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# Multiplicative Mappings of Gamma Rings

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**Abstract.** Let $\mathfrak{M}_i$ and $\Gamma_i$ $(i = 1,2)$ be abelian groups such that $\mathfrak{M}_i$ is a $\Gamma_i$-ring. An ordered pair $(\varphi, \phi)$ of mappings is called a *multiplicative isomorphism* of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ if they satisfy the following properties: *(i)* $\varphi$ is a bijective mapping from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$, *(ii)* $\phi$ is a bijective mapping from $\Gamma_1$ onto $\Gamma_2$ and *(iii)* $\varphi(x\gamma y) = \varphi(x)\phi(\gamma)\varphi(y)$ for every $x, y \in \mathfrak{M}_1$ and $\gamma \in \Gamma_1$. We say that the ordered pair $(\varphi, \phi)$ of mappings is *additive* when $\varphi(x + y) = \varphi(x) + \varphi(y)$, for all $x, y \in \mathfrak{M}_1$. In this paper we establish conditions on $\mathfrak{M}_1$ that assures that $(\varphi, \phi)$ is additive.

***Keywords***: Multiplicative mappings, Additivity, Gamma rings.

## Gamma Halkalarında Çarpımsal Dönüşümler

**Özet.** $\mathfrak{M}_i$ ve $\Gamma_i$ $(i = 1,2)$ değiştirmeli grup ve $\mathfrak{M}_i$ bir $\Gamma_i$-halka olsun. Aşağıdaki özellikler sağlanırsa dönüşümlerin $(\varphi, \phi)$ sıralı ikilisine $\mathfrak{M}_1$ den $\mathfrak{M}_2$ üzerine çarpımsal izomorfizm denir: *(i)* $\varphi$, $\mathfrak{M}_1$ den $\mathfrak{M}_2$, üzerine bijektif dönüşümdür. *(ii)* $\phi$, $\Gamma_1$ den $\Gamma_2$ üzerine bijektif dönüşümdür. *(iii)* Her $x, y \in \mathfrak{M}_1$ ve $\gamma \in \Gamma_1$.için $\varphi(x\gamma y) = \varphi(x)\phi(\gamma)\varphi(y)$ dir ve Her $x, y \in \mathfrak{M}_1$ için $\varphi(x + y) = \varphi(x) + \varphi(y)$, olduğunda dönüşümlerin $(\varphi, \phi)$ sıralı ikilisine toplamsaldır denir. Bu makalede $\mathfrak{M}_1$ üzerinde $(\varphi, \phi)$ nin toplamsallığını garanti edecek koşulları vereceğiz.

***Anahtar Kelimeler***: Çarpımsal dönüşümleri, Toplamsallık, Gamma halkaları.

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## 1. INTRODUCTION AND PRELIMINARIES

N. Nobusawa [1] introduced the concept of a $\Gamma$-ring which is called the $\Gamma$-ring in the sense of Nobusawa. He obtained an analogue of the Wedderburn's Theorem for $\Gamma$-rings with minimum condition on left ideals. W. E. Barnes [2] gave the definition of a $\Gamma$-ring as a generalization of a ring and he also developed some other concepts of $\Gamma$-rings such as $\Gamma$-homomorphism, prime and primary ideals, m-systems etc. $\Gamma$-rings are closely related to others ternary structures as ternary algebras, associative triple systems and associative pairs, which have been extensively studied see [3], [4] and [5].

Let $\mathfrak{M}$ and $\Gamma$ be two abelian groups. If there exists a mapping $\mathfrak{M} \times \Gamma \times \mathfrak{M} \to \mathfrak{M}$ (the image of $(x, \alpha, y)$ is denoted by $x\alpha y$ where $x, y \in \mathfrak{M}$ and $\alpha \in \Gamma$). We call $\mathfrak{M}$ a *$\Gamma$-ring* if the following conditions are satisfied:

1. $x\alpha y \in \mathfrak{M}$,

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2. $(x+y)\alpha z = x\alpha z + y\alpha z,\ x\alpha(y+z) = x\alpha y + x\alpha z,$

3. $x(\alpha+\beta)y = x\alpha y + x\beta y,$

4. $(x\alpha y)\beta z = x\alpha(y\beta z),$

for all $x, y, z \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$.

A nonzero element $1 \in \mathfrak{M}$ is called a multiplicative $\gamma$-identity of $\mathfrak{M}$ or *$\gamma$-unity element* (for some $\gamma \in \Gamma$) if $1\gamma x = x\gamma 1 = x$ for all $x \in \mathfrak{M}$. A nonzero element $e_1 \in \mathfrak{M}$ is called a *$\gamma_1$-idempotent* (for some $\gamma_1 \in \Gamma$) if $e_1\gamma_1 e_1 = e_1$ and a *nontrivial $\gamma_1$-idempotent* if it is a $\gamma_1$-idempotent different from multiplicative $\gamma_1$-identity element of $\mathfrak{M}$.

Let $\Gamma$ and $\mathfrak{M}$ be two abelian groups such that $\mathfrak{M}$ is a $\Gamma$-ring and $e_1 \in \mathfrak{M}$ a nontrivial $\gamma_1$-idempotent. Let us consider $e_2 : \Gamma \times \mathfrak{M} \to \mathfrak{M}$ and $e'_2 : \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps verifying the conditions $e_2(\gamma_1, a) = a - e_1\gamma_1 a$ and $e_{2'}(a, \gamma_1) = a - a\gamma_1 e_1$. Let us denote $e_2\alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_{2'}(a, \alpha)$, $1_1\alpha a = e_1\alpha a + e_2\alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$ and suppose $(a\alpha e_2)\beta b = a\alpha(e_2\beta b)$ for all $a, b \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$. Then $1_1\gamma_1 a = a\gamma_1 1_1 = a$ and $(a\alpha 1_1)\beta b = a\alpha(1_1\beta b)$, for all $a, b \in \mathfrak{M}$ and $\alpha, \beta \in \Gamma$, allowing us to write $1_1 = e_1 + e_2$ and $\mathfrak{M}$ as a direct sum of subgroups $\mathfrak{M} = \mathfrak{M}_{11} \oplus \mathfrak{M}_{12} \oplus \mathfrak{M}_{21} \oplus \mathfrak{M}_{22}$, where $\mathfrak{M}_{ij} = e_i\gamma_1\mathfrak{M}\gamma_1 e_j$ $(i, j = 1,2)$, called *Peirce decomposition* of $\mathfrak{M}$ relative to $e_1$, satisfying the multiplicative relations:

1. $\mathfrak{M}_{ij}\Gamma\mathfrak{M}_{kl} \subseteq \mathfrak{M}_{il}$ $(i, j, k, l = 1,2)$;
2. $\mathfrak{M}_{ij}\gamma_1\mathfrak{M}_{kl} = 0$ if $j \neq k$ $(i, j, k, l = 1,2)$.

For the reader interested in the Peirce decomposition of $\Gamma$-rings we indicate [6]. If $\mathfrak{A}$ and $\mathfrak{B}$ are subsets of a $\Gamma$-ring $\mathfrak{M}$ and $\Theta \subseteq \Gamma$, we denote $\mathfrak{A}\Theta\mathfrak{B}$ the subset of $\mathfrak{M}$ consisting of all finite sums of the form $\sum_i a_i\gamma_i b_i$ where $a_i \in \mathfrak{A}$, $\gamma_i \in \Theta$ and $b_i \in \mathfrak{B}$. A *right ideal (resp., left ideal)* of a $\Gamma$-ring $\mathfrak{M}$ is an additive subgroup $\mathfrak{I}$ of $\mathfrak{M}$ such that $\mathfrak{I}\Gamma\mathfrak{M} \subseteq \mathfrak{I}$ (resp., $\mathfrak{M}\Gamma\mathfrak{I} \subseteq \mathfrak{I}$). If $\mathfrak{I}$ is both a right and a left ideal of $\mathfrak{M}$, then we say that $\mathfrak{I}$ is an *ideal* or *two-side ideal* of $\mathfrak{M}$.

An ideal $\mathfrak{P}$ of a $\Gamma$-ring $\mathfrak{M}$ is called *prime* if for any ideals $\mathfrak{A}, \mathfrak{B} \subseteq \mathfrak{M}$, $\mathfrak{A}\Gamma\mathfrak{B} \subseteq \mathfrak{P}$ implies that $\mathfrak{A} \subseteq \mathfrak{P}$ or $\mathfrak{B} \subseteq \mathfrak{P}$. A $\Gamma$-ring $\mathfrak{M}$ is said to be *prime* if the zero ideal is prime.

**Theorem 1.1** *[7, Theorem 4] If $\mathfrak{M}$ is a $\Gamma$-ring, the following conditions are equivalent:*

1. $\mathfrak{M}$ is a prime $\Gamma$-ring;
2. if $a, b \in \mathfrak{M}$ and $a\Gamma\mathfrak{M}\Gamma b = 0$, then $a = 0$ or $b = 0$.

Let $\mathfrak{M}_i$ and $\Gamma_i$ $(i = 1,2)$ be abelian groups such that $\mathfrak{M}_i$ is a $\Gamma_i$-ring $(i = 1,2)$. An ordered pair $(\varphi, \phi)$ of mappings is called a *multiplicative isomorphism* of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ if they satisfy the following properties:

1. $\varphi$ is a bijective mapping from $\mathfrak{M}_1$ onto $\mathfrak{M}_2$;
2. $\phi$ is a bijective mapping from $\Gamma_1$ onto $\Gamma_2$;
3. $\varphi(x\gamma y) = \varphi(x)\phi(\gamma)\varphi(y)$ for all $x, y \in \mathfrak{M}_1$ and $\gamma \in \Gamma_1$.

We say that a multiplicative isomorphism $(\varphi, \phi)$ of $\mathfrak{M}_1$ onto $\mathfrak{M}_2$ is *additive* when $\varphi(x+y) = \varphi(x) + \varphi(y)$ for all $x, y \in \mathfrak{M}_1$.

## 2. GAMMA RINGS AND THE MULTIPLICATIVE ISOMORPHISMS

The study of the question of when a multiplicative isomorphism is additive has become an active research area in associative ring theory. In this case, one often tries to establish conditions on the ring which assures the additivity of every multiplicative isomorphism defined on it. The first result in this direction is due to Martindale III [8] who obtained a pioneer result in 1969, where in his condition requires that the ring possesses idempotents. In recent papers [9],[10] Ferreira has studied the additivity of elementary maps and multiplicative derivation on Gamma rings. This motivated us in the present paper we investigate the problem of when a multiplicative isomorphism is additive for the class of gamma rings.

Let us state our main theorem.

**Theorem 2.1** *Let* $\mathfrak{M}$ *be a* $\Gamma$*-ring containing a family* $\{e_\alpha|\alpha \in \Lambda\}$ *of nontrivial* $\gamma_\alpha$*-idempotents which satisfies:*

1. If $x \in \mathfrak{M}$ is such that $x\Gamma\mathfrak{M} = 0$, then $x = 0$;
2. If $x \in \mathfrak{M}$ is such that $e_\alpha\Gamma\mathfrak{M}\Gamma x = 0$ for all $\alpha \in \Lambda$, then $x = 0$ (and hence $\mathfrak{M}\Gamma x = 0$ implies $x = 0$);
3. For each $\alpha \in \Lambda$ and $x \in \mathfrak{M}$, if $(e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha)\Gamma\mathfrak{M}\Gamma(1_\alpha - e_\alpha) = 0$ then $e_\alpha\gamma_\alpha x\gamma_\alpha e_\alpha = 0$.

Then any multiplicative isomorphism $(\varphi, \phi)$ of $\mathfrak{M}$ onto an arbitrary gamma ring is additive.

The following lemmas have the same hypotheses of Theorem 2.1 and we need these lemmas for the proof of this theorem. Thus, let us consider $e_1 \in \{e_\alpha|\alpha \in \Lambda\}$ a nontrivial $\gamma_1$-idempotent of $\mathfrak{M}$ and .

**Lemma 2.1** $\varphi(0) = 0$.

*Proof.* Since $\varphi$ is onto, we can choose $x \in \mathfrak{M}$ such that $\varphi(x) = 0$. Thus $\varphi(0) = \varphi(0\gamma_1 x) = \varphi(0)\phi(\gamma_1)\varphi(x) = \varphi(0)\phi(\gamma_1)0 = 0$.

**Lemma 2.2** $\varphi(x_{ii} + x_{jk}) = \varphi(x_{ii}) + \varphi(x_{jk}), j \neq k$.

*Proof.* First assume that $i = j = 1$ and $k = 2$. Since $\varphi$ is onto, let $z$ be an element of $\mathfrak{M}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(x_{12})$. For arbitrary $\gamma \in \Gamma$ and $a_{1l} \in \mathfrak{M}_{1l}$ $(l = 1,2)$ we have

$$\varphi(z\gamma_1 e_1\gamma a_{1l}) = \varphi(z)\phi(\gamma_1)\varphi(e_1\gamma a_{1l}) = (\varphi(x_{11}) + \varphi(x_{12}))\phi(\gamma_1)\varphi(e_1\gamma a_{1l}) = \varphi(x_{11}\gamma_1 e_1\gamma a_{1l}) + \varphi(x_{12}\gamma_1 e_1\gamma a_{1l}) = \varphi(x_{11}\gamma_1 e_1\gamma a_{1l}) + \varphi(0) = \varphi((x_{11} + x_{12})\gamma_1 e_1\gamma a_{1l}).$$

Hence $(z - (x_{11} + x_{12}))\gamma_1 e_1\gamma a_{1l} = 0$. In a similar way, for $a_{2l} \in \mathfrak{M}_{2l}$ $(l = 1,2)$ we get that $(z - (x_{11} + x_{12}))\gamma_1 e_1\gamma a_{2l} = 0$. It follows that

$$(z - (x_{11} + x_{12}))\gamma_1 e_1\gamma a = 0, \tag{1}$$

where $a = a_{11} + a_{12} + a_{21} + a_{22}$. Next, for arbitraries $\gamma \in \Gamma$ and $a_{1l} \in \mathfrak{M}_{1l}$ $(l = 1,2)$ we have

$$\varphi(z\gamma_1 e_2\gamma a_{1l}) = \varphi(z)\phi(\gamma_1)\varphi(e_2\gamma a_{1l}) = (\varphi(x_{11}) + \varphi(x_{12}))\phi(\gamma_1)\varphi(e_2\gamma a_{1l}) = \varphi(x_{11}\gamma_1 e_2\gamma a_{1l}) + \varphi(x_{12}\gamma_1 e_2\gamma a_{1l}) = \varphi(0) + \varphi(x_{12}\gamma_1 e_2\gamma a_{1l}) = \varphi((x_{11} + x_{12})\gamma_1 e_2\gamma a_{1l})$$

which implies $(z-(x_{11}+x_{12}))\gamma_1 e_2\gamma a_{1l}=0$. In a similar way, we get that $(z-(x_{11}+x_{12}))\gamma_1 e_2\gamma a_{2l}=0$. Hence

$$(z-(x_{11}+x_{12}))\gamma_1 e_2\gamma a=0, \quad (2)$$

where $a=a_{11}+a_{12}+a_{21}+a_{22}$, by condition *(i)* of the Theorem. From (1) and (2), we have $(z-(x_{11}+x_{12}))\gamma_1 1_1\gamma a=0$, where $a=a_{11}+a_{12}+a_{21}+a_{22}$, which implies $(z-(x_{11}+x_{12}))\Gamma\mathfrak{M}=0$ and resulting in $z=x_{11}+x_{12}$, by condition *(i)* of the Theorem.

Now assume that $i=k=1$ and $j=2$. Again, we may find an element $z$ of $\mathfrak{M}$ such that $\varphi(z)=\varphi(x_{11})+\varphi(x_{21})$. For arbitraries $\gamma\in\Gamma$ and $a_{l1}\in\mathfrak{M}_{l1}$ $(l=1,2)$ we have
$\varphi(a_{l1}\gamma e_1\gamma_1 z)=\varphi(a_{l1}\gamma e_1)\phi(\gamma_1)\varphi(z)=\varphi(a_{l1}\gamma e_1)\phi(\gamma_1)(\varphi(x_{11})+\varphi(x_{21}))=$
$\varphi(a_{l1}\gamma e_1\gamma_1 x_{11})+\varphi(a_{l1}\gamma e_1\gamma_1 x_{21})=\varphi(a_{l1}\gamma e_1\gamma_1 x_{11})+\varphi(0)=\varphi(a_{l1}\gamma e_1\gamma_1(x_{11}+x_{21}))$.
It follows that $a_{l1}\gamma e_1\gamma_1(z-(x_{11}+x_{21}))=0$. In a similar way, for arbitraries $\gamma\in\Gamma$ and $a_{l2}\in\mathfrak{M}_{l2}$ $(l=1,2)$ we get that $a_{l2}\gamma e_1\gamma_1(z-(x_{11}+x_{21}))=0$. This implies

$$a\gamma e_1\gamma_1(z-(x_{11}+x_{21}))=0, \quad (3)$$

where $a=a_{11}+a_{12}+a_{21}+a_{22}$. Next, for arbitraries $\gamma\in\Gamma$ and $a_{l1}\in\mathfrak{M}_{l1}$ $(l=1,2)$ we have
$\varphi(a_{l1}\gamma e_2\gamma_1 z)=\varphi(a_{l1}\gamma e_2)\phi(\gamma_1)\varphi(z)=\varphi(a_{l1}\gamma e_2)\phi(\gamma_1)(\varphi(x_{11})+\varphi(x_{21}))=$
$\varphi(a_{l1}\gamma e_2\gamma_1 x_{11})+\varphi(a_{l1}\gamma e_2\gamma_1 x_{21})=\varphi(0)+\varphi(a_{l1}\gamma e_2\gamma_1 x_{11})=\varphi(a_{l1}\gamma e_2\gamma_1(x_{11}+x_{21}))$.
It follows that $a_{l1}\gamma e_2\gamma_1(z-(x_{11}+x_{21}))=0$. In a similar way, for arbitraries $\gamma\in\Gamma$ and $a_{l2}\in\mathfrak{M}_{l2}$ $(l=1,2)$ we get that $a_{l2}\gamma e_2\gamma_1(z-(x_{11}+x_{21}))=0$ which implies

$$a\gamma e_2\gamma_1(z-(x_{11}+x_{21}))=0, \quad (4)$$

where $a=a_{11}+a_{12}+a_{21}+a_{22}$, by condition *(i)* of the Theorem. From (3) and (4) we have $a\gamma 1_1\gamma_1(z-(x_{11}+x_{21}))=0$ which implies $\mathfrak{M}\Gamma(z-(x_{11}+x_{21}))=0$ resulting in $z=x_{11}+x_{21}$, by condition *(ii)* of the Theorem.

Similarly, we prove the remaining cases.

**Lemma 2.3** $\varphi(a_{1j}+b_{12}\gamma c_{1l})=\varphi(a_{1j})+\varphi(b_{12}\gamma c_{1l})$ $(j,l=1,2)$

*Proof.* First, let us note that

$$a_{1j}+b_{12}\gamma c_{1l}=(e_1+b_{12})\gamma_1(a_{1j}+e_2\gamma c_{1l}).$$

Hence

$\varphi(a_{1j}+b_{12}\gamma c_{1l})=\varphi((e_1+b_{12})\gamma_1(a_{1j}+e_2\gamma c_{1l}))=\varphi(e_1+b_{12})\phi(\gamma_1)\varphi(a_{1j}+e_2\gamma c_{1l})=(\varphi(e_1)+\varphi(b_{12}))\phi(\gamma_1)\varphi(a_{1j}+e_2\gamma c_{1l})=\varphi(e_1)\phi(\gamma_1)\varphi(a_{1j}+e_2\gamma c_{1l})+\varphi(b_{12})\phi(\gamma_1)\varphi(a_{1j}+e_2\gamma c_{1l})=\varphi(a_{1j})+\varphi(b_{12}\gamma a_{1j})$

, by Lemma 2.2.

**Lemma 2.4** $\varphi$ *is additive on* $\mathfrak{M}_{12}$.

*Proof.* Let $x_{12}, y_{12}\in\mathfrak{M}_{12}$ and choose $z\in\mathfrak{M}$ such that $\varphi(z)=\varphi(x_{12})+\varphi(y_{12})$, where $z=z_{11}+z_{12}+z_{21}+z_{22}$. For an arbitrary $a_{1l}\in\mathfrak{M}_{1l}$ $(l=1,2)$ we have

$\varphi(z\gamma_1 e_1\gamma a_{1l})=\varphi(z)\phi(\gamma_1)\varphi(e_1\gamma a_{1l})=(\varphi(x_{12})+\varphi(y_{12}))\phi(\gamma_1)\varphi(e_1\gamma a_{1l})=\varphi(x_{12}\gamma_1 e_1\gamma a_{1l})+\varphi(y_{12}\gamma_1 e_1\gamma a_{1l})=0$

which implies $z\gamma_1 e_1\gamma a_{1l}=0$. It follows that $(z-(x_{12}+y_{12}))\gamma_1 e_1\gamma a_{1l}=0$. In a similar way, for an arbitrary $a_{2l}\in\mathfrak{M}_{2l}$ $(l=1,2)$ we get that $(z-(x_{12}+y_{12}))\gamma_1 e_1\gamma a_{2l}=0$. Hence

$$(z-(x_{12}+y_{12}))\gamma_1 e_1\gamma a=0, \tag{5}$$

where $a=a_{11}+a_{12}+a_{21}+a_{22}$. Now, for an arbitrary element $a_{1l}\in\mathfrak{M}_{1l}$ $(l=1,2)$ we have

$\varphi(z\gamma_1 e_2\gamma a_{1l})=\varphi(z)\phi(\gamma_1)\varphi(e_2\gamma a_{1l})=(\varphi(x_{12})+\varphi(y_{12}))\phi(\gamma_1)\varphi(e_2\gamma a_{1l})=\varphi(x_{12}\gamma_1 e_2\gamma a_{1l})+\varphi(y_{12}\gamma_1 e_2\gamma a_{1l})=\varphi(x_{12}\gamma_1 e_2\gamma a_{1l}+y_{12}\gamma_1 e_2\gamma a_{1l})=\varphi((x_{12}+y_{12})\gamma_1 e_2\gamma a_{1l})$

, by Lemma 2.3. It follows that $(z-(x_{12}+y_{12}))\gamma_1 e_2\gamma a_{1l}=0$. Next, for an arbitrary element $a_{2l}\in\mathfrak{M}_{2l}$ $(l=1,2)$ we have

$\varphi(z\gamma_1 e_2\gamma a_{2l})=\varphi(z)\phi(\gamma_1)\varphi(e_2\gamma a_{2l})=(\varphi(x_{12})+\varphi(y_{12}))\phi(\gamma_1)\varphi(e_2\gamma a_{2l})=\varphi(x_{12})\phi(\gamma_1)\varphi(e_2\gamma a_{2l})+\varphi(y_{12}\phi(\gamma_1)\varphi(e_2\gamma a_{2l})=(\varphi(e_1)+\varphi(x_{12}))\phi(\gamma_1)(\varphi(e_2\gamma a_{2l})+\varphi(y_{12}\gamma_1 e_2\gamma a_{2l}))=\varphi(e_1+x_{12})\phi(\gamma_1)\varphi(e_2\gamma a_{2l}+y_{12}\gamma_1 e_2\gamma a_{2l})=\varphi(e_1+x_{12})\gamma_1(e_2\gamma a_{2l}+y_{12}\gamma_1 e_2\gamma a_{2l})=\varphi((x_{12}+y_{12})\gamma_1 e_2\gamma a_{2l})$

, by Lemma 2.2. It follows that $(z-(x_{12}+y_{12}))\gamma_1 e_2\gamma a_{2l}=0$. Hence

$$(z-(x_{12}+y_{12}))\gamma_1 e_2\gamma a=0, \tag{6}$$

where $a=a_{11}+a_{12}+a_{21}+a_{22}$, by condition *(i)* of the Theorem. From (5) and (6) we have $(z-(x_{12}+y_{12}))\gamma_1 1_1\gamma a=0$ which implies $(z-(x_{12}+y_{12}))\Gamma\mathfrak{M}=0$ and resulting in $z=x_{12}+y_{12}$, by condition *(i)* of the Theorem.

**Lemma 2.5** $\varphi$ *is additive on* $\mathfrak{M}_{11}$.

*Proof.* Let $x_{11}, y_{11} \in \mathfrak{M}_{11}$ and choose $z \in \mathfrak{M}$ such that $\varphi(z) = \varphi(x_{11}) + \varphi(y_{11})$, where $z = z_{11} + z_{12} + z_{21} + z_{22}$. Firstly, let us note that $\varphi(z) = \varphi(x_{11}\gamma_1 e_1) + \varphi(y_{11}\gamma_1 e_1) = (\varphi(x_{11}) + \varphi(y_{11}))\phi(\gamma_1)\varphi(e_1) = \varphi(z)\phi(\gamma_1)\varphi(e_1) = \varphi(z_{11} + z_{21})$

It follows that $z = z_{11} + z_{21}$ which results in $z_{12} = z_{22} = 0$. Similarly, we prove that $z_{21} = 0$ .This implies $z \in M_{11}$ which leads to $z - (x_{11} + y_{11}) \in M_{11}$. Next, for an arbitrary element $a_{ij} \in \mathfrak{M}_{ij}$ $(i, j = 1,2)$, applying Lemma 2.4 we get that

$$\begin{aligned}
&\varphi(z\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= \varphi(z)\phi(\alpha)\varphi(e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= (\varphi(x_{11}) + \varphi(y_{11}))\phi(\alpha)\varphi(e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= \varphi(x_{11})\phi(\alpha)\varphi(e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2) + \varphi(y_{11})\phi(\alpha)\varphi(e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= \varphi(x_{11}\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2) + \varphi(y_{11}\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= \varphi(x_{11}\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2 + y_{11}\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)\\
&= \varphi((x_{11} + y_{11})\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2)
\end{aligned}$$

$(k, l = 1,2)$ which implies $z\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2 = (x_{11} + y_{11})\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2$ and resulting in $(z - (x_{11} + y_{11}))\alpha e_k\gamma_1 a_{ij}\gamma_1 e_l\beta e_2 = 0$. It follows that

$$(z - (x_{11} + y_{11}))\alpha e_k\gamma_1 a\gamma_1 e_l\beta e_2 = 0 (k, l = 1,2), \tag{7}$$

where $a = a_{11} + a_{12} + a_{21} + a_{22}$, by condition *(i)* of the Theorem.

From (7) we have $(z - (x_{11} + y_{11}))\alpha 1_1\gamma_1 a\gamma_1 1_1\beta e_2 = 0$ which implies $(z - (x_{11} + y_{11}))\alpha a\beta e_2 = 0$. It follows that $(z - (x_{11} + y_{11}))\Gamma\mathfrak{M}\Gamma(1_1 - e_1) = 0$, that is,

$$(e_1\gamma_1(z - (x_{11} + y_{11})\gamma_1 e_1)\Gamma\mathfrak{M}\Gamma(1_1 - e_1) = 0.$$

By condition *(iii)* of the Theorem we conclude that $z = x_{11} + y_{11}$.

**Lemma 2.6** *$\varphi$ is additive on $e_1\Gamma\mathfrak{M}$.*

*Proof.* Let $x, y \in \mathfrak{M}$ and $\lambda, \mu \in \Gamma$ be arbitrary elements and let us write $x = x_{11} + x_{12} + x_{21} + x_{22}$ and $y = y_{11} + y_{12} + y_{21} + y_{22}$. It follows that $e_1\lambda x = e_1\lambda x_{11} + e_1\lambda x_{12} + e_1\lambda x_{21} + e_1\lambda x_{22}$ and $e_1\mu y = e_1\mu y_{11} + e_1\mu y_{12} + e_1\mu y_{21} + e_1\mu y_{22}$. Hence, by Peirce decomposition properties of $\mathfrak{M}$ and making use of the Lemmas 2.2, 2.4 and 2.5, we can see that

$$\varphi(e_1\lambda x + e_1\mu y) = \varphi((e_1\lambda x_{11} + e_1\lambda x_{12} + e_1\lambda x_{21} + e_1\lambda x_{22})$$

$$+(e_1\mu y_{11} + e_1\mu y_{12} + e_1\mu y_{21} + e_1\mu y_{22}))$$

$$= \varphi((e_1\lambda x_{11} + e_1\mu y_{11}) + (e_1\lambda x_{21} + e_1\mu y_{21})$$

$$+(e_1\lambda x_{12} + e_1\mu y_{12}) + (e_1\lambda x_{22} + e_1\mu y_{22}))$$

$$= \varphi((e_1\lambda x_{11} + e_1\mu y_{11}) + (e_1\lambda x_{21} + e_1\mu y_{21}))$$

$$+\varphi((e_1\lambda x_{12} + e_1\mu y_{12}) + (e_1\lambda x_{22} + e_1\mu y_{22}))$$

$$= \varphi(e_1\lambda x_{11} + e_1\lambda x_{21}) + \varphi(e_1\mu y_{11} + e_1\mu y_{21})$$

$$+\varphi(e_1\lambda x_{12} + e_1\lambda x_{22}) + \varphi(e_1\mu y_{12} + e_1\mu y_{22})$$

$$= \varphi(e_1\lambda x_{11} + e_1\lambda x_{21} + e_1\lambda x_{12} + e_1\lambda x_{22})$$

$$+\varphi(e_1\mu y_{11} + e_1\mu y_{21} + e_1\mu y_{12} + e_1\mu y_{22})$$

$$= \varphi(e_1\lambda x) + \varphi(e_1\mu y)$$

holds true, as desired.

**Proof of Theorem 2.1.** Suppose that $x, y \in \mathfrak{M}$ and choose $z \in \mathfrak{M}$ such that $\varphi(z) = \varphi(x) + \varphi(y)$. Since $\varphi$ is additive on $e_\alpha\Gamma\mathfrak{M}$ for all $\alpha \in \Lambda$, by Lemma 2.6, then for an arbitrary element $r \in \mathfrak{M}$ and elements $\lambda, \mu \in \Gamma$ we have

$$\varphi(e_\alpha\lambda r\mu z) = \varphi(e_\alpha)\phi(\lambda)\varphi(r)\phi(\mu)\varphi(z)$$

$$= \varphi(e_\alpha)\phi(\lambda)\varphi(r)\phi(\mu)(\varphi(x) + \varphi(y))$$

$$= \varphi(e_\alpha)\phi(\lambda)\varphi(r)\phi(\mu)\varphi(x) + \varphi(e_\alpha)\phi(\lambda)\varphi(r)\phi(\mu)\varphi(y)$$

$$= \varphi(e_\alpha \lambda r \mu x) + \varphi(e_\alpha \lambda r \mu y)$$

$$= \varphi(e_\alpha \lambda r \mu x + e_\alpha \lambda r \mu y)$$

$$= \varphi(e_\alpha \lambda r \mu (x + y)).$$

Hence $e_\alpha \lambda r \mu z = e_\alpha \lambda r \mu (x + y)$ which results in

$$e_\alpha \Gamma \mathfrak{M} \Gamma (z - (x + y)) = 0$$

for all $\alpha \in \Lambda$. From condition *(ii)* of the Theorem, we conclude that $z = x + y$. This shows that $\varphi$ is additive on $\mathfrak{M}$.

**Corollary 2.1** *Let $\mathfrak{M}$ be a prime $\Gamma$-ring containing a $\gamma_1$-idempotent $e_1$ ($\mathfrak{M}$ need not have a $\gamma_1$-identity element), where $\gamma_1 \in \Gamma$. Suppose $e_2: \Gamma \times \mathfrak{M} \to \mathfrak{M}$, $e'_2: \mathfrak{M} \times \Gamma \to \mathfrak{M}$ two $\mathfrak{M}$-additive maps such that $e_2(\gamma_1, a) = a - e_1\gamma_1 a$, $e_{2'}(a, \gamma_1) = a - a\gamma_1 e_1$, for all $a \in \mathfrak{M}$, and if we denote $e_2 \alpha a = e_2(\alpha, a)$, $a\alpha e_2 = e_{2'}(a, \alpha)$, $1_1 \alpha a = e_1 \alpha a + e_2 \alpha a$, $a\alpha 1_1 = a\alpha e_1 + a\alpha e_2$, then $(a\alpha e_2)\beta b = a\alpha(e_2 \beta b)$ for all $\alpha, \beta \in \Gamma$ and $a, b \in \mathfrak{M}$. Then any multiplicative isomorphism $(\varphi, \phi)$ of $\mathfrak{M}$ onto an arbitrary gamma ring is additive.*

*Proof.* The result follows directly from the Theorem 2.1.

**Corollary 2.2** *Let $\mathfrak{M}$ be a prime $\Gamma$-ring containing a $\gamma_1$-idempotent and a $\gamma_1$-unity element, where $\gamma_1 \in \Gamma$. Then any multiplicative isomorphism $(\varphi, \phi)$ of $\mathfrak{M}$ onto an arbitrary gamma ring is additive.*

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