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Inverse Nodal Problems for Dirac-Type Integro-Differential System with Boundary Conditions Polynomially Dependent on the Spectral Parameter

Baki KESKİN [1]* ***H. Dilara TEL*** [2]

1,2 Sivas Cumhuriyet University Faculty of Science Department of Mathematics, Sivas, TURKEY

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Abstract. In this work, we study the inverse nodal problem for Dirac type integro-differential operator with the boundary conditions dependent spectral parameter polynomially. We prove that dense subset of the nodal points determines the coefficients of differential part of operator and gives partial information for integral part of it.

Keywords: Dirac operator, Integro-differential operators, Inverse nodal problem, Parameter dependent boundary conditions.

Sınır Koşulları Spektral Parametreye Polinom Şeklinde Bağlı Olan Dirac Tipli İntegro-Diferansiyel Sistemi İçin Ters Nodal Problemler

Özet. Bu çalışmada sınır koşulunun spektral parametreye polinom şeklinde bağlı olduğu Dirac tipli integro diferansiyel operatörü ele aldık ve nodal noktaların yoğun bir alt kümesinin operatörün diferansiyel kısmının katsayılarını belirlediğini, integral parçası için de kısmi bilgi verdiğini gösterdik.

Anahtar Kelimeler: Dirac operatörü, integro diferansiyel operator, ters nodal problem, parametreye bağlı sınır koşulu.

1. INTRODUCTION

We consider the following boundary value problem $L(\Omega, M, a_1, a_2, b_1, b_2)$, generated by the Dirac-type integro-differential system

$$BY'(x) + \Omega(x)Y(x) + \int_0^x M(x,t)Y(t)dt = \lambda Y(x), \qquad x \in (0,\pi) \qquad (1)$$

with the spectral parameter dependent boundary conditions

$$\begin{cases} a_1(\lambda)y_1(0) + a_2(\lambda)y_2(0) = 0 \\ b_1(\lambda)y_1(\pi) + b_2(\lambda)y_2(\pi) = 0 \end{cases} \qquad (2)$$

* Corresponding author. *Email address: bkeskin@cumhuriyet.edu.tr*

where $a_v(\lambda)=\sum_{k=0}^{s_v} a_{vk}\lambda^k$ and $b_v(\lambda)=\sum_{k=0}^{r_v} b_{vk}\lambda^k$, $(v=1,2)$ are monic polynomial with real cofficients; $a_1(\lambda)$ and $a_2(\lambda)$ have no common zeros and same for $b_1(\lambda)$ and $b_2(\lambda)$, λ is the spectral parameter, $B=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}$, $\Omega(x)=\begin{pmatrix}V(x)+m & 0\\ 0 & V(x)-m\end{pmatrix}$, $M(x,t)=\begin{pmatrix}\chi_{11}(x,t) & \chi_{12}(x,t)\\ \chi_{21}(x,t) & \chi_{22}(x,t)\end{pmatrix}$, $Y(x)=\begin{pmatrix}y_1(x)\\ y_2(x)\end{pmatrix}$, $\Omega(x)$ and $M(x,t)$ are real-valued functions in the class of $W_2^1(0,\pi)$ where m is a real constant. Throughout this paper, we denote $p(x)=V(x)+m$, $r(x)=V(x)-m$.

Inverse nodal problems were first proposed and solved for Sturm-Liouville operator by McLaughlin in 1988 [1]. In this study, it has been shown that a dense subset of zeros of eigenfunctions, called nodal points, uniquely determines the potential of the Sturm Liouville operator. In 1989, Hald and McLaughlin gave some numerical schemes for reconstructing potential from nodal points for more general boundary conditions [2]. In 1997 Yang gave an algorithm to determine the coefficients of operator for the inverse nodal Sturm-Liouville problem [3]. Inverse nodal problems have been addressed by various researchers in several papers for different operators [4], [5], [6], [7], [8], [9], [10], [11] and [12]. The inverse nodal problems for Dirac operators with various boundary conditions have been studied and shown that the dense subsets of nodal points which are the first components of the eigenfunctions determines the coefficients of discussed operator by Yang C-F, Huang Z-Y [13]; Yang C-F, Pivovarchik VN [14] and Guo Y, Wei Y [15].

In recent years, perturbation of a differential operator by a Volterra type integral operator, namely the integro-differential operator has acquired significant popularity and major attention from several authors and take significant place in the literature [16], [17], [18], [19] *and* [20]. Integro-differential operators are nonlocal, and therefore they are more difficult for investigation, than local ones. New methods for solution of these problems are being developed. For Sturm-Liouville type integro-differential operators, there exist some studies about inverse problems but there is very little study for Dirac type integro-differential operators. The inverse nodal problem for Dirac type integro-differential operators was first studied by [21]. In their study, it is shown that the coefficients of the differential part of the operator can be determined by using nodal points and nodal points also gives the partial information about integral part. In [22] the authors considered boundary conditions depend on the spectral parameter linearly. In our study, we deal with an inverse nodal problem of reconstructing the Dirac type integro-differential operators with the spectral parameter in the boundary conditions polynomially. We have obtained asymptotic estimates of the solutions, eigenvalues and nodal points of considered problem. We have proved that the operator can be reconstructed by given dense subset of the nodal points.

2. MAIN RESULTS

Let $\varphi(x,\lambda)=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}$ be the solution of (1) under the the initial condition $\varphi(0,\lambda)=\begin{pmatrix}a_2(\lambda)\\ -a_1(\lambda)\end{pmatrix}$. It is easy to see that this solution is an entire function of λ for each fixed x and t. One can easily verify that the function $\varphi(x,\lambda)$ satisfies

$$\varphi_1(x,\lambda)=a_1(\lambda)sin\lambda x+a_2(\lambda)cos\lambda x$$

$$+\int_0^x p(t)\varphi_1(t)sin\lambda(x-t)dt+\int_0^x r(t)\varphi_2(t)cos\lambda(x-t)dt$$

$$+\int_0^x\int_0^t \{M_{11}(t,\xi)\varphi_1(\xi)+M_{12}(t,\xi)\varphi_2(\xi)\}sin\lambda(x-t)d\xi dt$$

$$+\int_0^x\int_0^t \{M_{21}(t,\xi)\varphi_1(\xi)+M_{12}t,\xi)\varphi_2(\xi)\}cos\lambda(x-t)\}d\xi dt$$

$$\varphi_2(x,\lambda)=-a_1(\lambda)cos\lambda x+a_2(\lambda)(\lambda)sin\lambda x$$

$$-\int_0^x p(t)\varphi_1(t)cos\lambda(x-t)dt+\int_0^x r(t)\varphi_2(t)sin\lambda(t-x)dt$$

$$-\int_0^x\int_0^t \{M_{11}(t,\xi)\varphi_1(\xi)+M_{12}(t,\xi)\varphi_2(\xi)\}cos\lambda(x-t)d\xi dt$$

$$+\int_0^x\int_0^t \{M_{21}(t,\xi)\varphi_1(\xi)+M_{12}t,\xi)\varphi_2(\xi)\}sin\lambda(x-t)d\xi dt$$

Theorem1. The functions $\varphi_1(x,\lambda)$ and $\varphi_2(x,\lambda)$ have the following asymptotic expansions:

$$\varphi_1(x,\lambda)=a_1(\lambda)sin[\lambda x-w(x)]+a_2(\lambda)cos[\lambda x-w(x)] \quad (3)$$

$$+\frac{a_1(\lambda)m}{\lambda}sin[\lambda x-w(x)]-\frac{a_1(\lambda)m^2x}{2\lambda}cos[\lambda x-w(x)]$$

$$+\frac{a_2(\lambda)m^2x}{2\lambda}sin[\lambda x-w(x)]-\frac{a_1(\lambda)}{2\lambda}K(x)sin[\lambda x-w(x)]$$

$$-\frac{a_2(\lambda)}{2\lambda}K(x)cos[\lambda x-w(x)]+\frac{a_1(\lambda)}{2\lambda}L(x)cos[\lambda x-w(x)]$$

$$-\frac{a_2(\lambda)}{2\lambda}L(x)sin[\lambda x-w(x)]+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

$$\varphi_2(x,\lambda)=-a_1(\lambda)cos[\lambda x-w(x)]+a_2(\lambda)sin[\lambda x-w(x)] \quad (4)$$

$$-\frac{a_2(\lambda)m}{\lambda}sin[\lambda x-w(x)]-\frac{a_1(\lambda)m^2x}{2\lambda}sin[\lambda x-w(x)]$$

$$-\frac{a_2(\lambda)m^2x}{2\lambda}cos[\lambda x-w(x)]+\frac{a_1(\lambda)}{2\lambda}K(x)cos[\lambda x-w(x)]$$

$$-\frac{a_2(\lambda)}{2\lambda}K(x)sin[\lambda x-w(x)]+\frac{a_1(\lambda)}{2\lambda}L(x)sin[\lambda x-w(x)]$$

$$+\frac{a_2(\lambda)}{2\lambda}L(x)cos[\lambda x-w(x)]+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

for sufficiently large $|\lambda|$, uniformly in x, where, $w(x)=\frac{1}{2}\int_0^x(p(t)+r(t))dt$,
$K(x)=\int_0^x(M_{11}(t,t)+M_{22}(t,t))dt$, $L(x)=\int_0^x(M_{12}(t,t)-M_{21}(t,t))dt$ and $\tau=Im\lambda$

Proof To apply the method of successive approximations to (4) and (5), put

$$\varphi_{1,0}(x,\lambda)=a_1(\lambda)sin\lambda x+a_2(\lambda)cos\lambda x$$

$$\varphi_{2,0}(x,\lambda)=-a_1(\lambda)cos\lambda x+a_2(\lambda)sin\lambda x$$

$$\varphi_{1,n+1}(x,\lambda)=\int_0^x p(t)\varphi_{1,n}(t)sin\lambda(x-t)dt+\int_0^x r(t)\varphi_{2,n}(t)cos\lambda(x-t)dt$$

$$+\int_0^x\int_0^t\{M_{11}(t,\xi)\varphi_{1,n}(\xi)+M_{12}(t,\xi)\varphi_{2,n}(\xi)\}sin\lambda(x-t)d\xi dt$$

$$+\int_0^x\int_0^t\{M_{21}(t,\xi)\varphi_{1,n}(\xi)+M_{22}(t,\xi)\varphi_{2,n}(\xi)\}cos\lambda(x-t)d\xi dt$$

$$\varphi_{2,n+1}(x,\lambda)=-\int_0^x p(t)\varphi_{1,n}(t)cos\lambda(x-t)dt+\int_0^x r(t)\varphi_{2,n}(t)sin\lambda(t-x)dt$$

$$-\int_0^x\int_0^t\{M_{11}(t,\xi)\varphi_{1,n}(\xi)+M_{12}(t,\xi)\varphi_{2,n}(\xi)\}cos\lambda(x-t)d\xi dt$$

$$+\int_0^x\int_0^t\{M_{21}(t,\xi)\varphi_{1,n}(\xi)+M_{22}t,\xi)\varphi_{2,n}(\xi)\}sin\lambda(x-t)d\xi dt$$

Then we have

$$\varphi_{1,1}(x,\lambda)=-a_1(\lambda)w(x)cos\lambda x+a_2(\lambda)w(x)sin\lambda x$$

$$+\frac{a_1(\lambda)m}{\lambda}sin\lambda x-\frac{a_1(\lambda)}{2\lambda}K(x)sin\lambda x$$

$$+\frac{a_1(\lambda)}{2\lambda}L(x)cos\lambda x-\frac{a_2(\lambda)}{2\lambda}K(x)cos\lambda x$$

$$-\frac{a_2(\lambda)}{2\lambda}L(x)sin\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

$$\varphi_{2,1}(x,\lambda)=-a_1(\lambda)w(x)sin\lambda x-a_2(\lambda)w(x)cos\lambda x$$

$$-\frac{a_2(\lambda)m}{\lambda}sin\lambda x+\frac{a_1(\lambda)}{2\lambda}K(x)cos\lambda x$$

$$+\frac{a_1(\lambda)}{2\lambda}L(x)sin\lambda x-\frac{a_2(\lambda)}{2\lambda}K(x)sin\lambda x$$

$$+\frac{a_2(\lambda)}{2\lambda}L(x)cos\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

and for $n\in\mathbb{Z}^+$

$$\varphi_{1,2n+1}(x,\lambda)=(-1)^{n+1}a_1(\lambda)\frac{w^{2n+1}(x)}{(2n+1)!}cos\lambda x+(-1)^n a_2(\lambda)\frac{w^{2n+1}(x)}{(2n+1)!}sin\lambda x$$

$$+(-1)^n\frac{a_1(\lambda)m}{\lambda}\frac{w^{2n}(x)}{(2n)!}sin\lambda x+(-1)^n\frac{a_1(\lambda)m^2x}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}sin\lambda x$$

$$+(-1)^n\frac{a_2(\lambda)m^2x}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}cos\lambda x+(-1)^{n+1}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}K(x)sin\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}K(x)cos\lambda x+(-1)^n\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}L(x)cos\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}L(x)sin\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

$$\varphi_{1,2n}(x,\lambda)=(-1)^n\frac{w^{2n}(x)}{(2n)!}sin\lambda x+(-1)^n\frac{w^{2n}(x)}{(2n)!}cos\lambda x$$

$$+(-1)^n\frac{a_1(\lambda)m}{\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}cos\lambda x+(-1)^n\frac{a_1(\lambda)m^2x}{2\lambda}\frac{w^{2n-2}(x)}{(2n-2)!}cos\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)m^2x}{2\lambda}\frac{w^{2n-2}(x)}{(2n-2)!}sin\lambda x+(-1)^{n+1}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}K(x)cos\lambda x$$

$$+(-1)^{n}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}K(x)sin\lambda x+(-1)^{n+1}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}L(x)sin\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}L(x)cos\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

$$\varphi_{2,2n+1}(x,\lambda)=(-1)^{n+1}\frac{w^{2n+1}(x)}{(2n+1)!}sin\lambda x+(-1)^{n+1}\frac{w^{2n+1}(x)}{(2n+1)!}cos\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)m}{\lambda}\frac{w^{2n}(x)}{(2n)!}sin\lambda x+(-1)^{n+1}\frac{a_1(\lambda)m^2x}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}cos\lambda x$$

$$+(-1)^{n}\frac{a_2(\lambda)m^2x}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}sin\lambda x+(-1)^{n}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}K(x)cos\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}K(x)sin\lambda x+(-1)^{n}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}L(x)sin\lambda x$$

$$+(-1)^{n}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n}(x)}{(2n)!}L(x)cos\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

$$\varphi_{2,2n}(x,\lambda)=(-1)^{n+1}a_1(\lambda)\frac{w^{2n}(x)}{(2n)!}cos\lambda x+(-1)^{n}a_2(\lambda)\frac{w^{2n}(x)}{(2n)!}sin\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)m}{\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}cos\lambda x+(-1)^{n}\frac{a_1(\lambda)m^2x}{2\lambda}\frac{w^{2n-2}(x)}{(2n-2)!}sin\lambda x$$

$$+(-1)^{n}\frac{a_2(\lambda)m^2x}{2\lambda}\frac{w^{2n-2}(x)}{(2n-2)!}cos\lambda x+(-1)^{n+1}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}K(x)sin\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}K(x)cos\lambda x+(-1)^{n}\frac{a_1(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}L(x)cos\lambda x$$

$$+(-1)^{n+1}\frac{a_2(\lambda)}{2\lambda}\frac{w^{2n-1}(x)}{(2n-1)!}L(x)sin\lambda x+o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

for sufficiently large $|\lambda|$, uniformly inx. Hence, the proof of the theorem1 is completed by successive approximations method.

For defineteness, below we suppose $s_1=s_2=s$ and $r_1=r_2=r$. The other cases can be treated similarly.

Define the entire function $\Delta(\lambda)$ by

$$\Delta(\lambda)=b_1(\lambda)\varphi_1(\pi,\lambda)+b_2(\lambda)\varphi_2(\pi,\lambda), \tag{5}$$

this function is called the characteristic function of the problem (1)-(3) and the zeros $\{\lambda_n\}_{n\in\mathbb{Z}}$. (counting with algebraic multipicities) coincide with the eigenvalues of the problem (1)-(3). The spectrum of the considered problem consist of eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}}$ up to $o(n^{-1})$. From (3) and (4), we get the characteristic function $\Delta(\lambda)$ has the following asymptotic relation for sufficiently large $|\lambda|$,

$$\Delta(\lambda)=\lambda^{s+r}\{2sin(\lambda\pi-w(\pi))-\frac{m^2\pi}{\lambda}cos(\lambda\pi-w(\pi)) \tag{6}$$

$$+\frac{L(\pi)}{\lambda}cos(\lambda\pi-w(\pi))-\frac{K(\pi)}{\lambda}sin(\lambda\pi-w(\pi))+o\left(\frac{e^{|\tau|\pi}}{\lambda}\right)\}$$

Using this asymptotic relation, for sufficiently large $n>0$, we get

$$\lambda_n = n - r - s + \frac{w(\pi)}{\pi} + \frac{m^2 - L(\pi)}{2n} + o\left(\frac{1}{n}\right) \quad (7)$$

Similarly, for n ≥ 1,

$$\lambda_{-n} = -n + \frac{w(\pi)}{\pi} - \frac{m^2 - L(\pi)}{2n} + o\left(\frac{1}{n}\right) \quad (8)$$

Denote the algebraic multiplicity of $\lambda_n \in \mathbb{Z}$ by ρ_n , then by virtue of (7) and (8) we have $\rho_n = 1$, for sufficiently large $|n|$.

Theorem2. For sufficiently large $|n|$, the first component $\varphi_1(x,\lambda_n)$of the eigenfunction $\varphi(x,\lambda_n)$ has exactly $n-r-s$ nodes ${x_n}^j$ $(j = 0, \dots, n-r-s-1)$ in the interval $(0,\pi)$ i.e.,

$0<{x_n}^0 < {x_n}^1 < \cdots < {x_n}^{n-r-s-1}<\pi$. The numbers $\{{x_n}^j\}$ satisfy the following asymptotic formula:

$$x_n{}^j = \frac{j\pi}{n-r-s} + \frac{w({x_n}^j) - 1}{n-r-s} - \frac{w(\pi)}{(n-r-s)\pi}\frac{j\pi}{n-r-s}$$

$$-\frac{(w({x_n}^j) - 1)w(\pi)}{(n-r-s)^2\pi} + \frac{m + m^2{x_n}^j - L({x_n}^j)}{(n-r-s)^2} + O\left(\frac{1}{n^3}\right)$$

Proof From (3), we can write

$$\varphi_1(x,\lambda_n) = a_1(\lambda_n)sin[\lambda_n x - w(x)] + a_2(\lambda_n)cos[\lambda_n x - w(x)]$$

$$+\frac{a_1(\lambda_n)m}{\lambda_n}sin[\lambda_n x - w(x)] - \frac{a_1(\lambda_n)m^2x}{2\lambda_n}cos[\lambda_n x - w(x)]$$

$$+\frac{a_2(\lambda_n)m^2x}{2\lambda_n}sin[\lambda_n x - w(x)] - \frac{a_1(\lambda_n)}{2\lambda_n}K(x)sin[\lambda_n x - w(x)]$$

$$-\frac{a_2(\lambda_n)}{2\lambda_n}K(x)cos[\lambda_n x - w(x)] + \frac{a_1(\lambda_n)}{2\lambda_n}L(x)cos[\lambda_n x - w(x)]$$

$$-\frac{a_2(\lambda_n)}{2\lambda_n}L(x)sin[\lambda_n x - w(x)] + o\left(\frac{e^{|\tau|x}}{\lambda^{1-s}}\right)$$

which is equivalent to

$$\varphi_1(x,\lambda_n) = (\lambda_n)^s\{\sin[\lambda_n x - w(x)] + \cos[\lambda_n x - w(x)] + \frac{m}{\lambda_n}\sin[\lambda_n x - w(x)]]$$

$$-\frac{m^2x}{2\lambda_n}cos[\lambda_n x - w(x)] + \frac{m^2x}{2\lambda_n}sin[\lambda_n x - w(x)]$$

$$-\frac{1}{2\lambda_n}K(x)sin[\lambda_n x - w(x)] \; -\frac{1}{2\lambda_n}K(x)cos[\lambda_n x - w(x)]$$

$$+\frac{1}{2\lambda_n}L(x)cos[\lambda_n x-w(x)]-\frac{1}{2\lambda_n}L(x)sin[\lambda_n x-w(x)]+o\left(\frac{e^{|\tau|x}}{\lambda_n}\right)$$

for sufficiently large |n|. From $\varphi_1(x_n{}^j,\lambda_n)=0$, we get

$$sin[\lambda_n x_n{}^j-w(x_n{}^j)]+cos[\lambda_n x_n{}^j-w(x_n{}^j)]+\frac{m}{\lambda_n}sin[\lambda_n x_n{}^j-w(x_n{}^j)]$$

$$-\frac{m^{2x_n{}^j}}{2\lambda_n}cos[\lambda_n x_n{}^j-w(x_n{}^j)]+\frac{m^{2x_n{}^j}}{2\lambda_n}sin[\lambda_n x_n{}^j-w(x_n{}^j)]$$

$$-\frac{1}{2\lambda_n}K(x_n{}^j)sin[\lambda_n x_n{}^j-w(x_n{}^j)]-\frac{1}{2\lambda_n}K(x_n{}^j)cos[\lambda_n x_n{}^j-w(x_n{}^j)]$$

$$+\frac{1}{2\lambda_n}L(x_n{}^j)cos[\lambda_n x_n{}^j-w(x_n{}^j)]-\frac{1}{2\lambda_n}L(x_n{}^j)sin[\lambda_n x_n{}^j-w(x_n{}^j)]+o\left(\frac{e^{|\tau|x}}{\lambda_n}\right)=0$$

divide both sides by $cos[\lambda_n x_n{}^j-w(x_n{}^j)$, we get

$$tan[\lambda_n x_n{}^j-w(x_n{}^j)]+1+\frac{m}{\lambda_n}tan[\lambda_n x_n{}^j-w(x_n{}^j)]-\frac{m^{2x_n{}^j}}{2\lambda_n}$$

$$+\frac{m^2x_n{}^j}{2\lambda_n}tan[\lambda_n x_n{}^j-w(x_n{}^j)]-\frac{1}{2\lambda_n}K(x_n{}^j)tan[\lambda_n x_n{}^j-w(x_n{}^j)]-\frac{1}{2\lambda_n}K(x_n{}^j)$$

$$+\frac{1}{2\lambda_n}L(x_n{}^j)-\frac{1}{2\lambda_n}L(x_n{}^j)tan[\lambda_n x_n{}^j-w(x_n{}^j)]+o\left(\frac{e^{|\tau|x}}{\lambda_n}\right)=0,$$

then, we get

$$tan[\lambda_n x_n{}^j-w(x_n{}^j)]=\left(1+\frac{m}{\lambda_n}+\frac{m^2x_n{}^j}{2\lambda_n}-\frac{K(x_n{}^j)}{2\lambda_n}-\frac{L(x_n{}^j)}{2\lambda_n}\right)^{-1}\times$$

$$\times\left(-1+\frac{m^2x_n{}^j}{2\lambda_n}+\frac{K(x_n{}^j)}{2\lambda_n}-\frac{L(x_n{}^j)}{2\lambda_n}+o\left(\frac{e^{|\tau|x}}{\lambda_n}\right)\right),$$

Taylor formula for the function arctangent yields

$$x_n{}^j=\frac{1}{\lambda_n}\left(j\pi+w(x_n{}^j)-1+\frac{m}{\lambda_n}+\frac{m^2x_n{}^j}{\lambda_n}-\frac{L(x_n{}^j)}{\lambda_n}+o\left(\frac{1}{\lambda_n}\right)\right).$$

If we put

$$\lambda_n{}^{-1}=\frac{1}{n-r-s}\left\{1-\frac{w(\pi)}{(n-r-s)\pi}+o\left(\frac{1}{n}\right)\right\};\ \frac{1}{\lambda_n{}^2}=\frac{1}{(n-r-s)^2}\left\{1-\frac{2w(\pi)}{(n-r-s)\pi}+o\left(\frac{1}{n}\right)\right\}$$

then we get

$$x_n{}^j = \frac{j\pi}{n-r-s} + \frac{w(x_n{}^j)-1}{n-r-s} - \frac{w(\pi)}{(n-r-s)\pi}\frac{j\pi}{n-r-s}$$

$$-\frac{(w(x_n{}^j)-1)w(\pi)}{(n-r-s)^2\pi} + \frac{m+m^2x_n{}^j - L(x_n{}^j)}{(n-r-s)^2} + O\left(\frac{1}{n^3}\right)$$

Fix x ∈ $(0,\pi)$. Let X be the set of nodal points. One can choose a sequence $x_n{}^j \subset X$ such that $x_n{}^j$converges to x. Then the following limits are exist and finite:

$$f(x) := \lim_{|n|\to\infty}\left(x_n{}^j - \frac{j\pi}{n-r-s}\right)(n-r-s) = w(x_n{}^j) - 1 + \frac{w(\pi)}{\pi}x \qquad (9)$$

and

$$g(x) := \lim_{|n|\to\infty}\left(x_n{}^j - \frac{j\pi}{n-r-s} - \frac{w(x_n{}^j)-1}{n-r-s} + \frac{w(\pi)}{(n-r-s)\pi}\frac{j\pi}{n-r-s}\right)(n-r-s)^2 \qquad (10)$$

$$= -\frac{(w(x_n{}^j)-1)w(\pi)}{\pi} + m + m^2x_n{}^j - L(x_n{}^j)$$

Now, we can formulate the following uniqueness theorem and establish a constructive procedure for reconstructing the potential of the considered problem. Without loss of generality, we assume

$$L(\pi) = \int_0^\pi (\chi_{12}(t,t) - \chi_{21}(t,t))dt = 0$$

Theorem3. The given dense subset of nodal set X uniquely determines the potential V(x) of the problem, the function $L'(x) = \chi_{12}(x,x) - \chi_{21}(x,x)$ of the partial imformation of the integral part, a.e. on $(0,\pi)$. Moreover, $V(x), L'(x)$, and m can be constructed by the following algorithm:

(1) fix $x \in (0,\pi)$, choose a sequence $\left(x_n{}^{j(n)}\right) \subset$ X such that $\lim_{|n|\to\infty} x_n{}^{j(n)} = x$;

(2) find the function $f(x)$ via (9) and calculate

$$w(\pi) = \frac{f(\pi)+1}{2}$$

$$V(x) = f'(x) - \frac{f(\pi)+1}{2\pi}$$

(3) find the function $g(x)$ via (10) and calculate

$$m = g(0) - \frac{f(\pi)+1}{2\pi}$$

$$L'(x) = -g'(x) - \frac{V(x)}{\pi} + m^2$$

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