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Comparison of estimators under different loss functions for two-parameter bathtub - shaped lifetime distribution

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Abstract

Chen is suggested a two-parameter distribution. This distribution can have increasing failure rate function or a bathtub-shaped that allows it to fit real lifetime data sets. The ML (Maximum Likelihood) and Bayes estimates of the parameters of Chen's distribution are constituted in this paper. The approximate values of Bayesian estimates are obtained by using the Tierney-Kadane approach. Two-parameter bathtub-shaped distribution's estimations are derived using Jeffrey's extension prior under General entropy, Squared and Linex loss functions. Besides, performances of ML and Bayes estimates are compared concerning MSE's (Mean Square Error) by using Monte Carlo simulation. As a result, it has been seen that approximate Bayes estimates obtained under linex loss function are better than others. Moreover, real data analysis for his distribution is presented.

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1. Introduction

In this study, we have studied parameter estimation for a two-parameter lifetime distribution with either bathtub-shaped or increasing failure rate investigated by Chen [1]. Moreover, some distributions have been proposed with models for bathtub-shaped failure rates, such as Hjorth [2] and Mudholkar and Srivastava [3]. This distribution has been studied by many authors such as Sarhan et al. [4], Selim [5], Jung and Yung [6] Javadkhani et al. [7] and Faizan and Sana [8]. The new two-parameter lifetime distribution with increasing failure rate function bathtub-shaped compared with other models has some desirable properties, which has two parameters. For more details, see Lee et al. [9], Chen [1] and Wang [10]. In this paper, the cumulative distribution function (CDF), probability density function (pdf), reliability and hazard function of an X random variable having Chen(α,β) are as follows.

$$f(x) = \alpha\beta x^{\beta-1}\exp\left[\alpha\left(1-\exp\left(x^{\beta}\right)\right)+x^{\beta}\right] \tag{1}$$

$$F(x) = 1-\exp\left[\alpha\left(1-\exp\left(x^{\beta}\right)\right)\right] \tag{2}$$

$$R(x) = \exp\left[\alpha\left(1-\exp\left(x^{\beta}\right)\right)\right], \tag{3}$$

$$h(x) = \alpha\beta x^{\beta-1}\exp\left(x^{\beta}\right), \tag{4}$$

and where

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- *if* $\beta < 1$, *h(t)* is bathtub function,

and

- *if* $\beta \geq 1$, *h(t)* is increasing function.

The distribution has increasing failure rate function when $\alpha > 1$ and $\beta < 1$. Figure 1 present the failure rate functions for different values $\alpha = 2, \beta = 0.5, 1, 1.5$

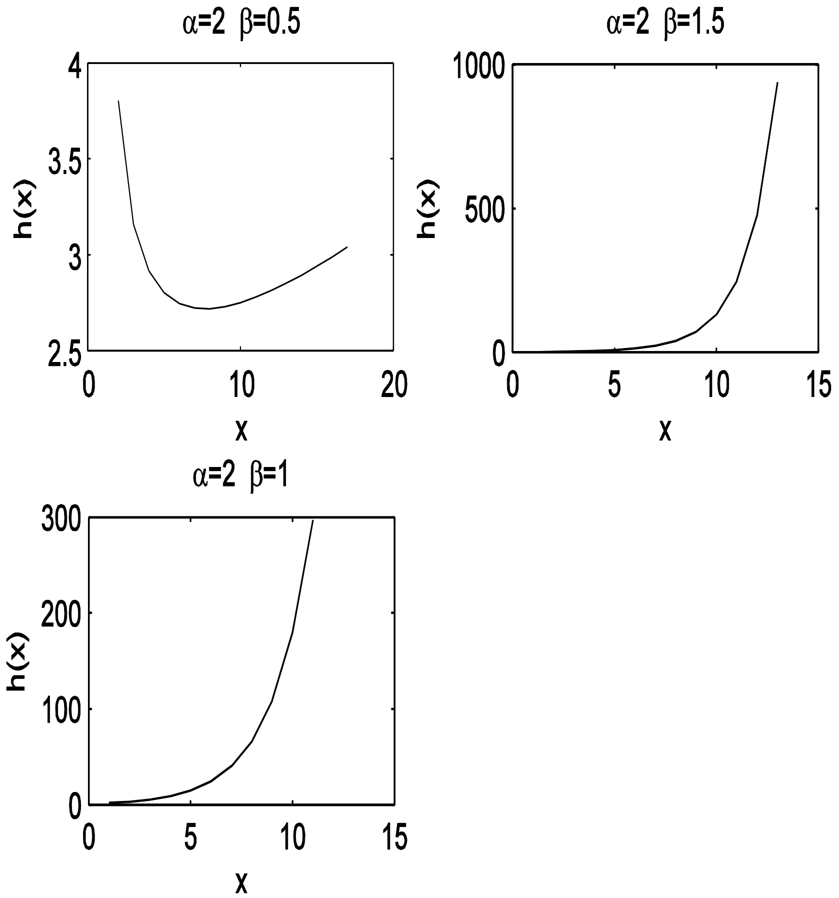

Figure 1. Failure rate functions of different parameters.

The primary objective of this study is to obtain the approximate Bayes estimators' samples under linex, general entropy and squared loss functions, following compare them in term of MSE's. The remaining text is arranged as follows. In Section 2, MLs for Chen distribution is given and the approximate Bayes estimators under different loss functions are derived by using Tierney's Kadane approximations. In section 4, using Monte Carlo simulation, Bayes estimations are compared with the ML in terms of MSE, and results are tabulated. A real data application is performed in Section 5. Finally, conclusion is given in the last section.

2. Methodology

2.1. Maximum likelihood estimation

Let $X_1, X_2, ..., X_n$ be the complete sample from independent random variables having Chen distribution with unknown α , β parameters. Then the log-likelihood function is given by,

$$L\left(\alpha,\beta\left|\underset{-}{x}\right.\right)=\prod_{i=1}^{n}\alpha\beta\exp\left(x_i^{\beta}\right)\exp\left[\alpha\left(1-e^{\left(x_i^{\beta}\right)}\right)\right]x_i^{\beta-1} \tag{5}$$

$$l(\alpha,\beta)=\ln\left(L\left(\alpha,\beta\left|\underset{-}{x}\right.\right)\right)==n\ln\alpha+n\ln\beta+(\beta-1)\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}x_i^{\beta}+\sum_{i=1}^{n}\alpha\left(1-e^{\left(x_i^{\beta}\right)}\right) \tag{6}$$

Differentiating the log-likelihood function $\ell\left(\alpha,\beta\left|\underset{-}{x}\right.\right)$ partially about unknown α, β parameters and after non-linear equations is attained. Newton-Raphson algorithm is one of the standard methods to determine the ML estimates of the two unknown parameters.

$$\frac{\partial l(\alpha,\beta)}{\partial\alpha}=0\Rightarrow\frac{n}{\alpha}+\sum_{i=1}^{n}\left(1-e^{\left(x_i^{\beta}\right)}\right)=0 \tag{7}$$

$$\frac{\partial l(\alpha,\beta)}{\partial\beta}=0\Rightarrow\frac{n}{\beta}+\sum_{i=1}^{n}\left(x_i^{\beta}\ln x_i\right)-\sum_{i=1}^{n}\left(\alpha x_i^{\beta}\ln x_i\exp\left(x_i^{\beta}\right)\right)+\sum_{i=1}^{n}(\ln x_i)=0 \tag{8}$$

2.2. Bayesian estimation

For estimation of the parameters, prior distributions for these parameters is needed. In this study, as the prior distributions, Jeffrey's extension prior is used, and these are as follows [11].

$$\pi_1(\alpha)\propto(\frac{1}{\alpha})^{d} \tag{9}$$

$$\pi_2(\beta)\propto(\frac{1}{\beta})^{d} \tag{10}$$

The joint priors and posterior distributions of $\alpha,\ \beta$ parameters are

$$\pi(\alpha,\beta)\propto(\frac{1}{\alpha\beta})^{d} \tag{11}$$

$$\pi\left(\alpha,\beta/\underset{-}{x}\right)=\frac{f\left((\alpha,\beta);\underset{-}{x}\right)}{f\left(\underset{-}{x}\right)}$$

$$=\frac{\alpha^{n}\beta^{n}\left(\exp\sum_{i=1}^{n}x_i^{\beta}\right)\exp\left[\sum_{i=1}^{n}\alpha\left(1-\exp\left(x_i^{\beta}\right)\right)\right]\prod_{i=1}^{n}x_i^{\beta-1}\left(\frac{1}{\alpha\beta}\right)^{d}}{\int_0^{\infty}\int_0^{\infty}\alpha^{n}\beta^{n}\left(\exp\sum_{i=1}^{n}x_i^{\beta}\right)\exp\left[\sum_{i=1}^{n}\alpha\left(1-\exp\left(x_i^{\beta}\right)\right)\right]\prod_{i=1}^{n}x_i^{\beta-1}\left(\frac{1}{\alpha\beta}\right)^{d}d\alpha d\beta} \tag{12}$$

Squared error loss function is a symmetric function and introduced by Legendre [12] and Gauss [13]. Let any function of α and β be $s(\alpha,\beta)=s$.

The SLF is as follows:

$$Loss_1(\hat{s}_{Squared},s)=(\hat{s}_{Squared}-s)^2 \tag{13}$$

The value which is minimize the expected value of SLF is expressed as,

$$\hat{s}_{Squared}(\alpha,\beta)=E\left[s(\alpha,\beta)\middle|\underline{x}\right] \tag{14}$$

In this case, Bayes estimator of $s(\alpha,\beta)$ under SLF is expressed as follows.

$$\hat{s}_{Squared}(\alpha,\beta)=E\left[s(\alpha,\beta)|\underline{x}\right]$$
$$=\frac{\int_0^\infty\int_0^\infty s\left(\alpha,\beta/\underline{x}\right)e^{\left[\ell\left(\alpha,\beta/\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\left[\ell\left(\alpha,\beta/\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta} \tag{15}$$

where $\ell\left(\alpha,\beta\middle|\underline{x}\right)$ is a log-likelihood function, $\rho\left(\alpha,\beta\middle|\underline{x}\right)$ is the logarithm of joint prior distribution. The Linex loss function (LLF), which is an asymmetric function organized by Varian [14] and Zellner [15]. Let any function of α and β be $s(\alpha,\beta)$. LLF is defined as follows.

$$Loss_2(\Delta)\ \alpha\ \exp(k\Delta)\text{-}k\Delta\text{-}1;\quad k\neq 0, \tag{16}$$

where, $\Delta=\hat{s}(\alpha,\beta)-s(\alpha,\beta)$. Then, posterior mean of the linex loss function is given as:

$$E_\theta\left[Loss_2\left(\hat{s}-s\right)\right]\propto\exp\left(k\hat{s}\right)E_\theta[\exp(-ks)]-k\left(\hat{s}-E_\theta(s)\right)-1 \tag{17}$$

where $\hat{s}=\hat{s}(\alpha,\beta)$ and $s=s(\alpha,\beta)$. $\hat{s}_{Linex}$, which minimizes this posterior mean, is Bayes estimator of s and is expressed as,

$$\hat{s}_{Linex}(\alpha,\beta)=-\frac{1}{k}\ln E\left[\exp\left(-ks(\alpha,\beta)\right)|\underline{x}\right]$$
$$=-\frac{1}{k}\ln\left(\frac{\int_0^\infty\int_0^\infty\exp\left(-ks(\alpha,\beta)\right)e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\left[\ell\left(\alpha,\beta|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}\right) \tag{18}$$

General entropy loss function (GLF) is an asymmetric function and suggested by Calabria and Pulcini [16]. Dey and Liao [17] studied with Bayes estimation under GLF. Let any function of α and β be $s(\alpha,\beta)$. GLF is denoted as,

$$Loss_3(\hat{s},s)\propto\left(\frac{\hat{s}}{s}\right)^a-a\,ln\left(\frac{\hat{s}}{s}\right)-1 \tag{19}$$

Then, posterior mean of GLF is given as:

$$E_\theta\left[Loss_3\left(\hat{s},s\right)\right]\propto E\left(\frac{\hat{s}}{s}\right)^a-aE\left[\ln(\hat{s})-\ln(s)\right]-1 \tag{20}$$

where $\hat{s}=\hat{s}(\alpha,\beta)$ and $s=s(\alpha,\beta)$. Then, $\hat{s}_{BGE}$, which minimizes this posterior mean, is Bayes estimator of s and is expressed as follows.

$$\hat{s}_{Entropy}(\alpha,\beta)=\left\{E\left\{\left[s(\alpha,\beta)\right]^{-a}|\underline{x}\right\}\right\}^{-\frac{1}{a}}$$
$$=\left\{\frac{\int_0^\infty\int_0^\infty\left[s(\alpha,\beta)\right]^{-a}e^{\left[\ell\left(\alpha,\beta\middle|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}{\int_0^\infty\int_0^\infty e^{\left[\ell\left(\alpha,\beta\middle|\underline{x}\right)+\rho(\alpha,\beta)\right]}d\alpha d\beta}\right\}^{-\frac{1}{a}} \quad (21)$$

It is complicated to solve the equations (15), (18) and (21) in closed-form. Due to this reason, the Bayes Estimators of $s(\alpha,\beta)$ can be attained using Tierney-Kadane's approximation.

2.3. Tierney Kadane' s approximation

Tierney and Kadane [18] are one of the most popular methods to find the approximate value of the mathematical explanations as to the odd of two integrals given in Equations (15), (18) and (21). This methods can be written as follows for a case with two parameters.

$$l(\alpha,\beta)=\frac{1}{n}\{\rho(\alpha,\beta)+\ell(\alpha,\beta)\} \quad (22)$$

$$l^*(\alpha,\beta)=\frac{1}{n}\log s(\alpha,\beta)+l(\alpha,\beta) \quad (23)$$

where $s(\alpha,\beta)$ is any function of α and β, $\ell\left(\alpha,\beta\middle|\underline{x}\right)$ is defined in Eq., (6), $\rho(\alpha,\beta)$ is logarithm joint prior distribution and defined as follows.

$$\rho(\alpha,\beta)=\ln\left(\pi(\alpha,\beta)\right)=-m\ln(\alpha)-m\ln(\beta) \quad (24)$$

$$\hat{s}_B(\alpha,\beta)=E\left(s(\alpha,\beta)\middle|\underline{x}\right)=\frac{\int e^{nl^*(\alpha,\beta)}d(\alpha,\beta)}{\int e^{nl(\alpha,\beta)}d(\alpha,\beta)}$$
$$=\left(\frac{\det\Sigma^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(l^*\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)-l\left(\hat{\alpha}_l,\hat{\beta}_l\right)\right)\right] \quad (25)$$

Where $\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)$ and $\left(\hat{\alpha}_l,\hat{\beta}_l\right)$ maximize $l^*(\alpha,\beta)$ and $l(\alpha,\beta)$, respectively. Σ^* And Σ are minus the inverse Hessians of $l^*(\alpha,\beta)$ and $l(\alpha,\beta)$ at $\left(\hat{\alpha}_{l^*},\hat{\beta}_{l^*}\right)$ and $\left(\hat{\alpha}_l,\hat{\beta}_l\right)$. Σ is defined as,

$$\Sigma=\begin{bmatrix}-\partial^2 l/\partial\alpha^2 & -\partial^2 l/\partial\alpha\partial\beta \\ -\partial^2 l/\partial\alpha\partial\beta & -\partial^2 l/\partial\beta^2\end{bmatrix}^{-1} \quad (26)$$

where l and partial derivatives are given as,

$$l(\alpha,\beta)=\frac{1}{n}\begin{bmatrix}n\ln\alpha+n\ln\beta+(\beta-1)\sum_{i=1}^{n}\ln x_i+\sum_{i=1}^{n}x_i^\beta+\sum_{i=1}^{n}\alpha\left(1-e^{\left(x_i^\beta\right)}\right) \\ -m\ln(\alpha)-m\ln(\beta)\end{bmatrix} \quad (27)$$

$$\frac{\partial^2\ell}{\partial\alpha^2}=\frac{1}{n}\left(-\frac{n}{\alpha^2}+\frac{m}{\alpha^2}\right) \quad (28)$$

$$\frac{\partial^2\ell}{\partial\alpha\partial\beta}=\frac{1}{n}\sum_{i=1}^{n}\left(-x_i^\beta\ ln\,x_i\ exp\left(x_i^\beta\right)\right) \quad (29)$$

$$\frac{\partial^2 \ell}{\partial \beta^2}=\frac{1}{n}\left(-\frac{n}{\beta^2}+\frac{m}{\beta^2}+\sum_{i=1}^{n}\begin{pmatrix}\left(x_i^{\beta}\ln x_i^2\right)\\ +\sum_{i=1}^{n}\left(-\alpha x_i^{\beta}\ln x_i^2 \exp\left(x_i^{\beta}\right)-\alpha\left(x_i^{\beta}\right)^2\ln x_i^2\exp\left(x_i^{\beta}\right)\right)\end{pmatrix}\right) \tag{30}$$

Bayes estimators for α,β parameters using Eq. (25) are found as follows.

i. If $s(\alpha,\beta)=\alpha$

$$\Sigma_1^* = \begin{bmatrix} -\partial^2 l_1^* / \partial\alpha^2 & -\partial^2 l_1^* / \partial\alpha\partial\beta \\ -\partial^2 l_1^* / \partial\alpha\partial\beta & -\partial^2 l_1^* / \partial\beta^2 \end{bmatrix}^{-1} \tag{31}$$

$$\hat{\alpha}_B = \left(\frac{\det\Sigma_1^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(l_1^*\left(\hat{\alpha}_{l_1^*},\hat{\beta}_{l_1^*}\right)-l\left(\hat{\alpha}_l,\hat{\beta}_l\right)\right)\right] \tag{32}$$

where $l_1^*(\alpha,\beta)=\frac{1}{n}\log\alpha + l(\alpha,\beta)$.

The partial derivatives related to l_1^* are given as,

$$\frac{\partial^2 \ell_1^*}{\partial\alpha^2} = -\frac{1}{n\alpha^2}+\frac{1}{n}\left(-\frac{n}{\alpha^2}+\frac{m}{\alpha^2}\right) \tag{33}$$

$$\frac{\partial^2 \ell_1^*}{\partial\alpha\partial\beta} = -\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{\beta}\ln x_i \exp\left(x_i^{\beta}\right)\right) \tag{34}$$

$$\frac{\partial^2 \ell_1^*}{\partial \beta^2}=\frac{1}{n}\begin{pmatrix}-\frac{n}{\beta^2}+\frac{m}{\beta^2}\\ +\sum_{i=1}^{n}\begin{pmatrix}\left(x_i^{\beta}\ln x_i^2\right)\\ +\sum_{i=1}^{n}\left(-\alpha x_i^{\beta}\ln x_i^2 \exp\left(x_i^{\beta}\right)-\alpha\left(x_i^{\beta}\right)^2\ln x_i^2\exp\left(x_i^{\beta}\right)\right)\end{pmatrix}\end{pmatrix} \tag{35}$$

ii. If $s(\alpha,\beta)=\beta$

$$\Sigma_2^* = \begin{bmatrix} -\partial^2 l_2^* / \partial\alpha^2 & -\partial^2 l_2^* / \partial\alpha\partial\beta \\ -\partial^2 l_2^* / \partial\alpha\partial\beta & -\partial^2 l_2^* / \partial\beta^2 \end{bmatrix}^{-1} \tag{36}$$

$$\hat{\beta}_B = \left(\frac{\det\Sigma_2^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(l_2^*\left(\hat{\alpha}_{l_2^*},\hat{\beta}_{l_2^*}\right)-l\left(\hat{\alpha}_l,\hat{\beta}_{ll}\right)\right)\right] \tag{37}$$

where $l_2^*(\alpha,\beta)=\frac{1}{n}\ln\beta + l(\alpha,\beta)$ .The partial derivatives related to l_2^* are given as,

$$\frac{\partial^2 \ell_2^*}{\partial \beta^2}=-\frac{1}{n\beta^2}+\frac{1}{n}\begin{pmatrix}-\frac{n}{\beta^2}+\frac{m}{\beta^2}\\ +\sum_{i=1}^{n}\begin{pmatrix}\left(x_i^{\beta}\ln x_i^2\right)\\ +\sum_{i=1}^{n}\left(-\alpha x_i^{\beta}\ln x_i^2 \exp\left(x_i^{\beta}\right)-\alpha\left(x_i^{\beta}\right)^2\ln x_i^2\exp\left(x_i^{\beta}\right)\right)\end{pmatrix}\end{pmatrix} \tag{38}$$

$$\frac{\partial^2 \ell_2^*}{\partial\alpha\partial\beta} = -\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{\beta}\ln x_i \exp\left(x_i^{\beta}\right)\right) \tag{39}$$

3. Simulation study

In this section, simulation study (based on 10000 replications) is performed to investigate the performance of the ML and Bayes estimators under loss functions in that their estimated risks. ML and approximate Bayes Estimators by Tierney-Kadane's approximation are attained under linex, general, and squared loss functions for Chen distribution. Finally, we obtained results to use Monte Carlo Simulation in the simulation study. It has been taken samples of size n=30, 50, and 100 from Chen Distribution. MSE is defined at follows:

Let θ be the true parameter value and $\hat{\theta}_i$ be the estimation value in i^{th} replication. Then the MSE can be written as,

$$MSE = \frac{1}{10000}\sum_{i=1}^{10000}\left(\hat{\theta}_i - \theta\right)^2 \tag{40}$$

The simulation steps are as follows.

Step 1 : It is generated data from Chen Distribution with $\alpha=0.3, \beta=0.6, d=1.5$, $\alpha=0.5, \beta=0.7, d=0.5$ parameters for the sample size n=30,50,100.

Step 2: ML estimates for α, β are computed by solution of non-linear Eqs.(7-8) by using Newton Raphson Method.

Step 3 : Tierney-Kadane Bayes estimates for α, β parameters under different loss functions.

Step 4 : MSE are computed over 10000 replications by using Eq.(40).

4. Real Data Application

Here we consider the real data of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic Center for the 50 years, from 1959 to 2009. (see the website of Los Angeles Almanac: www.laalmanac.com/ weather/we08aa.htm). This data set has been studied by [16]. This data set has been analyzed to compare the Chen distribution with other distributions such as, Exponential Poisson (EP) [17], ALT-Exponential [18]. Probability density functions of these distributions are given by,

$$f(x)_{ALT-Exp} = \begin{cases} \dfrac{\lambda \exp\left(-\dfrac{x}{\gamma}\right)}{\gamma \log(1+\lambda)\left(1+\lambda\left(1-\exp\left(-\dfrac{x}{\gamma}\right)\right)\right)} I_R(x), & \lambda > 0, \lambda \neq 0 \\ \dfrac{1}{\gamma}\exp\left(-\dfrac{x}{\gamma}\right), & \lambda = 0 \end{cases} \tag{41}$$

$$f(x)_{EP} = \frac{\lambda\beta}{1-\exp(-\lambda)}\exp\left(-\lambda - \beta x + \lambda \exp(-\beta x)\right) \tag{42}$$

The data is given in Table 1:

Table 1. Real data of the amount of annual rainfall (in inches) recorded at the Los Angeles Civic Center

8.180 4.850 18.790 8.380 7.930 13.680 20.440 22.000 16.580 27.470 7.740 12.320 7.170 21.260 14.920 14.350 7.210 12.300 33.440 19.670 26.980 8.960 10.710 31.280 10.430 12.820 17.860 7.660 2.480 8.081 7.350 11.990 21.000 7.360 8.110 24.350 12.440 12.400 31.010 9.090 11.570 17.940 4.420 16.420 9.250 37.960 13.190 3.210 13.530 9.080

AIC values and parameter estimates are given in Table 2.

Table 2. Parameter estimates and AIC values for amount of annual rainfall

Distributions	Parameter Estimations	AIC	-2ℓ
EP	$\hat{\lambda} = 5.6391$ $\hat{\beta} = 0.0139$	376.6237	372.6237
ALT-Exp	$\hat{\lambda} = -0.9659$ $\hat{\gamma} = 6.1265$	354.7464	350.7464
Chen	$\hat{\alpha} = 0.0228$ $\hat{\beta} = 0.4716$	352.2795	348.2795

Furthermore, fitted cdfs plots are presented Figure 2.

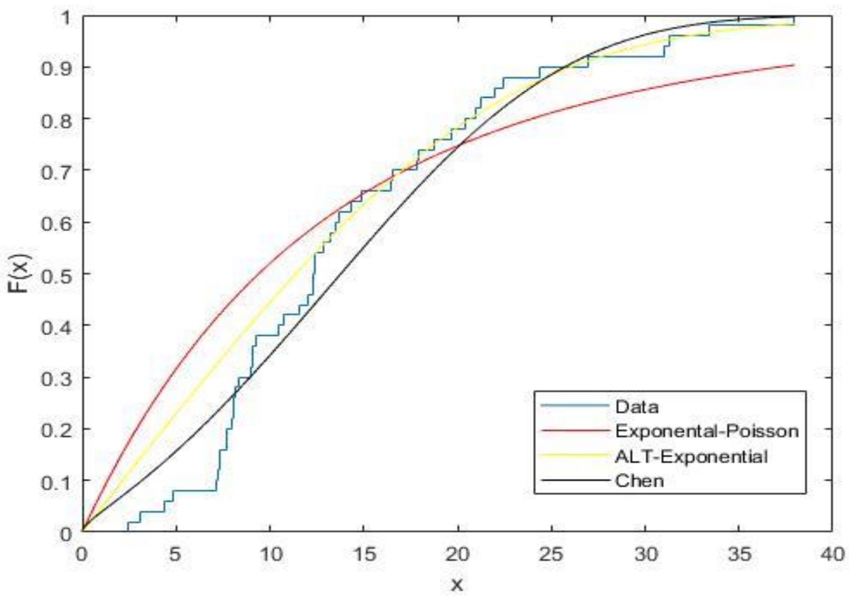

Figure 2. Fitted cdfs plots for amount of annual rainfall

5. Conclusion

As seen from Table 3-4, the performances of Bayes estimates for parameters for linex loss function are better than others regarding MSE's. Also, MSE's of ML and approximate Bayes estimates obtained under different loss functions are decreased when n is increased. Approximate Bayes estimators under LLF, GEL and SEL functions, obtained using the Tierney-Kadane method and ML's for Chen distribution with parameters are investigated. We found that Bayes estimates are superior to the corresponding ML's. The ML's of the unknown two parameters are computed by using the Newton Raphson method. The approximate estimators are compared with the ML's regarding MSE by using Monte Carlo simulation method. As a result, it has been seen that approximate Bayes estimates obtained under linex loss function are better than others. Moreover, a real data application is performed. We have concluded that the Chen distribution has to best fit other distributions according to AIC and -2ℓ .

Table 3. Mean Estimates and Mean Risk of ML's and Bayes Estimates for Chen Distribution (α=0.3,β=0.6,d=1.5)

n	$\hat{\alpha}$ $\hat{\beta}$	*ML*	*Sq*	*Lin*	*Ent*	*ML*	*Sq*	*Lin*	*Ent*
		$k=-0.2, a=-0.3$				$k=-0.2, a=-0.3$			
30	α_{MSE}	0.036496	0.034277	0.007991	0.008813	0.036566	0.034335	0.007285	0.008312
	α_{ME}	0.469334	0.464269	0.463338	0.451248	0.468806	0.463751	0.464725	0.456754
	β_{MSE}	0.045399	0.041710	0.002406	0.003297	0.044800	0.041153	0.002204	0.003034
	β_{ME}	0.722669	0.711246	0.710468	0.705441	0.722169	0.710779	0.711356	0.707676
50	α_{MSE}	0.032192	0.031096	0.007202	0.008886	0.031451	0.030382	0.006492	0.008187
	α_{ME}	0.468114	0.465287	0.464730	0.457545	0.466363	0.463568	0.464125	0.459411
	β_{MSE}	0.021948	0.020653	0.001176	0.001884	0.021787	0.020510	0.001111	0.001783
	β_{ME}	0.677582	0.670943	0.670573	0.667976	0.676676	0.670063	0.670338	0.668473
100	α_{MSE}	0.029581	0.029116	0.006724	0.008983	0.029940	0.029470	0.006307	0.008419
	α_{ME}	0.466930	0.465626	0.465353	0.461798	0.008835	0.466902	0.467179	0.464840
	β_{MSE}	0.008760	0.008454	0.000475	0.000895	0.638284	0.635006	0.635118	0.634328
	β_{ME}	0.638561	0.635295	0.635161	0.634029	0.008835	0.008528	0.000467	0.000886

Table 4. Mean Estimates and Mean Risk of ML's and Bayes Estimates for Chen Distribution (α=0.5,β=0.7,d=0.5)

n	$\hat{\alpha}$ $\hat{\beta}$	*ML*	*Sq*	*Lin*	*Ent*	*ML*	*Sq*	*Lin*	*Ent*
		$k=0.6, a=0.9$				$k=-0.6, a=-0.9$			
30	α_{MSE}	0.039525	0.035079	0.023698	0.097841	0.039666	0.035214	0.027175	0.116184
	α_{ME}	0.307212	0.319024	0.317330	0.303538	0.306991	0.318802	0.320275	0.317991
	β_{MSE}	0.024146	0.022595	0.008706	0.016173	0.023664	0.022156	0.007827	0.015459
	β_{ME}	0.736234	0.723560	0.721717	0.716852	0.735857	0.723191	0.724494	0.722846
50	α_{MSE}	0.036565	0.033940	0.022857	0.086401	0.036230	0.033619	0.025964	0.104714
	α_{ME}	0.312685	0.319802	0.318823	0.310570	0.012901	0.012708	0.004626	0.010706
	β_{MSE}	0.012534	0.012322	0.004575	0.010131	0.313620	0.320735	0.321634	0.320249
	β_{ME}	0.700241	0.692856	0.691968	0.689274	0.698462	0.691099	0.691799	0.690913
100	α_{MSE}	0.032944	0.031668	0.021317	0.074444	0.032971	0.031695	0.024481	0.093671
	α_{ME}	0.320800	0.324391	0.323904	0.319804	0.320732	0.324322	0.324781	0.324081
	β_{MSE}	0.007280	0.007476	0.002693	0.006893	0.007385	0.007579	0.002847	0.007647
	β_{ME}	0.665982	0.662368	0.662005	0.660786	0.665942	0.662327	0.662637	0.662245

ML:Maximum likelihood estimation. Sq:Bayes estimation under squared error loss function, Ent:Bayes estimation under general entropy loss function, Lin:Bayes estimation under linex loss function,

α_{MSE} : *MSEs for* α *parameter*, β_{MSE} : *MSEs for* β *parameter*

α_{ME} : *Mean estimate for* α *parameter* , β_{ME} : *Mean estimate for* β *parameter*

Conflicts of interest

There is no conflict of interest among the authors of the article.

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