Available online: April 26, 2020

Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat.
Volume 69, Number 1, Pages 847–853 (2020)
DOI: 10.31801/cfsuasmas.673191
ISSN 1303–5991 E-ISSN 2618-6470

http://communications.science.ankara.edu.tr

FOURIER-BESSEL TRANSFORMS OF DINI-LIPSCHITZ FUNCTIONS ON LEBESGUE SPACES $L_{p,\gamma}(\mathbb{R}^n_+)$

ISMAIL EKINCIOGLU, ESRA KAYA, AND S. ELIFNUR EKINCIOGLU

Abstract. In this paper, we prove a generalization of Titchmarsh's theorem for the Laplace-Bessel differential operator in the space $L_{p,\gamma}(\mathbb{R}^n_+)$ for functions satisfying the (ψ, p)-Laplace-Bessel Lipschitz condition for $1 < p \leq 2$ and $\gamma > 0$.

1. Introduction

Integral transforms and their inverse transforms are widely used to solve various problems in calculus, fourier analysis, mechanics, mathematical physics, and computational mathematics. Fourier transform is one of the most important integral transforms. Since it was introducted by Fourier in the early 1880s, it has become an important mathematical concept that is at the centre of the highly developed branch of mathematics called Fourier Analysis. It has many application areas. The Fourier transform of the kernel of singular integral operator is very important in applications of singular integral operator theory. The properties of the Fourier transform of the kernel give information about the existence of the solution of singular integral equations. Since singular integrals are convolution type operators, their Fourier transforms are the product of the Fourier transforms of two functions.

As it is well known that if Lipschitz conditions are applied on a function $f(x)$, then these conditions greatly affect the absolute convergence of the Fourier-Bessel series and behaviour of $F_\gamma f$ Fourier-Bessel transforms of f. In general, if $f(x)$ belongs to a certain function class, then the Lipschitz conditions have bearing as to the dual space to which the Fourier coefficients and Fourier-Bessel transforms of $f(x)$ belong. Younis (see [12]) worked the same phenomena for the wider Dini Lipschitz class for some classes of functions. Daher, El Quadih, Daher and El Hamma proved an analog Younis (see [12, Theorem 2.5]) in for the Fourier-Bessel transform for functions satisfies the Fourier-Bessel Dini Lipschitz condition in the

Received by the editors: January 10, 2020; Accepted: March 06, 2020.

2010 *Mathematics Subject Classification.* Primary 42B10; Secondary 26A16.

Key words and phrases. Laplace-Bessel differential operator, generalized shift operator, Laplace-Bessel Lipschitz function.

Lebesgue space $L^2_{\alpha,n}$ (see [10]). El Hamma and Daher proved a generalization of Titchmarsh's theorem for the Bessel transform in the space $L_{2,\gamma}(\mathbb{R}^n_+)$ (see [1]) .

In this paper we prove a generalization of Titchmarsh's theorem for the Laplace-Bessel transform in the space $L_{p,\gamma}(\mathbb{R}^n_+)$, where $1 < p \leq 2$ and $\gamma > 0$.

2. Preliminaries

Let $\mathbb{R}^n_+$ be the part of the Euclidean space $\mathbb{R}^n$ of points $x = (x_1, ..., x_n)$, defined by the inequality $x_n > 0$. We write $x = (x', x_n)$, $x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}_+$. S^n_+ denote the unit sphere on $\mathbb{R}^n_+$, which can be defined as $S^n_+ = \{x \in \mathbb{R}^n_+ : \ |x| = 1\}$. $\mathbb{S}_+ = \mathbb{S}(\mathbb{R}^n_+)$ be the space of functions which are the restrictions to $\mathbb{R}^n_+$ of the test functions of the Schwartz that are even with respect to x_n, decreasing sufficiently rapidly at infinity, together with all derivatives of the form

$$D^\alpha_\gamma = D^{\alpha'}_{x'} B^{\alpha_n}_n = D^{\alpha_1}_1 ... D^{\alpha_{n-1}}_{n-1} B^{\alpha_n}_n = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots\cdots \frac{\partial^{\alpha_{n-1}}}{\partial x_{n-1}^{\alpha_{n-1}}} B^{\alpha_n}_n,$$

i.e., for all $\varphi \in \mathbb{S}_+$, $\sup\limits_{x \in \mathbb{R}^n_+} \left| x^\beta D^\alpha_\gamma \varphi \right| < \infty$, where $\alpha = (\alpha_1, ..., \alpha_n)$ and $\beta = (\beta_1, ..., \beta_n)$ are multi-indexes, and $x^\beta = x_1^{\beta_1} \ldots x_n^{\beta_n}$ and $B_n = \dfrac{\partial^2}{\partial x_n^2} + \dfrac{\gamma}{x_n} \dfrac{\partial}{\partial x_n}$ is the Bessel differential expansion. For $\gamma \geq 0$, we introduce the Bessel normalized function of the first kind j_γ defined by

$$j_\gamma(z) = \Gamma(\gamma + 1) \sum_{n=0}^{\infty} \frac{(-1)^n}{n! \Gamma(n + \gamma + 1)} \left(\frac{z}{2}\right)^{2n}, \tag{1}$$

where Γ is the gamma-function (see [9]). Moreover, from (1) we see that

$$\lim_{z \to 0} \frac{j_{\frac{\gamma-1}{2}}(z) - 1}{z^2} \neq 0$$

by consequence, there exist $C > 0$ and $\eta > 0$ satisfying

$$|z| \leq \eta \Rightarrow \left| j_{\frac{\gamma-1}{2}}(z) - 1 \right| \geq C\,|z|^2. \tag{2}$$

The function $u = j_{\frac{\gamma-1}{2}}(z)$ satisfies the differential equation

$$B_{x_n} u(x, y) = B_{y_n} u(x, y)$$

with the initial conditions $u(x, 0) = f(x)$ and $u_y(x, 0) = 0$ is function infinitely differentiable, even, and, moreover entire analytic.

The Fourier-Bessel transformation and its inverse on $\mathbb{S}_+$ are defined by

$$F_\gamma f(x) = \int_{\mathbb{R}^n} f(y)\, e^{-i(x'y')} j_{\frac{\gamma-1}{2}}(x_n y_n)\; y_n^\gamma dy,$$
$$F_\gamma^{-1} f(x) = C_{n,\gamma} F_\gamma f(-x', x_n),$$

where $(x', y') = x_1y_1 + \ldots + x_{n-1}y_{n-1}$, j_γ, $\gamma > 0$, is the normalized Bessel function, and

$$C_{n,\gamma} = (2\pi)^{n-1}2^{\gamma-1}\Gamma^2((\gamma+1)/2),$$

(see [4, 9, 11]). This transform is associated to the Laplace-Bessel differential operator

$$\Delta_\gamma = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma}{x_n}\frac{\partial}{\partial x_n},\ \gamma > 0. \tag{3}$$

The expression (3) is a hybrid of the Hankel transform.

For a fixed parameter $\gamma > 0$, let $L_{p,\gamma} = L_{p,\gamma}(\mathbb{R}_+^n)$ be the space of measurable functions with a finite norm

$$\|f\|_{L_{p,\gamma}} = \left(\int_{\mathbb{R}_+^n} |f(x)|^p x_n^\gamma dx\right)^{1/p}, \quad 1 \le p < \infty.$$

The space of the essentially bounded measurable function on $\mathbb{R}_+^n$ is denoted by $L_{\infty,\gamma}(\mathbb{R}_+^n)$. For for $f \in L_{p,\gamma}$, I.A. Kipriyanov (for $n = 1$ B.M. Levitan [7, 8]) investigated the generalized convolution (Δ_γ-convolution)

$$(f \otimes g)(x) = \int_{\mathbb{R}_+^n} f(y)\, T^y g(x)\, y_n^\gamma dy,$$

associated with the Laplace-Bessel differential operator, where T^y is the generalized shift operator (Δ_γ-shift) defined by

$$T^y f(x) = C_\gamma \int_0^\pi f\left(x' - y', \sqrt{x_n^2 - 2x_ny_n\cos\theta + y_n^2}\right)\sin^{\gamma-1}\theta d\theta,$$

being $C_\gamma = \pi^{-\frac{1}{2}}\Gamma\left(\frac{\gamma+1}{2}\right)[\Gamma\left(\frac{\gamma}{2}\right)]^{-1}$ (see [5, 6, 7, 8]). We note that this convolution satisfies the property $(f \otimes g)(x) = (g \otimes f)(x)$ (see [2, 3]). The following relation connect the generalized shift operator and the Fourier-Bessel transform, we have

$$F_\gamma[T^y f(x)] = j_{\frac{\gamma-1}{2}}(x_ny_n)F_\gamma[f(x)]. \tag{4}$$

Given $1 < p \le 2$, $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $f \in L_{p,\gamma}$, we have the Hausdorff-Young inequality

$$\|F_\gamma f\|_{q,\gamma} \le C_q\, \|f\|_{p,\gamma}, \tag{5}$$

where and C_q is a positive constant.

3. Fourier-Bessel Transforms of Dini-Lipschitz Functions

In this section we give the main result of this paper. We need first to define (ψ, p)-Laplace Bessel Lipschitz class.

Definition 1. *A function $f \in L_{p,\gamma}(\mathbb{R}_+^n)$ is said to be in the (ψ, p)-Laplace Bessel Lipschitz class, denoted by Lip(ψ, γ, p), if*

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)) \quad as \quad y \to 0,$$

where $\psi(x)$ is a continuous increasing function on $\mathbb{R}^n_+$, $\psi(0) = 0$, and $\psi(xs) = \psi(x)\psi(s)$ for all $x, s \in \mathbb{R}^n_+$.

Theorem 2. *Let $f(x)$ belong to $Lip(\psi, \gamma, p)$. Then*

$$\int_{|\xi|\geq\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi = O(\psi(\tau^{-q})), \quad as \quad \tau \to +\infty.$$

Proof. Let $f \in Lip(\psi, \gamma, p)$. Then we have

$$\|T^y f(x) - f(x)\|_{p,\gamma} = O(\psi(y)) \quad \text{as} \quad y \to 0.$$

Now we consider Fourier-Bessel transform of generalized shift operator. We get

$$\begin{aligned}
F_\gamma[T^y f(x)](\xi) &= \int_{\mathbb{R}^n_+} T^y f(x) j_{\frac{\gamma-1}{2}}(x_n\xi_n) x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} T^y \left[j_{\frac{\gamma-1}{2}}(x_n\xi_n)\right] f(x) x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} j_{\frac{\gamma-1}{2}}(x_n\xi_n) j_{\frac{\gamma-1}{2}}(y_n\xi_n) f(x) x_n^\gamma dx \\
&= j_{\frac{\gamma-1}{2}}(y_n\xi_n) \int_{\mathbb{R}_n^+} f(x) j_{\frac{\gamma-1}{2}}(x_n\xi_n) x_n^\gamma dx \\
&= j_{\frac{\gamma-1}{2}}(y_n\xi_n) F_\gamma(f)(\xi),
\end{aligned}$$

where $T^y(j_p(\sqrt{\lambda}x)) = j_p(\sqrt{\lambda}y)j_p(\sqrt{\lambda}x)$. From formulas (4) and (5), we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n_+} F_\gamma |T^y f(x) - f(x)|^q x_n^\gamma dx &= \int_{\mathbb{R}^n_+} |F_\gamma T^y f(x) - F_\gamma f(x)|^q x_n^\gamma dx \\
&= \int_{\mathbb{R}^n_+} |j_{\frac{\gamma-1}{2}}(\xi y) F_\gamma f(\xi) - F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&= \int_{\mathbb{R}^n_+} |F_\gamma f(\xi)[1 - j_{\frac{\gamma-1}{2}}(\xi y)]|^q \xi_n^\gamma d\xi \\
&= \int_{\mathbb{R}^n_+} |1 - j_{\frac{\gamma-1}{2}}(\xi y)|^q |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&\leq C_q \int_{\mathbb{R}^n_+} |T^y f(x) - f(x)|^q \xi_n^\gamma d\xi \\
&\leq C_q \|T^y f(x) - f(x)\|^q_{p,\gamma}.
\end{aligned}$$

From (2), we have

$$\begin{aligned}
\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi &= C_q \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |1 - j_{\frac{\gamma-1}{2}}(\xi h)|^q |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\
&\geq C_q |h|^{-1} \int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi,
\end{aligned}$$

$0 < h \leq 1$. It follows from the above consideration that there exists a positive constant C such that

$$\int_{\frac{1}{h}\leq|\xi|\leq\frac{2}{h}} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C\psi^q(h) = C\psi(h^q).$$

Therefore, we get

$$\int_{\tau\leq|\xi|\leq 2\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C\psi(\tau^{-q}).$$

In fact, we have

$$\begin{aligned}\int_{\tau\leq|\xi|<\infty} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi &= \sum_{k=1}^{\infty}\int_{2^{k-1}\tau\leq|\xi|<2^k\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \\ &\leq C_q\psi(\tau^{-q}) + C_q\psi\left((2\tau)^{-q}\right) + C_q\psi\left((2^2\tau)^{-q}\right) + \ldots \\ &\leq C_q\psi(\tau^{-q})\left(1+\psi(2^{-q})+\psi^2(2^{-q})+\psi^3(2^{-q})+\ldots\right).\end{aligned}$$

Thus, we can write

$$\int_{\tau\leq|\xi|<\infty} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi \leq C_1\psi(\tau^{-q}),$$

where $C_1 = C_q\left(1-\psi(2^{-q})\right)^{-1}$ since $2^{-q} < 1$. Finally, we get

$$\int_{|\xi|\geq\tau} |F_\gamma f(\xi)|^q \xi_n^\gamma d\xi = O(\psi(\tau^{-q})) \quad \text{as} \quad \tau\to\infty.$$

Thus, the proof of theorem is completed. □

We can give the following result which is used for many the theorem given above. It is well known that

$$F_\gamma\left(B_n^{\alpha_n} f\right)(x) = (-x_n^2)^{\alpha_n} F_\gamma f(x), \tag{6}$$

$$F_\gamma\left(D_i^{2\alpha_i} f\right)(x) = (-x_i^2)^{\alpha_i} F_\gamma f(x), \quad i = 1,\ldots,n-1, \tag{7}$$

$$F_\gamma\left(\Delta_\gamma f\right)(x) = -|x|^2 F_\gamma f(x) \quad \text{and} \quad F_\gamma(f\otimes g) = F_\gamma f\, F_\gamma g, \tag{8}$$

$$F_\gamma\left(D_{x'}^{2\alpha'} B_n^{\alpha_n} f\right)(x) = (-1)^{|\alpha|} x^{2\alpha} F_\gamma f(x) \tag{9}$$

We can use the mathematical induction method for $k = 1$, we get

$$\begin{aligned}F_\gamma\big(\Delta_\gamma f\big)(x) &= C_{n,\gamma}\int_{\mathbb{R}_+^n} \Delta_\gamma f(y) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy \\ &= C_{n,\gamma}\int_{\mathbb{R}_+^n}\left(\sum_{k=1}^{n}\frac{\partial^2 f(y)}{\partial y_k^2} + \frac{\gamma}{y_n}\frac{\partial f(y)}{\partial y_n}\right) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^\gamma dy\end{aligned}$$

$$= C_{n,\gamma} \int_{\mathbb{R}^n_+} \left(\sum_{k=1}^{n} \frac{\partial^2 f(y)}{\partial y_k^2} e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^{\gamma} dy \right.$$

$$\left. - C_{n,\gamma} \int_{\mathbb{R}^n_+} \left(\sum_{k=1}^{n} \frac{\gamma}{y_n} \frac{\partial f(y)}{\partial y_n} \right) e^{-ix'y'} j_{\frac{\gamma-1}{2}}(x_n y_n) y_n^{\gamma} dy = I_1 + I_2. \right.$$

If we apply partial integration to the second term of I_1 and I_2, then we have

$$F_\gamma\big(\Delta_\gamma u\big)(x) = C_{n,\gamma} \int_{\mathbb{R}^n_+} f(y) e^{-ix'y'} \big(\Delta_\gamma j_{\frac{\gamma-1}{2}}(x_n y_n)\big) y_n^{\gamma} dy.$$

Here, if we use the following equality [8],

$$\int_0^\infty f(y) \Delta_\gamma j_{\frac{\gamma-1}{2}}(xy) y^\gamma dy = -|x|^2 \int_0^\infty f(y) j_{\frac{\gamma-1}{2}}(xy) y^\gamma dy$$

then we have

$$F_\gamma\left(\Delta_\gamma f\right)(x) = -|x|^2 F_\gamma f(x).$$

Since $f \in Lip(\psi, \gamma, p)$, it is clear that

$$||F_\gamma\left(\Delta_\gamma f\right)||_{L_{q,\gamma}(|\xi| \geq \tau)} \leq C_{n,\gamma} O(\psi(\tau^{-q}))$$

as $\tau \to +\infty$.

There are many examples. Here is one of them and a simple method to produce many more: $f(x) = |x|^{\frac{1}{p}}$ for $1 < p < \infty$, where $f(0) = 0$ is understood. These functions are uniformly continuous on all of $\mathbb{R}^n_+$. If $p = 2$, f belongs to the Lipschitz class at $\mathbb{R}_+$.

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Current address: Ismail Ekincioglu: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.

E-mail address: ismail.ekincioglu@dpu.edu.tr

ORCID Address: https://orcid.org/0000-0002-5636-1214

Current address: Esra Kaya: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.

E-mail address: kayaesra.e.k@gmail.com

ORCID Address: https://orcid.org/0000-0002-6971-0452

Current address: S. Elifnur Ekincioglu: Department of Mathematics, Dumlupinar University, Kutahya, Turkey.

E-mail address: ekinciogluelifnur@gmail.com

ORCID Address: https://orcid.org/0000-0001-6700-8232