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Equality of internal angles and vertex points in conformal hyperbolic triangles

Ümit TOKEŞER[1,*] ***Ömer ALSAN***[2]

[1] *Department of Mathematics, Faculty of Science and Arts, Kastamonu University, 37100, Kastamonu, Turkey*

[2] *Zeytinburnu Anatolian High School, 34100, Istanbul, Turkey*

Abstract

In this article, by using the conformal structure in Euclidean space, the conformal structures in hyperbolic space and the equality of the internal angles and vertex points of conformal triangles in hyperbolic space are given. Especially in these special conformal triangles, the conformal hyperbolic equilateral triangle and the conformal hyperbolic isosceles triangle, the internal angles and vertices are shown.

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1. Introduction

The set $H_0^n = \left\{ x \in R_1^{n+1} : < x, x >= -1 \right\}$ is also called the n-dimensional unit pseudo-hyperbolic space. Two connected components of space H_0^n are $H_{0,+}^n$ and $H_{0,-}^n$; each of these components can be taken as the model of n-dimensional hyperbolic space. Based on the literature, we will consider the positive component as a model of hyperbolic space; that is $H_{0,+}^n = H^n \subset R_1^{n+1}$ [1,2,8].

First, we remember the concepts of lines and triangles in the hyperbolic plane.

As for $\alpha : IR \to H^n \text{ and } x, y \in H^n$, curve

$$\alpha(t) = (\cosh t)x + (\sinh t)\frac{(y + \langle x, y\rangle x)}{\|y + \langle x, y\rangle x\|}$$

is called *line through* x, y of H^n [9].

Similarly for $\alpha : IR \to H^n \text{ and } x, y \in H^n$,

$$\alpha(t) = (\cosh t)x + (\sinh t)\frac{(y - \cosh t_1 x)}{\sinh t_1}, \quad t \in [0, t_1]$$

curve segment is called *the line segment of* H^n *limited to* x, y [9].

x, y, z, three of which are three points on the same hyperbolic line;

*Corresponding author. *Email address: utokeser@kastamonu.edu.tr*

$$\alpha(t) = (\cosh t)x + (\sinh t)\frac{(y - \cosh t_1\, x)}{\sinh t_1}, \quad t \in [0, t_1]$$
$$\beta(s) = (\cosh s)y + (\sinh s)\frac{(z - \cosh s_1\, y)}{\sinh s_1}, \quad s \in [0, s_1]$$
$$\gamma(u) = (\cosh u)z + (\sinh u)\frac{(x - \cosh u_1\, z)}{\sinh u_1}, \quad u \in [0, u_1]$$

the combination of the $\alpha(t_1) = \beta(0), \beta(s_1) = \gamma(0)$ *ve* $\gamma(u_1) = \alpha(0)$ segmented line segments is called the hyperbolic triangle, and the hyperbolic zone bounded by the triangle is called the *hyperbolic triangular zone* [9].

Ω is hyperbolic triangle with P_1, P_2, P_3 vertex points;

$$M = \begin{bmatrix} -1 & -\cosh\varphi_{12} & -\cosh\varphi_{13} \\ -\cosh\varphi_{12} & -1 & -\cosh\varphi_{23} \\ -\cosh\varphi_{13} & -\cosh\varphi_{23} & -1 \end{bmatrix}$$

matrix is called *egde matrix* of Ω [4].

P_i, P_j two vertices of Ω;

$$\cosh\varphi_{ij} = -\langle P_i, P_j \rangle$$

the real number φ_{ij} in the property $\cosh\varphi_{ij} = -\langle P_i, P_j \rangle$ is called *edge length limited by* P_i, P_j of Ω [4].

Definition 1. The edges of the P_i, P_j, P_k -pointed Ω hyperbolic triangle through P_k point are also

$\alpha : IR \to H^n$,
$\beta : IR \to H^n$;

the θ_{ij} angle, which is to be $\langle \alpha'(t)|_{P_k}, \langle \beta'(s)|_{P_k} \rangle = \cos\theta_{ij}$, is called *the internal angle of* Ω *at point* P_k [9].

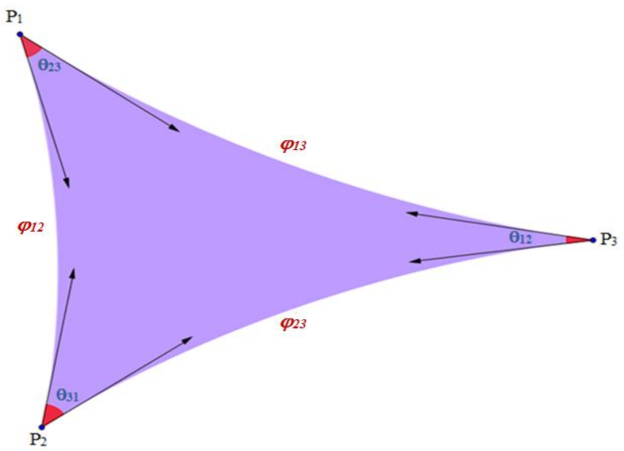

Figure 1. Triangle in Hyperbolic Space

2. Conformal Triangles in Hyperbolic Space

Definition 2. The set $\{P \in H^2 : \langle m, P\rangle = -\cosh r\}$, as $m \in H^2$ and $r \in IR^+$, is called *the m-centered r hyperbolic circle in* H^2 [9].

Definition 3. Let Ω be the hyperbolic triangle with P_1, P_2, P_3 vertex points. If there are real numbers $r_1, r_2, r_3 \in IR^+$ as $\varphi_{ij} = r_i + r_j$ with an edge length φ_{ij} limited to P_i, P_j; Ω is called *conformal hyperbolic triangle* [9].

Theorem 4. Let Ω be hyperbolic triangle with P_1, P_2, P_3 vertex points. Ω to be conformal if and only if

$$r_i > \ln\sqrt{2}\ , \quad i = 1,2,3 \tag{2.1}$$

where $r_1, r_2, r_3 \in IR^+$ [9].

Now, we give egde matricies for conformal hyperbolic triangles. These matricies play very important roles throughout the paper for calculations.

Lemma 5. Edge matrix of conformal hyperbolic triangles, edge matrix of conformal hyperbolic equilateral triangles and edge matrix of conformal hyperbolic isosceles triangles as follows

$$M = \begin{bmatrix} -1 & -\cosh(r_1+r_2) & -\cosh(r_1+r_3) \\ -\cosh(r_1+r_2) & -1 & -\cosh(r_2+r_3) \\ -\cosh(r_1+r_3) & -\cosh(r_2+r_3) & -1 \end{bmatrix} \tag{2.2}$$

$$\tilde{M} = \begin{bmatrix} -1 & -\cosh(r_1+r_2) & -\cosh(r_1+r_2) \\ -\cosh(r_1+r_2) & -1 & -\cosh(r_1+r_2) \\ -\cosh(r_1+r_2) & -\cosh(r_1+r_2) & -1 \end{bmatrix} \tag{2.3}$$

$$\widehat{M} = \begin{bmatrix} -1 & -\cosh(r_1+r_2) & -\cosh(r_1+r_2) \\ -\cosh(r_1+r_2) & -1 & -\cosh(r_2+r_3) \\ -\cosh(r_1+r_2) & -\cosh(r_2+r_3) & -1 \end{bmatrix} \tag{2.4}$$

respectively [9].

From [4]

$$\cos\theta_{ij} = \frac{M_{ij}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3 \tag{2.5}$$

and from equation (8) in [5], we can define

$$\sin\theta_{ij} = \frac{\sqrt{-|M|}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3\ . \tag{2.6}$$

3. Equality of Internal Angles and Vertex Points in Conformal Hyperbolic Triangles

In this section, using the expressions of the internal angles and vertex points, we defined in Definition 1,equality of internal angles to vertex points of the conformal hyperbolic triangle and special conformal hyperbolic triangles will be shown.

Now, in Eq. 2.5

$$\cos\theta_{ij} = \frac{M_{ij}}{\sqrt{M_{ii}M_{jj}}} \quad , i \neq j;\ i, j = 1,2,3$$

was given.

As $$sin P_k = \frac{\sqrt{-|M|}}{\sqrt{(-M_{ii})(-M_{jj})}} \quad ,i \neq j, i \neq k, j \neq k;\ i, j, k = 1,2,3 \quad . \tag{3.1}$$

It is

$$cos\,\theta_{12} = \frac{M_{12}}{\sqrt{M_{11}M_{22}}}$$

if M_{11}, M_{12} and M_{22} from Eq. 2.2 are calculated and replaced,

$$\cos\theta_{12} = \frac{\cosh(r_1+r_3)\cosh(r_2+r_3) - \cosh(r_1+r_2)}{\sqrt{\sinh^2(r_2+r_3)\sinh^2(r_1+r_3)}}$$

is obtained.

Similarly, if M_{11}, M_{12} and $|M|$ are used at Eq 3.1, calculated from Eq 2.2,

$$sin\,P_3 = \frac{\sqrt{-|M|}}{\sqrt{M_{11}M_{22}}}$$

$$sin\,P_3 = \frac{\sqrt{4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh(r_1+r_2+r_3)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_3)}}$$

would be. From here

$$\theta_{12} = \arccos\left(\frac{\cosh(r_1+r_3)\cosh(r_2+r_3) - \cosh(r_1+r_2)}{\sqrt{\sinh^2(r_2+r_3)\sinh^2(r_1+r_3)}}\right),$$

$$P_3 = arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh(r_1+r_2+r_3)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_3)}}\right) \tag{3.2}$$

are obtained.

We calculate the cosine of the right side of Eq 3.2. It would be

$$cos\left(arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh\left(r_1+r_2+r_3\right)}}{\sqrt{sinh^2\left(r_2+r_3\right)sinh^2\left(r_1+r_3\right)}}\right)\right)$$

$$=\sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh\left(r_1+r_2+r_3\right)}}{\sqrt{sinh^2\left(r_2+r_3\right)sinh^2\left(r_1+r_3\right)}}\right)\right)}$$

$$=\sqrt{1-\left(\frac{\sqrt{4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh\left(r_1+r_2+r_3\right)}}{\sqrt{sinh^2\left(r_2+r_3\right)sinh^2\left(r_1+r_3\right)}}\right)^2}$$

$$=\frac{\sqrt{sinh^2\left(r_1+r_2\right)sinh^2\left(r_1+r_3\right)-4\,sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh\left(r_1+r_2+r_3\right)}}{sinh\left(r_1+r_2\right)sinh\left(r_1+r_3\right)}.$$

When necessary calculations are made, we get

$$sinh^2\left(r_1+r_2\right)sinh^2\left(r_1+r_3\right)-4sinh\,r_1\,sinh\,r_2\,sinh\,r_3\,sinh\left(r_1+r_2+r_3\right)=\left(cosh\left(r_1+r_3\right)cosh\left(r_2+r_3\right)-cosh\left(r_1+r_2\right)\right)^2$$

Thus,

$\theta_{12}=P_3$

equation is obtained. By using similar method

$\theta_{23}=P_1$

and

$\theta_{13}=P_2$

are obtained [6].

3.1. Equality of internal angles and vertex points in the conformal hyperbolic equilateral triangle

Definition 6. Let Ω be a hyperbolic triangle with P_1,P_2,P_3 vertex points, $\theta_{12},\theta_{13},\theta_{23}$ dihedral angles and $\varphi_{12},\varphi_{13},\varphi_{23}$ edge lengths. Let $\Omega\in H^2$; if $\theta_{12}=\theta_{13}=\theta_{23}$, $\varphi_{12}=\varphi_{13}=\varphi_{23}$ and $\theta_{12}<\frac{\pi}{3}$, Ω is called *equilateral hyperbolic triangle* [7].

Now, in Eq. 2.5

$$\cos\theta_{ij}=\frac{\tilde{M}_{ij}}{\sqrt{\tilde{M}_{ii}\tilde{M}_{jj}}}\quad, i\neq j;\ i,j=1,2,3$$

was given.

Including

$$sin\,P_k=\frac{\sqrt{-\left|\tilde{M}\right|}}{\sqrt{\left(-\tilde{M}_{ii}\right)\left(-\tilde{M}_{jj}\right)}}\quad ,i\neq j,i\neq k,j\neq k;\ i,j,k=1,2,3\ . \tag{3.3}$$

If $\tilde{M}_{11},\tilde{M}_{12}$ and $\tilde{M}_{22}$ are calculated and replaced from Eq. 2.3;

$$cos\,\theta_{12}=\frac{cosh(r_1+r_2)\left(cosh(r_1+r_2)-1\right)}{\sqrt{sinh^4(r_1+r_2)}}$$

is obtained.

Similarly, if $\tilde{M}_{11},\tilde{M}_{12}$ and $\left|\tilde{M}\right|$ calculated from Eq. 2.3 used in Eq. 3.3 , it becomes as

$$sin\,P_3=\frac{\sqrt{-\left|\tilde{M}\right|}}{\sqrt{\tilde{M}_{11}\tilde{M}_{22}}}$$

$$sin\,P_3=\frac{\sqrt{\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{\sqrt{sinh^4(r_1+r_2)}}.$$

Here,

$$\theta_{12}=\arccos\left(\frac{\cosh(r_1+r_2)\left(\cosh(r_1+r_2)-1\right)}{\sqrt{\sinh^4(r_1+r_2)}}\right),$$

$$P_3=arcsin\left(\frac{\sqrt{\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{\sqrt{sinh^4(r_1+r_2)}}\right) \tag{3.4}$$

are obtained.

We calculate the cosine of the right side of Eq. 3.4 as follow,

$$cos\left(arcsin\left(\frac{\sqrt{\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{\sqrt{sinh^4(r_1+r_2)}}\right)\right)$$

$$=\sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{\sqrt{sinh^4(r_1+r_2)}}\right)\right)}$$

$$=\sqrt{1-\left(\frac{\sqrt{\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{\sqrt{sinh^4(r_1+r_2)}}\right)^2}$$

$$=\frac{\sqrt{sinh^2(r_1+r_2)-\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)}}{sinh^2(r_1+r_2)}.$$

We get

$$sinh^2(r_1+r_2)-\left(cosh(r_1+r_2)-1\right)^2\left(cosh(r_1+r_2)+1\right)=cosh^2(r_1+r_2)\left(cosh(r_1+r_2)-1\right)^2$$

when necessary calculations are made. Thus

$\theta_{12} = P_3$

equality is obtained. By using similar method

$\theta_{23} = P_1$

and

$\theta_{13} = P_2$

are obtained [6].

3.2. Equality of internal angles and vertex points in the conformal hyperbolic isosceles triangle

Definition 7. Let Ω be a hyperbolic triangle with P_1, P_2, P_3 vertex points, $\theta_{12}, \theta_{13}, \theta_{23}$ dihedral angles and $\varphi_{12}, \varphi_{13}, \varphi_{23}$ edge lengths. Let $\Omega \in H^2$; if $\theta_{12} = \theta_{13}$ and $2\theta_{12} < \pi - \theta_{23}$, Ω is called *isosceles hyperbolic triangle* [7].

Now, in Eq. 2.5

$$\cos\theta_{ij} = \frac{\widehat{M}_{ij}}{\sqrt{\widehat{M}_{ii}\widehat{M}_{jj}}} \quad , i \neq j;\ i, j = 1,2,3$$

was given.

Including

$$sin\,P_k = \frac{\sqrt{-\left|\widehat{M}\right|}}{\sqrt{\left(-\widehat{M}_{ii}\right)\left(-\widehat{M}_{jj}\right)}} \quad ,i \neq j, i \neq k, j \neq k;\ i, j, k = 1,2,3 \quad . \tag{3.5}$$

If $\widehat{M}_{11}, \widehat{M}_{22}$ and $\widehat{M}_{22}$ are calculated and replaced from Eq. 2.4;

$$cos\,\theta_{12} = \frac{cosh(r_1 + r_2)\left(cosh(r_2 + r_3) - 1\right)}{\sqrt{sinh^2(r_2 + r_3)\,sinh^2(r_1 + r_2)}}$$

is obtained.

Similarly, if $\widehat{M}_{11}, \widehat{M}_{22}$ and $\left|\widehat{M}\right|$ calculated from Eq. 2.4 used in Eq. 3.5, it becomes as

$$sin\,P_3 = \frac{\sqrt{-\left|\widehat{M}\right|}}{\sqrt{\widehat{M}_{11}\widehat{M}_{22}}}$$

$$sin\,P_3 = \frac{\sqrt{4\,sinh\,r_1\,sinh^2 r_2\,sinh(r_1 + r_2)}}{\sqrt{sinh^2(r_2 + r_3)\,sinh^2(r_1 + r_2)}}.$$

Here,

$$\theta_{12} = \arccos\left(\frac{\cosh(r_1 + r_2)\left(\cosh(r_2 + r_3) - 1\right)}{\sqrt{\sinh^2(r_2 + r_3)\sinh^2(r_1 + r_2)}}\right),$$

$$P_3 = arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)}}\right) \tag{3.6}$$

are obtained.

We calculate the cosine of the right side of Eq. 3.6 as follow,

$$cos\left(arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)}}\right)\right)$$

$$=\sqrt{1-sin^2\left(arcsin\left(\frac{\sqrt{4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)}}\right)\right)}$$

$$=\sqrt{1-\left(\frac{\sqrt{4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)}}{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)}}\right)^2}$$

$$=\frac{\sqrt{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)-4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)}}{sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)}.$$

We get

$$sinh^2(r_2+r_3)\,sinh^2(r_1+r_2)-4\,sinh\,r_1\,sinh^2\,r_2\,sinh(r_1+r_2)=cosh^2(r_1+r_2)\big(cosh(r_2+r_3)-1\big)^2$$

when necessary calculations are made. Thus

$\theta_{12} = P_3$

equality is obtained. By using similar method,

$\theta_{23} = P_1$

and

$\theta_{13} = P_2$

are obtained [6].

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