector product with the aid of this matrix. We examine certain properties of this vector product. Furthermore, we define vector products for Fibonacci 4-vectors and Fibonacci 7-vectors. We also give in the same vein the corresponding anti-symmetric matrix for Fibonacci 7-vector and redefine the vector product by using this matrix. In the final instance we investigate the Lorentzian inner products, Lorentzian vector products and Lorentzian triple scalar products for Fibonacci 3-vectors, Fibonacci 4-vectors and Fibonacci 7-vectors.

Keywords and 2010 Mathematics Subject Classification
Keywords: Fibonacci vectors — vector product — anti-symmetric matrix — Lorentzian space
MSC: 11B39, 97G70

[1,2] *Yildiz Technical University, Faculty of Arts and Sciences, Department of Mathematics, Davutpasa Campus, 34220, Esenler, Istanbul, TURKEY*
[1] ✉kubracetinberk@gmail.com, [2] ✉sayuce@yildiz.edu.tr
Corresponding author: Kübra Çetinberk
Article History: Received 30 July 2020; Accepted 27 August 2020

1. Introduction

The Fibonacci numbers are popular topics in Linear Algebra and the Fibonacci vectors have become an important subject of Geometry. In the literature, some identities of Fibonacci 3-vectors are given by Atanassov et. al. [1]. By the n-th Fibonacci (respectively Lucas) vector of length r, this means that the vector whose components are the n-th through $(n+r-1)$-st Fibonacci (respectively Lucas) numbers are defined by Salter [2]. For every integer r, n-th Fibonacci r-vector denoted by $\vec{F}_n$ and can be written as

$$\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & \dots & F_{n+r-2} & F_{n+r-1} \end{bmatrix}^T_{1\times r} = \begin{bmatrix} F_n \\ F_{n+1} \\ F_{n+2} \\ \vdots \\ F_{n+r-2} \\ F_{n+r-1} \end{bmatrix}_{r\times 1},$$

where F_n is n-th Fibonacci number. Also, for arbitrary r, Salter [2] expressed the inner product of any two Fibonacci vectors, any two Lucas vectors, and any Fibonacci vector and any Lucas vector in terms of the Fibonacci and Lucas numbers. Moreover, Salter used these formulas to deduce a number of identities involving the Fibonacci and Lucas numbers [2]. Güven & Nurkan [3] defined new vectors which are called dual Fibonacci vectors and they gave properties of these dual Fibonacci vectors to use in the geometry of dual space. Furthermore, a definition of generalized dual Fibonacci vectors, the inner product and cross product of two generalized dual Fibonacci vectors and the triple scalar product of three generalized dual Fibonacci vectors given by Yüce & Torunbalcı Aydın [4]. Vector products of considering two Fibonacci 3-vectors, two Lucas 3-vectors and one of each vector by using vector version of the Binet's formula are investigated by Kaya & Önder [5].

In this paper, firstly the corresponding anti-symmetric matrix for Fibonacci 3-vector is described and the vector product is given by using this anti-symmetric matrix. Then, the properties of the vector product is given by using anti-symmetric matrix are given. Also, vector products for the Fibonacci 4-vectors and the Fibonacci 7-vectors are defined. Similar to Fibonacci 3-vectors, the corresponding anti-symmetric matrix for Fibonacci 7-vector is described and the vector product is re-examined by using this anti-symmetric matrix. Furthermore, properties of vector product for Fibonacci 7-vectors are given. Moreover, vector product for Fibonacci 7-vectors by using Binet's Formula are obtained. Lastly, the Lorentzian inner products, vector products, and triple scalar products for Fibonacci 3-vectors, Fibonacci 4-vectors, and Fibonacci 7-vectors are investigated.

2. Preliminaries

2.1 Fibonacci Numbers

n-th Fibonacci number F_n is defined for all positive integers by the second order recurrence relation and initial conditions as follows:

$$\begin{aligned} F_{n+2} &= F_{n+1}+F_n, &(1)\\ F_1 &= F_2 = 1, &(2)\end{aligned}$$

respectively. The Fibonacci sequence is

$$1,1,2,3,5,8,13,21,34,55,89,144,233,\ldots,F_n,\ldots. \quad (3)$$

For the Fibonacci sequence, we can give the following identities, ([6]-[10]):

$$\begin{aligned} F_{-n} &= (-1)^{n+1}F_n,\\ F_{n+1}^2-F_n^2 &= F_{2n},\\ F_{n-1}F_{n+1}-F_n^2 &= (-1)^n, \text{ (Cassini Identity)},\\ F_nF_m+F_{n+1}F_{m+1} &= F_{n+m+1},\\ F_nF_{m+r}-F_{n+r}F_m &= (-1)^m F_rF_{n-m},\\ F_nF_{m+1}-F_{n+1}F_m &= (-1)^m F_{n-m},\\ F_n &= \frac{(\alpha^n-\beta^n)}{\alpha-\beta}, \text{ (Binet's Formula)}, \end{aligned} \quad (4)$$

where $\alpha=\frac{1+\sqrt{5}}{2}$, $\beta=\frac{1-\sqrt{5}}{2}$ are roots of $x^2-x-1=0$, it follows that $\alpha+\beta=1$, $\alpha-\beta=\sqrt{5}$ and $\alpha\beta=-1$. Also, α is called the golden ratio.

Table 1. Fibonacci numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
F_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233
F_{-n}	0	1	-1	2	-3	5	-8	13	-21	34	-55	89	-144	233

2.2 Fibonacci Vectors

2.2.1 Fibonacci 3-Vectors

Definition 1. *For all integers n,* **Fibonacci 3-vector** *is defined by*

$$\vec{F}_n = \begin{bmatrix}F_n & F_{n+1} & F_{n+2}\end{bmatrix}^T,$$

where F_n is n-th Fibonacci number, [1, 2].

Theorem 2. *For all integers n, the vector version of the Binet's formula every Fibonacci 3-vector $\vec{F}_n$ can be defined by*

$$\vec{F}_n = \frac{1}{\alpha-\beta}\left(\alpha^n\vec{a}-\beta^n\vec{b}\right), \quad (5)$$

where $\vec{a}=\begin{bmatrix}1 & \alpha & \alpha^2\end{bmatrix}^T$ and $\vec{b}=\begin{bmatrix}1 & \beta & \beta^2\end{bmatrix}^T$, [2].

Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. Then, ***Euclidean inner product*** between these vectors can be defined as follows, [3]:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = F_nF_m + F_{n+1}F_{m+1} + F_{n+2}F_{m+2} = F_nF_m + F_{n+m+3}. \tag{6}$$

In other viewpoint:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = \frac{1}{5}\left(L_3L_{n+m+2} - (-1)^n L_{m-n}\right), [2]. \tag{7}$$

We can also write:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = -F_nF_{m-1} + F_{n+2}F_{m+3}. \tag{8}$$

Thus, the ***norm*** of $\vec{F}_n$ can be written in the following ways:

$$\left\| \vec{F}_n \right\| = \sqrt{F_n^2 + F_{2n+3}}, [3], \tag{9}$$

$$\left\| \vec{F}_n \right\| = \sqrt{\frac{1}{5}\left(L_3L_{2n+2} - (-1)^n L_0\right)}, [2] \tag{10}$$

or

$$\left\| \vec{F}_n \right\| = \sqrt{-F_nF_{n-1} + F_{n+2}F_{n+3}}. \tag{11}$$

Furthermore, for any Fibonacci 3-vector $\vec{F}_n$, the following equation can be given:

$$\left\langle \vec{F}_n, \vec{F}_{n+1} \right\rangle = -{F_n}^2 + F_{n+2}F_{n+4}. \tag{12}$$

Definition 3. *Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. Then,* **vector product** *of these two vectors is defined by*

$$\vec{F}_n \wedge \vec{F}_m = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \end{vmatrix}, \tag{13}$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ are orthonormal basis vectors of $\mathbb{R}^3$, [1, 3].

Theorem 4. *[3] For all Fibonacci 3-vectors $\vec{F}_n$ and $\vec{F}_m$, the vector product of these Fibonacci vectors is*

$$\begin{aligned} \vec{F}_n \wedge \vec{F}_m &= (-1)^m F_{n-m}\left(-\vec{e}_1 - \vec{e}_2 + \vec{e}_3\right) \\ &= (-1)^m F_{n-m}\begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T. \end{aligned} \tag{14}$$

Moreover, ***the scalar triple product*** for Fibonacci 3-vectors can be given by the following theorem.

Theorem 5. *[3] Let $\vec{F}_n$, $\vec{F}_m$ and $\vec{F}_k$ be Fibonacci 3-vectors. The scalar product of these three vectors is zero,* i.e.,

$$\left\langle \vec{F}_n \wedge \vec{F}_m, \vec{F}_k \right\rangle = 0.$$

Corollary 6. *A parallelepiped can not be constructed by Fibonacci vectors, [3].*

Vector Product of Fibonacci 3-Vectors by using Binet's Formula

Theorem 7. *[5] Let $\vec{a}$ and $\vec{b}$ be vectors given in Theorem 2. The vector product of $\vec{a}$ and $\vec{b}$ is*

$$\vec{a} \wedge \vec{b} = (\alpha - \beta)\begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T. \tag{15}$$

2.2.2 Fibonacci 4-Vectors

Definition 8. *For all integers n,* **Fibonacci 4-vector** *is defined by*

$$\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{bmatrix}^T, \tag{16}$$

where F_n is n-th Fibonacci number, [2].

Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 4-vectors. Then, ***Euclidean inner product*** between these vectors can be written as follows:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = F_4 F_{n+m+3}, [2], \tag{17}$$

and

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = -F_n F_{m-1} + F_{n+4} F_{m+3}. \tag{18}$$

Furthermore, the ***norm*** of $\vec{F}_n$:

$$\left\| \vec{F}_n \right\| = \sqrt{F_4 F_{2n+3}}, [2] \tag{19}$$

and

$$\left\| \vec{F}_n \right\| = \sqrt{F_{n+3} F_{n+4} - F_{n-1} F_n}. \tag{20}$$

Also, for all Fibonacci 4-vectors $\vec{F}_n$, we can write:

$$\left\langle \vec{F}_n, \vec{F}_{n+1} \right\rangle = -F_n{}^2 + F_{n+4}{}^2. \tag{21}$$

2.2.3 Fibonacci 7-Vectors

Definition 9. *For all integers n,* **Fibonacci 7-vector** *is defined by*

$$\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} & F_{n+4} & F_{n+5} & F_{n+6} \end{bmatrix}^T, \tag{22}$$

where F_n is n-th Fibonacci number, [2].

Similar to Fibonacci 3-vectors, there is a vector version of the Binet's formula for Fibonacci 7-vectors.

Theorem 10. *For all integers n, vector version of the Binet's formula for Fibonacci 7-vector $\vec{F}_n$ is*

$$\vec{F}_n = \frac{1}{\alpha - \beta} \left(\alpha^n \vec{a} - \beta^n \vec{b} \right),$$

where $\vec{a} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^5 & \alpha^6 \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} 1 & \beta & \beta^2 & \dots & \beta^5 & \beta^6 \end{bmatrix}^T$, [2].

Let $\vec{F}_n$ and $\vec{F}_m$ be two Fibonacci 7-vectors. In that case, ***Euclidean inner product*** between these vectors can be written as follows:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = \frac{1}{5} \left(L_7 L_{n+m+6} - (-1)^n L_{m-n} \right), [2] \tag{23}$$

and

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = -F_n F_{m-1} + F_{n+6} F_{m+7}. \tag{24}$$

Also, for all Fibonacci 7-vectors $\vec{F}_n$, the ***norm*** of $\vec{F}_n$:

$$\left\| \vec{F}_n \right\| = \sqrt{\frac{1}{5} \left(L_7 L_{2n+6} - (-1)^n L_0 \right)}, [2] \tag{25}$$

and

$$\left\| \vec{F}_n \right\| = \sqrt{F_{n+6} F_{n+7} - F_{n-1} F_n}. \tag{26}$$

Furthermore, for any Fibonacci 7-vector $\vec{F}_n$, we can write:

$$\left\langle \vec{F}_n, \vec{F}_{n+1} \right\rangle = -F_n{}^2 + F_{n+6} F_{n+8}. \tag{27}$$

2.2.4 Fibonacci r-Vectors

Definition 11. *For all integers n,* **Fibonacci r-vector** $\vec{F}_n$ *is defined by*

$$\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & \dots & F_{n+r-2} & F_{n+r-1} \end{bmatrix}^T$$

where F_n is n-th Fibonacci number, [2].

Also, for every Fibonacci r-vector $\vec{F}_n$, recurrence relation is provided i.e.,

$$\vec{F}_{n+2} = \vec{F}_{n+1} + \vec{F}_n.$$

Theorem 12. *(Vector version of the Binet's formula) For all integers n, every Fibonacci r-vector $\vec{F}_n$ can be defined by*

$$\vec{F}_n = \frac{1}{\alpha - \beta}\left(\alpha^n \vec{a} - \beta^n \vec{b}\right), \tag{28}$$

where $\vec{a} = \begin{bmatrix} 1 & \alpha & \alpha^2 & \dots & \alpha^{r-2} & \alpha^{r-1} \end{bmatrix}^T$ and $\vec{b} = \begin{bmatrix} 1 & \beta & \beta^2 & \dots & \beta^{r-2} & \beta^{r-1} \end{bmatrix}^T$, [2].

For all Fibonacci r-vectors $\vec{F}_n$ and $\vec{F}_m$, ***Euclidean inner product*** is defined by as follows, [2]:

$$\begin{aligned} \left\langle \vec{F}_n, \vec{F}_m \right\rangle &= \left(\vec{F}_n\right)^T \vec{F}_m \\ &= \sum_{i=0}^{r-1} F_{n+i} F_{m+i} \\ &= F_n F_m + F_{n+1} F_{m+1} + \dots + F_{n+r-2} F_{m+r-2} + F_{n+r-1} F_{m+r-1}. \end{aligned}$$

Then, for all $\vec{F}_n$ and $\vec{F}_m$ Fibonacci r-vectors, the Euclidean inner product can be written as follows, [2]:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = \begin{cases} F_r F_{n+m+r-1}, & \text{if } r \text{ is even,} \\ \frac{1}{5}\left(L_r L_{n+m+r-1} - (-1)^n L_{m-n}\right), & \text{if } r \text{ is odd,} \end{cases} \tag{29}$$

where, F_n is n-th Fibonacci number and L_n is n-th Lucas number.[1]

Also, for all Fibonacci r-vectors, the Euclidean inner product can be defined by a taking a new perspective such that:

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle = \begin{cases} -F_n F_{m-1} + F_{n+r} F_{m+r-1}, & \text{if } r \text{ is even,} \\ -F_n F_{m-1} + F_{n+r-1} F_{m+r}, & \text{if } r \text{ is odd.} \end{cases} \tag{30}$$

3. Vector Product of Fibonacci 3-Vectors by Using Anti-Symmetric Matrix

Definition 13. *Let $\vec{F}_n$ be a Fibonacci 3-vector i.e. $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$. In this case 3×3 anti-symmetric matrix which corresponds to $\vec{F}_n$ can be defined as follows:*

$$S_{\vec{F}_n} = \mathbb{F}_n = \begin{bmatrix} 0 & -F_{n+2} & F_{n+1} \\ F_{n+2} & 0 & -F_n \\ -F_{n+1} & F_n & 0 \end{bmatrix}. \tag{31}$$

Theorem 14. *For all $\lambda, \mu \in \mathbb{R}$ and for all Fibonacci 3-vectors $\vec{F}_n$and $\vec{F}_m$, we can write new vector as $\lambda\vec{F}_n + \mu\vec{F}_m$. Then, 3×3 anti-symmetric matrix which corresponds to this vector is $\lambda\mathbb{F}_n + \mu\mathbb{F}_m$.*

Let compute the vector product of Fibonacci 3-vectors by using anti-symmetric matrix given in eq. (31).

Theorem 15. *For all Fibonacci 3-vectors $\vec{F}_n$ and $\vec{F}_m$,*

$$\mathbb{F}_n \vec{F}_m = (-1)^m F_{n-m} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T. \tag{32}$$

Proof. Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. Then, let us find the matrix product between the anti-symmetric matrix which corresponding to the $\vec{F}_n$ with $\vec{F}_m$.

$$\mathbb{F}_n \vec{F}_m = \begin{bmatrix} 0 & -F_{n+2} & F_{n+1} \\ F_{n+2} & 0 & -F_n \\ -F_{n+1} & F_n & 0 \end{bmatrix} \begin{bmatrix} F_m \\ F_{m+1} \\ F_{m+2} \end{bmatrix} = (-1)^m F_{n-m} \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}.$$

■

[1] The Lucas numbers L_n are defined for all integers n by using the same Fibonacci recurrence relation as $L_{n+1} = L_n + L_{n-1}$ but initial conditions $L_1 = 1$ and $L_2 = 3$, [8]

From eq. (14) and eq. (32), we can simply obtain the following corollary.

Corollary 16. *Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. The vector product of these two vectors equals the matrix product between the anti-symmetric matrix which corresponding to the first vector given in eq. (31) with the second Fibonacci 3-vector. i.e.,*

$$\vec{F}_n \wedge \vec{F}_m = \mathbb{F}_n \vec{F}_m. \tag{33}$$

Example 17. *Let $\vec{F}_5$ and $\vec{F}_9$ be Fibonacci 3-vectors. So we can write these Fibonacci 3-vectors as follows:*

$$\vec{F}_5 = \begin{bmatrix} F_5 & F_6 & F_7 \end{bmatrix}^T = \begin{bmatrix} 5 & 8 & 13 \end{bmatrix}^T \quad , \quad \vec{F}_9 = \begin{bmatrix} F_9 & F_{10} & F_{11} \end{bmatrix}^T = \begin{bmatrix} 34 & 55 & 89 \end{bmatrix}^T.$$

The vector product of the Fibonacci 3-vectors $\vec{F}_5$ and $\vec{F}_9$ is

$$\vec{F}_5 \wedge \vec{F}_9 = (-1)^9 F_{5-9} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T = -F_{-4} \begin{bmatrix} -1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} -3 & -3 & 3 \end{bmatrix}^T.$$

On the other hand, we can see that

$$\mathbb{F}_5 \vec{F}_9 = \begin{bmatrix} 0 & -13 & 8 \\ 13 & 0 & -5 \\ -8 & 5 & 0 \end{bmatrix} \begin{bmatrix} 34 \\ 55 \\ 89 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ 3 \end{bmatrix}.$$

Therefore, we get $\vec{F}_5 \wedge \vec{F}_9 = \mathbb{F}_5 \vec{F}_9$.

3.1 Properties of Fibonacci 3-Vectors Vector Product by Using Anti-Symmetric Matrix

For all Fibonacci 3-vectors $\vec{F}_n$, $\vec{F}_m$, $\vec{F}_k$ and $\vec{F}_l$, following properties are provided:

1. $\vec{F}_n \left(\vec{F}_n\right)^T = \left\|\vec{F}_n\right\|^2 \mathrm{I}_3 + \mathbb{F}_n{}^2$ where I_3 is a 3×3 identity matrix,
2. $\mathbb{F}_n \vec{F}_m = -\mathbb{F}_m \vec{F}_n$,
3. $\mathbb{F}_n = -\mathbb{F}_n^T$,
4. $\vec{F}_n \wedge \vec{F}_n = \mathbb{F}_n \vec{F}_n = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T$,
5. $\mathbb{F}_n \mathbb{F}_m = \vec{F}_m \left(\vec{F}_m\right)^T - \left(\left(\vec{F}_m\right)^T \vec{F}_m\right) \mathrm{I}_3$, where I_3 is a 3×3 identity matrix,
6. $(\mathbb{F}_n \mathbb{F}_m)^T = \mathbb{F}_m \mathbb{F}_n$ where the notation "T" represents transpose of matrix,
7. $\mathbb{F}_n \mathbb{F}_m \, \mathbb{F}_k = \mathbb{F}_n \vec{F}_k \left(\vec{F}_m\right)^T - \left(\left(\vec{F}_m\right)^T \vec{F}_k\right) \mathbb{F}_n$,
8. $\mathbb{F}_n \mathbb{F}_m \, \mathbb{F}_n = -\left(\left(\vec{F}_n\right)^T \vec{F}_m\right) \mathbb{F}_n$,
9. $\mathbb{F}_n^3 = -\left\|\vec{F}_n\right\|^2 \mathbb{F}_n$,
10. $\left(\vec{F}_k\right)^T \mathbb{F}_n \vec{F}_m = \left(\vec{F}_n\right)^T \mathbb{F}_m \vec{F}_k = \left(\vec{F}_m\right)^T \mathbb{F}_k \vec{F}_n$,
11. $\mathbb{F}_n \mathbb{F}_m \, \vec{F}_k = \left(\left(\vec{F}_n\right)^T \vec{F}_k\right) \vec{F}_m - \left(\left(\vec{F}_n\right)^T \vec{F}_m\right) \vec{F}_k = \vec{F}_n \wedge \left(\vec{F}_m \wedge \vec{F}_k\right)$,
12. $\left(\mathbb{F}_n \vec{F}_m\right)^T \mathbb{F}_k \vec{F}_l = \left(\left(\vec{F}_n\right)^T \vec{F}_k\right) \left(\left(\vec{F}_m\right)^T \vec{F}_l\right) - \left(\left(\vec{F}_n\right)^T \vec{F}_l\right) \left(\left(\vec{F}_m\right)^T \vec{F}_k\right) = \left(\vec{F}_n \wedge \vec{F}_m\right)^T \left(\vec{F}_k \wedge \vec{F}_l\right)$,
13. Let S be an anti-symmetric matrix which corresponds to $\mathbb{F}_n \vec{F}_m$ vector. Then, S can be calculated as follows:
$S = \vec{F}_m \left(\vec{F}_n\right)^T - \vec{F}_n \left(\vec{F}_m\right)^T = \mathbb{F}_n \mathbb{F}_m - \mathbb{F}_m \mathbb{F}_n$.

Proof. Proofs can be shown by using anti-symmetric matrix which is given eq. (31). ■

By the Corollary 16 and Theorem 7 , we can give the following corollary:

Corollary 18. *Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. Another way of stating vector product of these Fibonacci 3-vectors is as follows:*

$$\vec{F}_n \wedge \vec{F}_m = \mathbb{F}_n \vec{F}_m = (-1)^{m+1} F_{n-m} \frac{\vec{a} \wedge \vec{b}}{\alpha - \beta}. \tag{34}$$

4. Vector Product for Fibonacci 4-Vectors

Definition 19. *Let $\vec{F}_n, \vec{F}_m$ and $\vec{F}_k$ be Fibonacci 4-vectors.* **The vector product** *of these three vectors is defined by as follows:*

$$\vec{F}_n \otimes \vec{F}_m \otimes \vec{F}_k = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_m & F_{m+1} & F_{m+2} & F_{m+3} \\ F_k & F_{k+1} & F_{k+2} & F_{k+3} \end{vmatrix}, \tag{35}$$

where $\{\vec{e}_1, \vec{e}_2, \vec{e}_3, \vec{e}_4\}$ is a orthonormal basis of $\mathbb{R}^4$.

Theorem 20. *For all Fibonacci 4-vectors $\vec{F}_n, \vec{F}_m$ and $\vec{F}_k$, the vector product of these three vector is zero. i.e.,*

$$\vec{F}_n \otimes \vec{F}_m \otimes \vec{F}_k = 0. \tag{36}$$

Proof. The proof can be easily seen by using the usual properties of determinant function. ■

Corollary 21. *For all Fibonacci 4-vectors $\vec{F}_n, \vec{F}_m, \vec{F}_k$ and $\vec{F}_l$,*

$$\det\left(\vec{F}_n, \vec{F}_m, \vec{F}_k, \vec{F}_l\right) = 0. \tag{37}$$

Corollary 22. *Let $\vec{F}_n, \vec{F}_m, \vec{F}_k$ and $\vec{F}_l$ be Fibonacci 4-vectors.* **The triple scalar product** *of these vectors is zero, i.e.,*

$$\left\langle \vec{F}_n \otimes \vec{F}_m \otimes \vec{F}_k, \vec{F}_l \right\rangle = 0. \tag{38}$$

5. Vector Product for Fibonacci 7-Vectors

Definition 23. *For all Fibonacci 7-vectors $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} & F_{n+4} & F_{n+5} & F_{n+6} \end{bmatrix}^T$ and $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} & F_{m+3} & F_{m+4} & F_{m+5} & F_{m+6} \end{bmatrix}^T$,* **vector product** *of these two vectors is defined by as follows:*

$$\vec{F}_n \times \vec{F}_m = \begin{bmatrix} F_{n+2}F_{m+1} + F_{n+1}F_{m+2} - F_{n+4}F_{m+3} + F_{n+3}F_{m+4} - F_{n+5}F_{m+6} + F_{n+6}F_{m+5} \\ F_nF_{m+2} + F_{n+2}F_m - F_{n+5}F_{m+3} + F_{n+3}F_{m+5} - F_{n+6}F_{m+4} + F_{n+4}F_{m+6} \\ F_{n+1}F_m + F_nF_{m+1} - F_{n+6}F_{m+3} + F_{n+3}F_{m+6} - F_{n+4}F_{m+5} + F_{n+5}F_{m+4} \\ F_nF_{m+4} + F_{n+4}F_m - F_{n+2}F_{m+6} + F_{n+6}F_{m+2} - F_{n+1}F_{m+5} + F_{n+5}F_{m+1} \\ F_{n+3}F_m + F_nF_{m+3} - F_{n+1}F_{m+6} + F_{n+6}F_{m+1} - F_{n+5}F_{m+2} + F_{n+2}F_{m+5} \\ F_{n+6}F_m + F_nF_{m+6} - F_{n+3}F_{m+1} + F_{n+1}F_{m+3} - F_{n+2}F_{m+4} + F_{n+4}F_{m+2} \\ F_nF_{m+5} + F_{n+5}F_m - F_{n+4}F_{m+1} + F_{n+1}F_{m+4} - F_{n+3}F_{m+2} + F_{n+2}F_{m+3} \end{bmatrix} \tag{39}$$

Theorem 24. *Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 7-vectors. Then, vector product of these two vectors is*

$$\vec{F}_n \times \vec{F}_m = (-1)^m F_{n-m} \begin{bmatrix} -1 & -1 & -2 & -3 & 9 & 6 & -6 \end{bmatrix}^T. \tag{40}$$

5.1 Vector Product of Fibonacci 7-vectors by Using Anti-Symmetric Matrix

Definition 25. *For any Fibonacci 7-vector $\vec{F}_n$, there is an anti-symmetric matrix size of 7×7 corresponds to $\vec{F}_n$. This anti-symmetric matrix can be given as follows:*

$$S_{\vec{F}_n}=\begin{bmatrix} 0 & -F_{n+2} & F_{n+1} & -F_{n+4} & F_{n+3} & F_{n+6} & -F_{n+5} \\ F_{n+2} & 0 & -F_n & -F_{n+5} & -F_{n+6} & F_{n+3} & F_{n+4} \\ -F_{n+1} & F_n & 0 & -F_{n+6} & F_{n+5} & -F_{n+4} & F_{n+3} \\ F_{n+4} & F_{n+5} & F_{n+6} & 0 & -F_n & -F_{n+1} & -F_{n+2} \\ -F_{n+3} & F_{n+6} & -F_{n+5} & F_n & 0 & F_{n+2} & -F_{n+1} \\ -F_{n+6} & -F_{n+3} & F_{n+4} & F_{n+1} & -F_{n+2} & 0 & F_n \\ F_{n+5} & -F_{n+4} & -F_{n+3} & F_{n+2} & F_{n+1} & -F_n & 0 \end{bmatrix}. \tag{41}$$

Theorem 26. *For every $\lambda,\mu\in\mathbb{R}$ and for every Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, the anti-symmetric matrix of the vector of $\lambda\vec{F}_n+\mu\vec{F}_m$ is equal to $\lambda S_{\vec{F}_n}+\mu S_{\vec{F}_m}$.*

Theorem 27. *For all Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, we have*

$$S_{\vec{F}_n}\vec{F}_m=(-1)^m F_{n-m}\begin{bmatrix}-1 & -1 & -2 & -3 & 9 & 6 & -6\end{bmatrix}^T. \tag{42}$$

Form eq. (40) and eq. (42), we can give this corollary:

Corollary 28. *Given any Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, the vector product of these two Fibonacci vectors is*

$$\vec{F}_n\times\vec{F}_m=S_{\vec{F}_n}\vec{F}_m. \tag{43}$$

Hence, for any two Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, vector product of these two vectors equals the matrix product between the anti-symmetric matrix which corresponding to the first vector given in eq. (41) with the second Fibonacci 7-vector.

Example 29. *For Fibonacci 7-vectors $\vec{F}_1$ and $\vec{F}_4$, let us find $\vec{F}_1\times\vec{F}_4$. With $\vec{F}_1=\begin{bmatrix}1 & 1 & 2 & 3 & 5 & 8 & 13\end{bmatrix}^T$ and $\vec{F}_4=\begin{bmatrix}3 & 5 & 8 & 13 & 21 & 34 & 55\end{bmatrix}^T$*

$$\begin{aligned}\vec{F}_1\times\vec{F}_4 &= (-1)^4F_{(1-4)}\begin{bmatrix}-1 & -1 & -2 & -3 & 9 & 6 & -6\end{bmatrix}^T \\ &= \begin{bmatrix}-2 & -2 & -4 & -6 & 18 & 12 & -12\end{bmatrix}^T.\end{aligned}$$

Also,

$$S_{\vec{F}_1}\vec{F}_4=\begin{bmatrix} 0 & -2 & 1 & -5 & 3 & 13 & -8 \\ 2 & 0 & -1 & -8 & -13 & 3 & 5 \\ -1 & 1 & 0 & -13 & 8 & -5 & 3 \\ 5 & 8 & 13 & 0 & -1 & -1 & -2 \\ -3 & 13 & -8 & 1 & 0 & 2 & -1 \\ -13 & -3 & 5 & 1 & -2 & 0 & 1 \\ 8 & -5 & -3 & 2 & 1 & -1 & 0 \end{bmatrix}\begin{bmatrix}3\\5\\8\\13\\21\\34\\55\end{bmatrix}=\begin{bmatrix}-2\\-2\\-4\\-6\\18\\12\\-12\end{bmatrix}.$$

Hence, we can see $\vec{F}_1\times\vec{F}_4=S_{\vec{F}_1}\vec{F}_4$.

5.2 Properties of Vector Product for Fibonacci 7-Vectors

For every Fibonacci 7-vectors $\vec{F}_n$, $\vec{F}_m$ and $\vec{F}_k$, following properties are provided:

1. $\vec{F}_n\times\left(\vec{F}_m\times\vec{F}_k\right)\neq\left\langle\vec{F}_n,\vec{F}_k\right\rangle\vec{F}_m-\left\langle\vec{F}_n,\vec{F}_m\right\rangle\vec{F}_k$,
2. $\vec{F}_n\times\vec{F}_n=S_{\vec{F}_n}\vec{F}_n=\vec{0}$,
3. $\vec{F}_n\times\vec{F}_m=-\vec{F}_m\times\vec{F}_n$,
4. $\left\langle\vec{F}_n,\vec{F}_n\times\vec{F}_m\right\rangle=\left\langle\vec{F}_m,\vec{F}_n\times\vec{F}_m\right\rangle=0$,
5. $\left\|\vec{F}_n\times\vec{F}_m\right\|^2=\left\|\vec{F}_n\right\|^2\left\|\vec{F}_m\right\|^2-\left(\left\langle\vec{F}_n,\vec{F}_m\right\rangle\right)^2$,

6. Since all Fibonacci 7-vectors $\vec{F_n}$ and $\vec{F_m}$ are linearly independent, it follows that $\vec{F_n} \times \vec{F_m} \neq \vec{0}$,
7. $\left\langle \vec{F_n} \times \vec{F_m}, \vec{F_k} \right\rangle = \left\langle \vec{F_m} \times \vec{F_k}, \vec{F_n} \right\rangle = \left\langle \vec{F_k} \times \vec{F_n}, \vec{F_m} \right\rangle = 0,$
8. $\vec{F_n} \times \left(\vec{F_n} \times \vec{F_m} \right) = \left\langle \vec{F_n}, \vec{F_m} \right\rangle \vec{F_n} - \left\langle \vec{F_n}, \vec{F_n} \right\rangle \vec{F_m},$
9. $\left\langle \vec{F_m}, \vec{F_n} \times \vec{F_k} \right\rangle = \left\langle \vec{F_n}, \vec{F_k} \times \vec{F_m} \right\rangle = 0,$
10. $\vec{F_n} \times \left(\vec{F_m} \times \vec{F_k} \right) + \vec{F_m} \times \left(\vec{F_n} \times \vec{F_k} \right) = \left\langle \vec{F_m}, \vec{F_k} \right\rangle \vec{F_n} + \left\langle \vec{F_n}, \vec{F_k} \right\rangle \vec{F_m} - 2\left\langle \vec{F_n}, \vec{F_m} \right\rangle \vec{F_k},$

Proof. Proofs can be shown by using anti-symmetric matrix which is given eq. (41). ■

5.3 Vector Product of Fibonacci 7-Vectors by using Binet's Formula

Theorem 30. *Let $\vec{a}$ and $\vec{b}$ be vectors given in Theorem 10. The vector product of $\vec{a}$ and $\vec{b}$ is*

$$\vec{a} \wedge \vec{b} = (\alpha - \beta) \begin{bmatrix} 1 & 1 & 2 & 3 & -9 & -6 & 6 \end{bmatrix}^T. \tag{44}$$

Proof. It is easy to check that by using property that is given by Salter in [2], $\alpha^{n_1}\beta^{n_2} - \alpha^{n_2}\beta^{n_1} = (-1)^{n_1+1}(\alpha - \beta) F_{n_2-n_1}$. ■

By the Corollary 5.1 and Theorem 30, we easily obtain the following corollary:

Corollary 31. *Let $\vec{F_n}$ and $\vec{F_m}$ be Fibonacci 7-vectors. Here is another way of stating vector product of these Fibonacci 7-vectors is*

$$\vec{F_n} \times \vec{F_m} = S_{\vec{F_n}} \vec{F_m} = (-1)^{m+1} F_{n-m} \frac{\vec{a} \times \vec{b}}{\alpha - \beta}. \tag{45}$$

6. Lorentzian Geometry of Fibonacci Vectors

6.1 Lorentzian Geometry of Fibonacci 3-Vectors

6.1.1 Lorentzian Inner Product for Fibonacci 3-Vectors

Theorem 32. *For any Fibonacci 3-vectors $\vec{F_n}$ and $\vec{F_m}$,* **the Lorentzian inner product**[2] *of these two vectors is*

$$\left\langle \vec{F_n}, \vec{F_m} \right\rangle_L = -F_n F_m + F_{n+1} F_{m+1} + F_{n+2} F_{m+2} = F_{n+2} F_{m+1} + F_{n+1} F_{m+2}.$$

Proof. Let $\vec{F_n}$ and $\vec{F_m}$ be Fibonacci 3-vectors. Then,

$$\begin{aligned} \left\langle \vec{F_n}, \vec{F_m} \right\rangle_L &= -F_n F_m + F_{n+1} F_{m+1} + F_{n+2} F_{m+2} \\ &= -F_n F_m + F_{n+1} F_{m+1} + (F_n + F_{n+1})(F_m + F_{m+1}) \\ &= -F_n F_m + F_{n+1} F_{m+1} + F_n F_m + F_n F_{m+1} + F_{n+1} F_m + F_{n+1} F_{m+1} \\ &= F_{n+1}(F_m + F_{m+1}) + F_{m+1}(F_n + F_{n+1}) \\ &= F_{n+1} F_{m+2} + F_{n+2} F_{m+1}. \end{aligned}$$

■

Also, for any Fibonacci 3-vector $\vec{F_n} = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$, ***the Lorentzian norm*** of $\vec{F_n}$ is

$$\left\| \vec{F_n} \right\|_L = \sqrt{\left| \left\langle \vec{F_n}, \vec{F_n} \right\rangle_L \right|} = \sqrt{|2F_{n+1}F_{n+2}|}.$$

Furthermore, we have $\left\langle \vec{F_n}, \vec{F_{n+1}} \right\rangle_L = {F_{n+2}}^2 + F_{n+1} F_{n+3}$.

[2](Ratcliffe, 2006) Let x and y be vectors in $\mathbb{R}^n$. The Lorentzian inner product of x and y is defined to be the real number,

$$\langle x, y \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3 + \ldots + x_n y_n.$$

6.1.2 Lorentzian Vector Product for Fibonacci 3-Vectors

Definition 33. *Let $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$ and $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$ be Fibonacci 3-vectors and let*

$$\mathbb{J}_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{46}$$

The Lorentzian vector product *of $\vec{F}_n$ and $\vec{F}_m$ is defined by,*

$$\vec{F}_n \wedge_L \vec{F}_m = \mathbb{J}_3 . \left(\vec{F}_n \wedge \vec{F}_m \right) \tag{47}$$

where, $\wedge$ is the Euclidean vector product which is given in eq. (34).

Theorem 34. *The Lorentzian vector product of $\vec{F}_n$ and $\vec{F}_m$ can be calculated with following determinant:*

$$\begin{aligned} \vec{F}_n \wedge_L \vec{F}_m &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \end{vmatrix} \\ &= (F_{n+2}F_{m+1} - F_{n+1}F_{m+2}, F_{n+2}F_m - F_nF_{m+2}, F_nF_{m+1} - F_{n+1}F_m), \end{aligned}$$

where $\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ $\quad \vec{e}_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}) \in \mathbb{R}^3, \vec{e}_1 \wedge \vec{e}_2 = \vec{e}_3, \vec{e}_2 \wedge \vec{e}_3 = -\vec{e}_1, \vec{e}_3 \wedge \vec{e}_1 = \vec{e}_2.$

Theorem 35. *For any Fibonacci 3-vectors $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$ and $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$, the Lorentzian vector product of $\vec{F}_n$ and $\vec{F}_m$ is*

$$\vec{F}_n \wedge_L \vec{F}_m = (-1)^m F_{n-m} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T. \tag{48}$$

Proof. Let $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$ and $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$ be Fibonacci 3-vectors. Then, the Lorentzian vector product of $\vec{F}_n$ and $\vec{F}_m$ is

$$\begin{aligned} \vec{F}_n \wedge_L \vec{F}_m &= \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \end{vmatrix} \\ &= (F_{n+2}F_{m+1} - F_{n+1}F_{m+2}, F_{n+2}F_m - F_nF_{m+2}, F_nF_{m+1} - F_{n+1}F_m), \\ &= (-(-1)^{m+1}F_{n-m}, -(-1)^m F_{n-m}, (-1)^m F_{n-m}) \\ &= (-1)^m F_{n-m} \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}^T. \end{aligned}$$

■

Observe that,

$$\left\langle \vec{F}_n, \vec{F}_n \wedge_L \vec{F}_m \right\rangle_L = \left\langle \vec{F}_n, \mathbb{J}_3 . (\vec{F}_n \wedge \vec{F}_m) \right\rangle_L = \left\langle \vec{F}_n, \vec{F}_n \wedge \vec{F}_m \right\rangle = 0, \tag{49}$$

$$\left\langle \vec{F}_m, \vec{F}_n \wedge_L \vec{F}_m \right\rangle_L = \left\langle \vec{F}_m, \mathbb{J}_3 . (\vec{F}_n \wedge \vec{F}_m) \right\rangle_L = \left\langle \vec{F}_m, \vec{F}_n \wedge \vec{F}_m \right\rangle = 0, \tag{50}$$

where, $\langle , \rangle$ is Euclidean inner product which is given in eq, (6) and $\wedge$ is the Euclidean vector product which is given in eq. (34). Therefore $\vec{F}_n \wedge_L \vec{F}_m$ is Lorentzian orthogonal to both $\vec{F}_n$ and $\vec{F}_m$.

Theorem 36. *Let $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$ and $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$ be Fibonacci 3-vectors. Then, the Euclidean vector product of Fibonacci 3-vectors can be written*

$$\vec{F}_n \wedge_L \vec{F}_m = \mathbb{J}_3 . \left(\vec{F}_m \right) \wedge \mathbb{J}_3 . \left(\vec{F}_n \right), \tag{51}$$

where, $\wedge$ is the Euclidean vector product which is given in eq. (34).

Theorem 37. *If $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$, $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$, $\vec{F}_k = \begin{bmatrix} F_k & F_{k+1} & F_{k+2} \end{bmatrix}^T$ and $\vec{F}_t = \begin{bmatrix} F_t & F_{t+1} & F_{t+2} \end{bmatrix}^T$ Fibonacci 3-vectors, in this case following properties are provided:*

1. $\vec{F}_n \wedge_L \vec{F}_m = -\vec{F}_m \wedge_L \vec{F}_n$,
2. $\left\langle \vec{F}_n \wedge_L \vec{F}_m, \vec{F}_k \right\rangle_L = \begin{vmatrix} F_n & F_{n+1} & F_{n+2} \\ F_m & F_{m+1} & F_{m+2} \\ F_k & F_{k+1} & F_{k+2} \end{vmatrix} = 0$,
3. $\vec{F}_n \wedge_L (\vec{F}_m \wedge_L \vec{F}_k = \left\langle \vec{F}_n, \vec{F}_m \right\rangle_L \vec{F}_k - \left\langle \vec{F}_k, \vec{F}_n \right\rangle_L \vec{F}_m$,
4. $\left\langle (\vec{F}_n \wedge_L \vec{F}_m), (\vec{F}_k \wedge_L \vec{F}_t) \right\rangle_L = \begin{vmatrix} \left\langle \vec{F}_n, \vec{F}_t \right\rangle_L & \left\langle \vec{F}_n, \vec{F}_k \right\rangle_L \\ \left\langle \vec{F}_m, \vec{F}_t \right\rangle_L & \left\langle \vec{F}_m, \vec{F}_k \right\rangle_L \end{vmatrix}$.

Moreover, for all Fibonacci 3-vectors $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} \end{bmatrix}^T$, $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} \end{bmatrix}^T$ and $\vec{F}_k = \begin{bmatrix} F_k & F_{k+1} & F_{k+2} \end{bmatrix}^T$, ***the Lorentzian triple scalar product*** of these vectors is zero i.e.

$$\left\langle \vec{F}_n \wedge_L \vec{F}_m, \vec{F}_k \right\rangle_L = 0.$$

Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 3-vectors. Then, we can rewrite the Lorentzian vector product between $\vec{F}_n$ and $\vec{F}_m$ by using anti-symmetric matrix which is given in eq. (31) and matrix $\mathbb{J}_3$ which is given in eq. (46).

Theorem 38. *Let $\vec{a}$ and $\vec{b}$ be vectors given in Theorem 2, the Lorentzian vector product of these vectors is*

$$\vec{a} \wedge_L \vec{b} = (\alpha - \beta) \begin{bmatrix} -1 & 1 & -1 \end{bmatrix}^T. \tag{52}$$

Corollary 39. *(Lorentzian Vector Product by Using Anti-Symmetric Matrix) For all Fibonacci 3-vectors $\vec{F}_n$ and $\vec{F}_m$, the Lorentzian vector product of $\vec{F}_n$ and $\vec{F}_m$ is can be defined by*

$$\vec{F}_n \wedge_L \vec{F}_m = \mathbb{J}_3 . \left(\mathbb{F}_n \vec{F}_m \right). \tag{53}$$

Corollary 40. *(Lorentzian Vector Product by Using Binet's Formula)*

- *Considering eq. (15), the Lorentzian vector product of the two Fibonacci 3-vectors can be written as:*

$$\vec{F}_n \wedge_L \vec{F}_m = (-1)^{m+1} F_{n-m} \frac{\mathbb{J}_3 . \left(\vec{a} \wedge \vec{b} \right)}{\alpha - \beta}.$$

- *Considering eq. (52), the Lorentzian vector product of the two Fibonacci 3-vectors also can be written as:*

$$\vec{F}_n \wedge_L \vec{F}_m = (-1)^{m+1} F_{n-m} \frac{\vec{a} \wedge_L \vec{b}}{\alpha - \beta}.$$

Example 41. *Let $\vec{F}_5$ and $\vec{F}_9$ be Fibonacci 3-vectors. Then, let us find $\vec{F}_5 \wedge_L \vec{F}_9$.*

$$\vec{F}_5 \wedge_L \vec{F}_9 = \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ F_5 & F_6 & F_7 \\ F_9 & F_{10} & F_{11} \end{vmatrix} = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}.$$

Also,

$$\begin{aligned} \mathbb{J}_3 . \left(\mathbb{F}_n \vec{F}_m \right) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} 0 & -13 & 8 \\ 13 & 0 & -5 \\ -8 & 5 & 0 \end{bmatrix} \begin{bmatrix} 34 \\ 55 \\ 89 \end{bmatrix} \right) \\ &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left(\begin{bmatrix} -3 \\ 3 \\ -3 \end{bmatrix} \right) = \begin{bmatrix} 3 \\ -3 \\ 3 \end{bmatrix}. \end{aligned}$$

So, we obtain $\vec{F}_5 \wedge_L \vec{F}_9 = \mathbb{J}_3 . \left(\mathbb{F}_5 \vec{F}_9 \right)$.

Note that, similar to Euclidean vector product properties of Fibonacci 3-vectors, Lorentzian vector product properties of Fibonacci 3-vectors can be simply examined.

6.2 Lorentzian Geometry of Fibonacci 4-Vectors

6.2.1 Lorentzian Inner Product for Fibonacci 4-Vectors

Theorem 42. *For any Fibonacci* 4*-vectors* $\vec{F}_n$ *and* $\vec{F}_m$*, the Lorentzian inner product of these two vectors is*

$$\left\langle \vec{F}_n,\ \vec{F}_m \right\rangle_L = -F_nF_{m+2} + F_{n+4}F_{m+3}.$$

Proof. Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 4-vectors. Then,

$$\begin{aligned}
\left\langle \vec{F}_n, \vec{F}_m \right\rangle_L &= -F_nF_m + F_{n+1}F_{m+1} + (F_n + F_{n+1})(F_m + F_{m+1}) + (F_{n+4} - F_{n+2})F_{m+3} \\
&= F_{n+1}(F_m + F_{m+1}) + F_{m+1}(F_n + F_{n+1}) + F_{n+4}F_{m+3} - F_{n+2}F_{m+3} \\
&= F_{n+2}(F_{m+1} - F_{m+3}) + F_{n+1}F_{m+2} + F_{n+4}F_{m+3} \\
&= -F_{n+2}F_{m+2} + F_{n+1}F_{m+2} + F_{n+4}F_{m+3} \\
&= F_{m+2}(F_{n+1} - F_{n+2}) + F_{n+4}F_{m+3} \\
&= -F_nF_{m+2} + F_{n+4}F_{m+3}.
\end{aligned}$$

■

Also, ***the Lorentzian norm*** of $\vec{F}_n$ is

$$\left\|\vec{F}_n\right\|_L = \sqrt{\left|\left\langle \vec{F}_n, \vec{F}_n \right\rangle_L\right|} = \sqrt{\left|-F_nF_{n+2} + F_{n+4}F_{n+3}\right|}.$$

Furthermore, we have $\left\langle \vec{F}_n,\ \vec{F}_{n+1} \right\rangle_L = -F_nF_{n+3} + {F_{n+4}}^2.$

6.2.2 Lorentzian Vector Product for Fibonacci 4-Vectors

Definition 43. *For any Fibonacci* 4*-vectors* $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{bmatrix}^T$, $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} & F_{m+3} \end{bmatrix}^T$ *and* $\vec{F}_k = \begin{bmatrix} F_k & F_{k+1} & F_{k+2} & F_{k+3} \end{bmatrix}^T$, **the Lorentzian vector product** *of* $\vec{F}_n$, $\vec{F}_m$ *and* $\vec{F}_k$ *is defined by,*

$$\vec{F}_n \otimes_L \vec{F}_m \otimes_L \vec{F}_k \quad = \quad - \begin{vmatrix} -\vec{e}_1 & \vec{e}_2 & \vec{e}_3 & \vec{e}_4 \\ F_n & F_{n+1} & F_{n+2} & F_{n+3} \\ F_m & F_{m+1} & F_{m+2} & F_{m+3} \\ F_k & F_{k+1} & F_{k+2} & F_{k+3} \end{vmatrix},$$

where $\delta_{ij} = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases}$ $\quad e_i = (\delta_{i1}, \delta_{i2}, \delta_{i3}, \delta_{i4}) \in \mathbb{R}^4,\ \vec{e}_1 \otimes_L \vec{e}_2 \otimes_L \vec{e}_3 = \vec{e}_4,$
$\vec{e}_2 \otimes_L \vec{e}_3 \otimes_L \vec{e}_4 = \vec{e}_1, \vec{e}_3 \otimes_L \vec{e}_4 \otimes_L \vec{e}_1 = \vec{e}_2,\ \vec{e}_4 \otimes_L \vec{e}_1 \otimes_L \vec{e}_2 = -\vec{e}_3.$

Theorem 44. *For all Fibonacci* 4*-vectors* $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{bmatrix}^T$, $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} & F_{m+3} \end{bmatrix}^T$ *and* $\vec{F}_k = \begin{bmatrix} F_k & F_{k+1} & F_{k+2} & F_{k+3} \end{bmatrix}^T$, **the Lorentzian vector product** *of these vectors is zero vector i.e.*

$$\vec{F}_n \otimes_L \vec{F}_m \otimes_L \vec{F}_k = \vec{0}.$$

Proof. Proof of above theorem is elementary. Using usual determinant function properties, it's clear to see that. ■

Corollary 45. *For all Fibonacci* 4*-vectors* $\vec{F}_n = \begin{bmatrix} F_n & F_{n+1} & F_{n+2} & F_{n+3} \end{bmatrix}^T$, $\vec{F}_m = \begin{bmatrix} F_m & F_{m+1} & F_{m+2} & F_{m+3} \end{bmatrix}^T$, $\vec{F}_k = \begin{bmatrix} F_k & F_{k+1} & F_{k+2} & F_{k+3} \end{bmatrix}^T$ *and* $\vec{F}_t = \begin{bmatrix} F_t & F_{t+1} & F_{t+2} & F_{t+3} \end{bmatrix}^T$, **the Lorentzian triple scalar product** *of* $\vec{F}_n$, $\vec{F}_m$, $\vec{F}_k$ *and* $\vec{F}_t$ *is zero i.e.*

$$\left\langle \vec{F}_n \otimes_L \vec{F}_m \otimes_L \vec{F}_k, \vec{F}_t \right\rangle_L = 0.$$

6.3 Lorentzian Geometry of Fibonacci 7-Vectors

6.3.1 Lorentzian Inner Product for Fibonacci 7-Vectors

Theorem 46. *For any Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, the Lorentzian inner product of these two vectors is*

$$\left\langle \vec{F}_n, \vec{F}_m \right\rangle_L = -F_{m+2}F_n + F_{n+6}F_{m+7}.$$

Proof. Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 7-vectors. Then,

$$\begin{aligned} \left\langle \vec{F}_n, \vec{F}_m \right\rangle_L &= -F_nF_m + F_{n+1}F_{m+1} + (F_n + F_{n+1})(F_m + F_{m+1}) \\ &\quad + (F_{n+4} - F_{n+2})F_{m+3} + F_{n+4}F_{m+4} + F_{n+5}F_{m+5} + F_{n+6}F_{m+6} \\ &= F_{n+1}(F_{m+1} + F_m) + F_{m+1}(F_{n+1} + F_n) - F_{n+2}F_{m+3} + F_{n+4}(F_{m+3} + F_{m+4}) + F_{n+5}F_{m+5} + F_{n+6}F_{m+6} \\ &= F_{n+1}F_{m+2} + F_{n+2}(F_{m+1} - F_{m+3}) + F_{n+6}(F_{m+5} + F_{m+6}) \\ &= F_{n+1}F_{m+2} - F_{n+2}F_{m+2} + F_{n+6}F_{m+7} \\ &= -F_{m+2}F_n + F_{n+6}F_{m+7}. \end{aligned}$$

■

Also, ***the Lorentzian norm*** of $\vec{F}_n$ is

$$\left\| \vec{F}_n \right\|_L = \sqrt{\left| \left\langle \vec{F}_n, \vec{F}_n \right\rangle_L \right|} = \sqrt{|-F_{n+2}F_n + F_{n+6}F_{n+7}|}.$$

Furthermore, we have $\left\langle \vec{F}_n,\ \vec{F}_{n+1} \right\rangle_L = -F_{n+3}F_n + F_{n+6}F_{n+8}$.

6.3.2 Lorentzian Vector Product by Using Anti-Symmetric Matrix

Definition 47. *Let $\vec{F}_n$ and $\vec{F}_m$ be Fibonacci 7-vectors and let*

$$\mathbb{J}_7 = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{54}$$

The Lorentzian vector product *of $\vec{F}_n$ and $\vec{F}_m$ is defined by as follows:*

$$\vec{F}_n \times_L \vec{F}_m = \mathbb{J}_7.\left(S_{\vec{F}_n}\vec{F}_m\right), \tag{55}$$

where $S_{\vec{F}_n}$ is anti-symmetric matrix which given eq. (41).

Theorem 48. *For all Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, the Lorentzian vector product of $\vec{F}_n$ and $\vec{F}_m$ is*

$$\vec{F}_n \times_L \vec{F}_m = \mathbb{J}_7.\left(S_{\vec{F}_n}\vec{F}_m\right) = (-1)^m F_{n-m} \begin{bmatrix} 1 & -1 & -2 & -3 & 9 & 6 & -6 \end{bmatrix}^T. \tag{56}$$

Similar to eq. (49) and eq. (50), for Fibonacci 7-vectors $\vec{F}_n$ and $\vec{F}_m$, we can also observed that

$$\left\langle \vec{F}_n, \vec{F}_n \times_L \vec{F}_m \right\rangle_L = \left\langle \vec{F}_n, \mathbb{J}_7.(\vec{F}_n \times \vec{F}_m) \right\rangle_L = \left\langle \vec{F}_n, \vec{F}_n \times \vec{F}_m \right\rangle = 0, \tag{57}$$

$$\left\langle \vec{F}_m, \vec{F}_n \times_L \vec{F}_m \right\rangle_L = \left\langle \vec{F}_m, \mathbb{J}_7.(\vec{F}_n \times \vec{F}_m) \right\rangle_L = \left\langle \vec{F}_m, \vec{F}_n \times \vec{F}_m \right\rangle = 0. \tag{58}$$

Hence, $\vec{F}_n \wedge_L \vec{F}_m$ is Lorentzian orthogonal to both $\vec{F}_n$ and $\vec{F}_m$.

Theorem 49. *Let $\vec{F}_n$, $\vec{F}_m$ and $\vec{F}_k$ be Fibonacci 7-vectors. Then, following properties are provided:*

1. $\vec{F}_n \wedge_L \vec{F}_m = -\vec{F}_m \wedge_L \vec{F}_n$,
2. $\vec{F}_n \wedge_L \vec{F}_m \neq \mathbb{J}_7.(\vec{F}_m) \wedge \mathbb{J}_7.(\vec{F}_n)$,

3. $\left\langle \vec{F}_n \wedge_L \vec{F}_m, \vec{F}_k \right\rangle_L = 0.$

Corollary 50. *(Lorentzian Vector Product by Using Binet's Formula) Eq. (44) is taken into account, the Lorentzian vector product of two Fibonacci* 7*-vectors can be considered as follows:*

$$\vec{F}_n \wedge_L \vec{F}_m = (-1)^{m+1} F_{n-m} \frac{\mathbb{J}_7 \left(\vec{a} \wedge \vec{b}\right)}{\alpha - \beta}.$$

Note that, similar to the the Euclidean vector product properties for Fibonacci 7-vectors, Lorentzian vector product properties can be easily examined.

Example 51. *For Fibonacci* 7*-vectors* $\vec{F}_1$ *and* $\vec{F}_4$ *, let us find* $\vec{F}_1 \times_L \vec{F}_4$.

$$\vec{F}_1 \times_L \vec{F}_4 \quad = \quad \mathbb{J}_7.\left(S_{\vec{F}_1} \vec{F}_4\right) = \mathbb{J}_7.\left(\begin{bmatrix} 0 & -2 & 1 & -5 & 3 & 13 & -8 \\ 2 & 0 & -1 & -8 & -13 & 3 & 5 \\ -1 & 1 & 0 & -13 & 8 & -5 & 3 \\ 5 & 8 & 13 & 0 & -1 & -1 & -2 \\ -3 & 13 & -8 & 1 & 0 & 2 & -1 \\ -13 & -3 & 5 & 1 & -2 & 0 & 1 \\ 8 & -5 & -3 & 2 & 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 8 \\ 13 \\ 21 \\ 34 \\ 55 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -2 \\ -4 \\ -6 \\ 18 \\ 12 \\ -12 \end{bmatrix}.$$

7. Conclusions

In this study, the corresponding anti-symmetric matrix for Fibonacci 3-vectors were described and the vector product by using this matrix was reconsidered. After that, the properties of vector product by using anti-symmetric matrix were given. Also, vector product for Fibonacci 3-vectors by using Binet's Formula was given. Furthermore, the vector product for Fibonacci 4-vectors was defined. The vector product for Fibonacci 7-vectors was defined and similar to Fibonacci 3-vectors, the vector product was rewritten using by the anti-symmetric matrix. Moreover, properties of vector product by using anti-symmetric matrix for Fibonacci 7-vectors were given. In addition to these vector product for Fibonacci 7-vectors by using Binet's Formula were given. Finally, Lorentzian inner product, Lorentzian vector product and Lorentzian triple scalar product for Fibonacci 3-vectors, Fibonacci 4-vectors and Fibonacci 7-vectors were given.

8. Acknowledgment

We sincerely thank anonymous reviewers whose comments and suggestions helped improve and clarify this manuscript.

References

[1] Atanassov, K. T. (2002). *New visual perspectives on Fibonacci numbers*. World Scientific.

[2] Salter, E. (2005). *Fibonacci Vectors*. Graduate Theses and Dissertations, University of South Florida, USA.

[3] Güven, İ. A., & Nurkan, S. K. (2015). A new approach to Fibonacci, Lucas numbers and dual vectors. *Advances in Applied Clifford Algebras, 25*(3), 577-590, https://doi.org/10.1007/s00006-014-0516-7.

[4] Yüce, S., & Torunbalcı Aydın, F. (2016). Generalized dual Fibonacci sequence. *The International Journal of Science & Technoledge, 4*(9), 193-200.

[5] Kaya, O., & Önder, M. (2018). On Fibonacci and Lucas Vectors and Quaternions. *Universal Journal of Applied Mathematics, 6*(5), 156-163.

[6] Knuth, D. (2008). *NegaFibonacci numbers and the hyperbolic plane*. In San Jose-Meeting of the Mathematical Association of America, (Vol. 5).

[7] Struyk, A. (1970). One Curiosum Leads to Another. *Scripta Mathematica. 17*, 230.

[8] Vajda, S. (1989). *Fibonacci and Lucas Numbers, and the Golden Section: Theory and Applications*. Ellis Horwood Series. Mathematics and Applications.

[9] Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. John Wiley and Sons, Proc., Toronto, New York.

[10] Weisstein, E.W., *Fibonacci Number*. MathWorld, (online mathematics reference work).

[11] Ratcliffe, J. G., Axler, S., & Ribet, K. A. (2006). *Foundations of hyperbolic manifolds*. (Vol. 149), New York: Springer.