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Lacunary $\mathcal{I}$-invariant convergence

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Abstract

In this study, firstly, we introduce the notion of lacunary invariant uniform density of any subset E of the set $\mathbb{N}$ (the set of all natural numbers). Then, as associated with this notion, we give the definition of lacunary $\mathcal{I}_\sigma$-convergence for real sequences. Furthermore, we examine relations between this new type convergence notion and the notions of lacunary invariant summability, lacunary strongly q-invariant summability and lacunary σ-statistical convergence which are studied in this area before. Finally, introducing the notions of lacunary $\mathcal{I}_\sigma^*$-convergence and $\mathcal{I}_\sigma$-Cauchy sequence, we give the relations between these notions and the notion of lacunary $\mathcal{I}_\sigma$-convergence.

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1. Introduction and Background

Let σ be a mapping such that $\sigma: \mathbb{N}^+ \to \mathbb{N}^+$ (the set of all positive integers). A continuous linear functional φ on ℓ_∞, the space of real bounded sequences, is said to be an invariant mean or a σ-mean if it satisfies the following conditions:

i. $\varphi(x_n) \geq 0$, when the sequence (x_n) has $x_n \geq 0$ for all $n \in \mathbb{N}$,

ii. $\varphi(e) = 1$, where $e = (1,1,1,\dots)$ and

iii. $\varphi(x_{\sigma(n)}) = \varphi(x_n)$ for all $(x_n) \in \ell_\infty$.

The mappings σ are assumed to be one-to-one and such that $\sigma^m(n) \neq n$ for all $m, n \in \mathbb{N}^+$, where $\sigma^m(n)$ denotes the m th iterate of the mapping σ at n. Thus, φ extends the limit functional on c, the space of convergent sequences, in the sense that $\varphi(x_n) = \lim x_n$ for all $(x_n) \in c$.

In the case σ is translation mappings $\sigma(n) = n + 1$, the σ-mean is often called a Banach limit.

The space V_σ, the set of bounded sequences whose invariant means are equal, can be shown that

$$V_\sigma = \left\{ (x_k) \in \ell_\infty : \lim_{m\to\infty} \frac{1}{m} \sum_{k=1}^{m} x_{\sigma^k(n)} = L, \text{ uniformly in } n \right\}.$$

Several authors studied on the notions of invariant mean and invariant convergent sequence (for examples, see [1-8]).

The notion of strongly σ-convergence (it is denoted by $[V_\sigma]$) was introduced by Mursaleen [9]. Then this notion, using a positive real number p, was generalized by Savaş [10] (it is denoted by $[V_\sigma]_p$).

By a lacunary sequence, we mean an increasing integer sequence $\theta = \{k_r\}$ such that

$$k_0 = 0 \text{ and } h_r = k_r - k_{r-1} \to \infty \text{ as } r \to \infty.$$

The intervals determined by θ is denoted by $I_r = (k_{r-1}, k_r]$ (see, [11]).

Throughout the study, $\theta = \{k_r\}$ will be taken as a lacunary sequence.

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The set of lacunary strongly σ-convergence sequences was defined by Savaş [12] as below:

$$L_\theta = \left\{ (x_k) \colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(n)} - L| = 0, \text{ uniformly in } n \right\}.$$

Recently, Pancaroğlu and Nuray [13] defined the notions of lacunary invariant summability and lacunary strongly q-invariant summability as follows.

A sequence (x_k) is said to be lacunary invariant summable to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} x_{\sigma^k(n)} = L,$$

uniformly in n.

A sequence (x_k) is said to be lacunary strongly q-invariant summable $(0 < q < \infty)$ to L if

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} |x_{\sigma^k(n)} - L|^q = 0,$$

uniformly in n and it is denoted by $x_k \to L([V_{\sigma\theta}]_q)$.

The idea of statistical convergence was introduced by Fast [14] and then studied by several authors (for example, see [15-17]). In one of these studies, Savaş and Nuray [18] defined the notion of lacunary σ-statistical convergence as below.

A sequence (x_k) is said to be lacunary σ-statistical convergent to L if for every $\varepsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r \colon |x_{\sigma^k(n)} - L| \geq \varepsilon\}| = 0,$$

uniformly in n, where the vertical bars denote the number of elements in the enclosed set.

The idea of $\mathcal{I}$-convergence which is a generalization of the statistical convergence notion was introduced by Kostyrko et al. [19]. Some properties of this notion and similar notions which are noted following studied by several authors (for examples, see [20-22]).

A family of sets $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal iff

i. $\emptyset \in \mathcal{I}$,

ii. For each $E, F \in \mathcal{I}$, we have $E \cup F \in \mathcal{I}$,

iii. For each $E \in \mathcal{I}$ and each $F \subseteq E$, we have $F \in \mathcal{I}$.

An ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called non-trivial if $\mathbb{N} \notin \mathcal{I}$ and a non-trivial ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called admissible if $\{n\} \in \mathcal{I}$ for each $n \in \mathbb{N}$.

All ideals in this study will be assumed to be admissible in $2^{\mathbb{N}}$ (the power set of $\mathbb{N}$).

An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ has property (AP) if for every countable family of mutually disjoint sets $\{E_1, E_2, ...\}$ belonging to $\mathcal{I}$, there exists a countable family of sets $\{F_1, F_2, ...\}$ such that the symmetric differences $E_i \Delta F_i$ is a finite for each $i \in \mathbb{N}$ and $F = \bigcup_{i=1}^{\infty} F_i \in \mathcal{I}$.

A family of sets $\mathcal{F} \subseteq 2^{\mathbb{N}}$ is called a filter iff

i. $\emptyset \notin \mathcal{F}$,

ii. For each $E, F \in \mathcal{F}$, we have $E \cap F \in \mathcal{F}$,

iii. For each $E \in \mathcal{F}$ and each $F \supseteq E$, we have $F \in \mathcal{F}$.

There is a filter $\mathcal{F}(\mathcal{I})$ corresponding with $\mathcal{I}$ such that $\mathcal{F}(\mathcal{I}) = \{M \subset \mathbb{N} \colon (\exists E \in \mathcal{I})(M = \mathbb{N} \backslash E)\}$ for any ideal $\mathcal{I} \subseteq 2^{\mathbb{N}}$.

A sequence (x_k) is said to be $\mathcal{I}$-convergent to L if for every $\varepsilon > 0$, the set

$$E(\varepsilon) = \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}$ and it is denoted by $\mathcal{I} - \lim x_k = L$.

A sequence (x_k) is said to be $\mathcal{I}^*$-convergent to L if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k\to\infty} x_{m_k} = L$$

and it is denoted by $\mathcal{I}^* - \lim x_k = L$.

The notions of $\mathcal{I}$-Cauchy sequence and $\mathcal{I}^*$-Cauchy sequence were introduced by Nabiev et al. [23]. Similar notions were studied in [24], too.

A sequence (x_k) is called an $\mathcal{I}$-Cauchy sequence if for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that

$$F(\varepsilon) = \{k \in \mathbb{N}: |x_k - x_N| \geq \varepsilon\} \in \mathcal{I}.$$

A sequence (x_k) is called an $\mathcal{I}^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I})$ such that

$$\lim_{k,p\to\infty} |x_{m_k} - x_{m_p}| = 0.$$

Lately, Nuray et al. [25] introduced the notions of $\mathcal{I}_\sigma$-convergence and $\mathcal{I}_\sigma^*$-convergence for real sequences. Also, they gave some relations between these notions and the notions which are studied in this area before.

2. Main Results

In this section, firstly, we introduce the notion of lacunary invariant uniform density of any subset E of the set $\mathbb{N}$. After that, associate with this notion, we give the definition of lacunary $\mathcal{I}_\sigma$-convergence for real sequences. Furthermore, we examine relations between this new type convergence notion and the notions of lacunary invariant summability, lacunary strongly q-invariant summability and lacunary σ-statistical convergence which are studied in this area before.

Definition 2.1 Let $\theta = \{k_r\}$ be a lacunary sequence, $E \subseteq \mathbb{N}$ and

$$s_r := \min_n \{|E \cap \{\sigma^m(n): m \in I_r\}|\}, \quad S_r := \max_n \{|E \cap \{\sigma^m(n): m \in I_r\}|\}.$$

If the following limits exist

$$\underline{V_\theta}(E) := \lim_{r\to\infty} \frac{s_r}{h_r} \quad \text{and} \quad \overline{V_\theta}(E) := \lim_{r\to\infty} \frac{S_r}{h_r},$$

then they are called a lower lacunary invariant uniform density and an upper lacunary invariant uniform density of the set E, respectively. If $\underline{V_\theta}(E) = \overline{V_\theta}(E)$, then $V_\theta(E) = \underline{V_\theta}(E) = \overline{V_\theta}(E)$ is called the lacunary invariant uniform density of the set E.

The class of all $E \subset \mathbb{N}$ with $V_\theta(E) = 0$ will be denoted by $\mathcal{I}_{\sigma\theta}$. Note that $\mathcal{I}_{\sigma\theta}$ is an admissible ideal.

Definition 2.2 A sequence (x_k) is lacunary $\mathcal{I}_\sigma$-convergent to L if for every $\varepsilon > 0$, the set

$$E(\varepsilon) := \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\}$$

belongs to $\mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(E(\varepsilon)) = 0$ and we write $\mathcal{I}_{\sigma\theta} - \lim x_k = L$.

The class of all lacunary $\mathcal{I}_\sigma$-convergent sequences will be denoted by $\mathfrak{I}_{\sigma\theta}$.

It can be easily verified that if $\mathcal{I}_{\sigma\theta} - \lim x_k = L_1$ and $\mathcal{I}_{\sigma\theta} - \lim y_k = L_2$, then

i. $\mathcal{I}_{\sigma\theta} - \lim (x_k + y_k) = L_1 + L_2$ and

ii. $\mathcal{I}_{\sigma\theta} - \lim (\alpha x_k) = \alpha L_1$ (α is a constant).

Theorem 2.1 Let $(x_k) \in \ell_\infty$. If (x_k) is lacunary $\mathcal{I}_\sigma$-convergent to L, then this sequence is lacunary invariant summable to L.

Proof. Let $n \in \mathbb{N}$ be arbitrary and $\varepsilon > 0$. Also, we assume that $(x_k) \in \ell_\infty$ and (x_k) is lacunary $\mathcal{I}_\sigma$-convergent to L.

Now, we calculate

$$T_\theta(n) := \left| \frac{1}{h_r} \sum_{m \in I_r} x_{\sigma^m(n)} - L \right|.$$

For every $n = 1,2, \ldots$, we have

$$T_\theta(n) \leq T_\theta^{(1)}(n) + T_\theta^{(2)}(n),$$

where

$$T_\theta^{(1)}(n) := \frac{1}{h_r} \sum_{\substack{m \in I_r \\ |x_{\sigma^m(n)} - L| \geq \varepsilon}} \left| x_{\sigma^m(n)} - L \right|$$

and

$$T_\theta^{(2)}(n) := \frac{1}{h_r} \sum_{\substack{m \in I_r \\ |x_{\sigma^m(n)} - L| < \varepsilon}} |x_{\sigma^m(n)} - L|.$$

For every $n = 1,2, \ldots$, it is obvious that $T_\theta^{(2)}(n) < \varepsilon$. Since $(x_k) \in \ell_\infty$, there exists a $\lambda > 0$ such that

$$|x_{\sigma^m(n)} - L| \leq \lambda \; (m \in I_r,\; n = 1,2, \ldots)$$

and so we have

$$T_\theta^{(1)}(n) = \frac{1}{h_r} \sum_{\substack{m \in I_r \\ |x_{\sigma^m(n)} - L| \geq \varepsilon}} \left| x_{\sigma^m(n)} - L \right| \leq \frac{\lambda}{h_r} |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}|$$

$$\leq \lambda \, \frac{\max_n \left\{ |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}| \right\}}{h_r}$$

$$= \lambda \, \frac{S_r}{h_r}.$$

Hence, due to our assumption, the sequence (x_k) is lacunary invariant summable to L.

In general, the converse of Theorem 2.1 does not hold. For example, let (x_k) be the sequence defined as follows:

$$x_k := \begin{cases} 1 \quad , & \text{if } \; k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r}\right] \; \text{ and } k \text{ is an even integer,} \\ 0 \quad , & \text{if } \; k_{r-1} < k < k_{r-1} + \left[\sqrt{h_r}\right] \; \text{ and } k \text{ is an odd integer.} \end{cases}$$

When $\sigma(n) = n + 1$, this sequence is lacunary invariant summable to $\frac{1}{2}$ but it is not lacunary $\mathcal{I}_\sigma$-convergent.

In [25], Nuray et al. gave some relations between the notions of $\mathcal{I}_\sigma$-convergence and $[V_\sigma]_p$-convergence and they showed that these notions are equivalent for bounded sequences. Now, we will give analogous theorems which are state relations between the notions of lacunary $\mathcal{I}_\sigma$-convergence and lacunary strongly q-invariant summability, and we will show that these notions are equivalent for bounded sequences.

Theorem 2.2 If a sequence (x_k) is lacunary strongly q-invariant summable to L, then this sequence is lacunary $\mathcal{J}_\sigma$-convergent to L.

Proof. Let $0 < q < \infty$ and $\varepsilon > 0$. Also, we assume that $x_k \to L([V_{\sigma\theta}]_q)$. For every $n = 1,2,...$, we have

$$\sum_{m\in I_r} |x_{\sigma^m(n)} - L|^q \geq \sum_{\substack{m\in I_r \\ |x_{\sigma^m(n)}-L|\geq\varepsilon}} |x_{\sigma^m(n)} - L|^q$$

$$\geq \varepsilon^q \, |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}|$$

$$\geq \varepsilon^q \max_n \left\{ |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}| \right\}$$

and so

$$\frac{1}{h_r}\sum_{m\in I_r} |x_{\sigma^m(n)} - L|^q \quad \geq \quad \varepsilon^q \, \frac{\max_n \left\{ |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}| \right\}}{h_r}$$

$$= \varepsilon^q \, \frac{S_r}{h_r}.$$

Hence, due to our assumption, $\mathcal{J}_{\sigma\theta} - \lim x_k = L$.

Theorem 2.3 Let $(x_k) \in \ell_\infty$. If (x_k) is lacunary $\mathcal{J}_\sigma$-convergent to L, then this sequence is lacunary strongly q-invariant summable to L.

Proof. Let $0 < q < \infty$ and $\varepsilon > 0$. Also, we assume that $(x_k) \in \ell_\infty$ and $\mathcal{J}_{\sigma\theta} - \lim x_k = L$. Since $(x_k) \in \ell_\infty$, there exists a $\lambda > 0$ such that $|x_{\sigma^m(n)} - L| \leq \lambda \ (m \in I_r,\ n = 1,2,\dots)$ and so we have

$$\frac{1}{h_r}\sum_{m\in I_r} |x_{\sigma^m(n)} - L|^q = \frac{1}{h_r} \sum_{\substack{m\in I_r \\ |x_{\sigma^m(n)}-L|\geq\varepsilon}} |x_{\sigma^m(n)} - L|^q + \frac{1}{h_r} \sum_{\substack{m\in I_r \\ |x_{\sigma^m(n)}-L|<\varepsilon}} |x_{\sigma^m(n)} - L|^q$$

$$\leq \lambda \, \frac{\max_n \left\{ |\{m \in I_r : |x_{\sigma^m(n)} - L| \geq \varepsilon\}| \right\}}{h_r} + \varepsilon^q$$

$$\leq \lambda \, \frac{S_r}{h_r} + \varepsilon^q.$$

Hence, due to our assumption, $x_k \to L([V_{\sigma\theta}]_q)$.

Theorem 2.4 Let $(x_k) \in \ell_\infty$. Then, (x_k) is lacunary $\mathcal{J}_\sigma$-convergent to L if and only if this sequence is lacunary strongly q-invariant summable to L.

Proof. This is an immediate consequence of Theorem 2.2 and Theorem 2.3.

Now, without proof, we will state a theorem that gives a relation between the notions of lacunary $\mathcal{J}_\sigma$-convergence and lacunary σ-statistical convergence.

Theorem 2.5 A sequence (x_k) is lacunary $\mathcal{J}_\sigma$-convergent to L if and only if this sequence is lacunary σ-statistical convergent to L.

Remark 2.1 By combining Theorem 2 in [18] and Theorem 5 in [25], we obtain that $\mathfrak{J}_{\sigma\theta} = \mathfrak{J}_\sigma$ for every lacunary sequence $\theta = \{k_r\}$, where $\mathfrak{J}_\sigma$ is the class of all $\mathcal{J}_\sigma$-convergent sequences.

Finally, introducing the notions of lacunary $\mathcal{J}_\sigma^*$-convergence and lacunary $\mathcal{J}_\sigma$-Cauchy sequence, we will give the relations between these notions and the notion of lacunary $\mathcal{J}_\sigma$-convergence.

Definition 2.3 A sequence (x_k) is lacunary $\mathcal{I}_\sigma^*$-convergent to L if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that

$$\lim_{k\to\infty} x_{m_k} = L.$$

In this case, we write $\mathcal{I}_{\sigma\theta}^* - \lim x_k = L$.

Theorem 2.6 If a sequence (x_k) is lacunary $\mathcal{I}_\sigma^*$-convergent to L, then this sequence is lacunary $\mathcal{I}_\sigma$-convergent to L.

Proof. Let $\varepsilon > 0$. Also, we assume that $\mathcal{I}_{\sigma\theta}^* - \lim x_k = L$. Then, there exists a set $H \in \mathcal{I}_{\sigma\theta}$ such that for $M = \mathbb{N}\backslash H = \{m_1 < m_2 < \cdots < m_k < \cdots\}$ we have

$$\lim_{k\to\infty} x_{m_k} = L$$

and so there exists a $k_0 \in \mathbb{N}$ such that $|x_{m_k} - L| < \varepsilon$ for every $k > k_0$. Hence, it is obvious that for every $\varepsilon > 0$

$$E(\varepsilon) = \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \subset H \cup \{ m_1 < m_2 < \cdots < m_{k_0}\}.$$

Since $\mathcal{I}_{\sigma\theta} \subset 2^{\mathbb{N}}$ is admissible,

$$H \cup \{ m_1 < m_2 < \cdots < m_{k_0}\} \in \mathcal{I}_{\sigma\theta}$$

and so we have $E(\varepsilon) \in \mathcal{I}_{\sigma\theta}$. Consequently, $\mathcal{I}_{\sigma\theta} - \lim x_k = L$.

The converse of Theorem 2.6 holds if the ideal $\mathcal{I}_{\sigma\theta}$ has the property (AP).

Theorem 2.7 Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP). If a sequence (x_k) is lacunary $\mathcal{I}_\sigma$-convergent to L, then this sequence is lacunary $\mathcal{I}_\sigma^*$-convergent to L.

Proof. Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP) and $\varepsilon > 0$. Also, we assume that $\mathcal{I}_{\sigma\theta} - \lim x_k = L$. Then, for every $\varepsilon > 0$ we have

$$E(\varepsilon) = \{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \in \mathcal{I}_{\sigma\theta}.$$

Denote $E_1, E_2, \ldots, E_{\mathrm{n}}$ as following

$$E_1 := \{k \in \mathbb{N}: |x_k - L| \geq 1\} \quad \text{and} \quad E_n := \left\{k \in \mathbb{N}: \frac{1}{n} \leq |x_k - L| < \frac{1}{n-1}\right\},$$

where $n \geq 2$ $(n \in \mathbb{N})$. Note that $E_i \cap E_j = \emptyset$ $(i \neq j)$ and $E_i \in \mathcal{I}_{\sigma\theta}$ (for each $i \in \mathbb{N}$). Since $\mathcal{I}_{\sigma\theta}$ has the property (AP), there exists a set sequence $\{F_n\}_{n\in\mathbb{N}}$ such that the symmetric differences $E_i \Delta F_i$ are finite (for each $i \in \mathbb{N}$) and $F = \bigcup_{j=1}^{\infty} F_j \in \mathcal{I}_{\sigma\theta}$. Now, to complete the proof, it is enough to prove that

$$\lim_{\substack{k\to\infty \\ k\in M}} x_k = L, \tag{2.1}$$

where $M = \mathbb{N}\backslash F$. Let $\zeta > 0$. Choose $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \zeta$. Then, we have

$$\{k \in \mathbb{N}: |x_k - L| \geq \zeta\} \subset \bigcup_{i=1}^{n+1} E_i.$$

Since the symmetric differences $E_i \Delta F_i$ $(i = 1,2, \ldots, n+1)$ are finite, there exists a $k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{i=1}^{n+1} E_i\right) \cap \{k \in \mathbb{N}: k > k_0\} = \left(\bigcup_{i=1}^{n+1} F_i\right) \cap \{k \in \mathbb{N}: k > k_0\}. \tag{2.2}$$

If $k > k_0$ and $k \notin F$, then

$$k \notin \bigcup_{i=1}^{n+1} F_i \quad \text{and by (2.2)} \quad k \notin \bigcup_{i=1}^{n+1} E_i.$$

This implies that

$$|x_k - L| < \frac{1}{n+1} < \zeta$$

and so (2.1) holds. Consequently, $\mathcal{I}_{\sigma\theta}^* - \lim x_k = L$.

Definition 2.4 A sequence (x_k) is a lacunary $\mathcal{I}_\sigma$-Cauchy sequence if for every $\varepsilon > 0$, there exists an $N = N(\varepsilon) \in \mathbb{N}$ such that the set

$$B(\varepsilon) = \{k \in \mathbb{N}: |x_k - x_N| \geq \varepsilon\}$$

belongs to $\mathcal{I}_{\sigma\theta}$, i.e., $V_\theta(B(\varepsilon)) = 0$.

Definition 2.5 A sequence (x_k) is a lacunary $\mathcal{I}_\sigma^*$-Cauchy sequence if there exists a set $M = \{m_1 < m_2 < \cdots < m_k < \cdots\} \in \mathcal{F}(\mathcal{I}_{\sigma\theta})$ such that

$$\lim_{k,p\to\infty} |x_{m_k} - x_{m_p}| = 0.$$

The proof of the following theorems are similar to the proof of theorems in [23], so we omit them.

Theorem 2.8 If a sequence (x_k) is lacunary $\mathcal{I}_\sigma$-convergent, then this sequence is a lacunary $\mathcal{I}_\sigma$-Cauchy sequence.

Theorem 2.9 If a sequence (x_k) is a lacunary $\mathcal{I}_\sigma^*$-Cauchy sequence, then this sequence is a lacunary $\mathcal{I}_\sigma$-Cauchy sequence.

Theorem 2.10 Let the ideal $\mathcal{I}_{\sigma\theta}$ be with the property (AP). Then, the notions of lacunary $\mathcal{I}_\sigma$-Cauchy sequence and lacunary $\mathcal{I}_\sigma^*$-Cauchy sequence coincide.

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Conflicts of interest

The authors state that did not have conflict of interests.

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