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3-Zero-Divisor Hypergraph with Respect to an Element in Multiplicative Lattice

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Abstract. Let L be a multiplicative lattice and z be a proper element of L. We introduce the 3-zero-divisor hypergraph of L with respect to z which is a hypergraph whose vertices are elements of the set $\left\{x_1 \in L-\{z\} \middle| \begin{matrix} x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z,\ x_2x_3 \nleq z\ and\ x_1x_3 \nleq z \\ for\ some\ x_2, x_3 \in L-\{z\} \end{matrix} \right\}$ where distinct vertices x_1, x_2 and x_3 are adjacent, that is, $\{x_1, x_2, x_3\}$ is a hyperedge if and only if $x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z, x_2x_3 \nleq z\ and\ x_1x_3 \nleq z$. Throughout this paper, the hypergraph is denoted by $H_3(L,z)$. We investigate many properties of the hypergraph over a multiplicative lattice. Moreover, we find a lower bound of diameter of $H_3(L,z)$ and obtain that $H_3(L,z)$ is connected.

Keywords: 3-Zero-Divisor Hypergraph, Complete n-partite Hypergraph.

Çarpımsal Kafeslerde Bir Eleman ile İlgili 3-lü Sıfır Bölen Hipergrafı

Özet. L bir çarpımsal kafes ve z, L nin bir has elemanı olsun. z ile ilgili L nin 3-lü sıfır bölen hipergrafını tanıttık öyle ki bu hipergrafın köşeleri $\left\{x_1 \in L-\{z\} \middle| \begin{matrix} x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z,\ x_2x_3 \nleq z\ ve\ x_1x_3 \nleq z \\ herhangi\ x_2, x_3 \in L-\{z\}\ için \end{matrix} \right\}$ kümesinin elemanlarıdır ki burada x_1, x_2 ve x_3 komşudur, yani, $\{x_1, x_2, x_3\}$ bu hipergarfın bir hiperkenarıdır ancak ve ancak $x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z, x_2x_3 \nleq z\ ve\ x_1x_3 \nleq z$. Bu çalışma boyunca, bu hipergrafı $H_3(L,z)$ ile göstereceğiz. Çarpımsal bir kafes üzerinde bu hipergrafın birçok özelliğini araştırdık. Ayrıca, $H_3(L,z)$ nin diametresinin bir alt sınırını bulduk ve bu hipergrafın bağlantılı olduğunu gösterdik.

Anahtar Kelimeler: 3-lü Sıfır Bölen Hipergraf, n-parçalı Tam Hipergraf.

1. INTRODUCTION

A complete lattice L is called multiplicative lattice if there exists a commutative, associative, completely join distributive product on the lattice with the compact greatest element 1_L, which is the multiplicative identity, and the least element 0_L. It can be easily seen that $L/a = \{b \in L | a \leq b\}$ is a multiplicative lattice with the product $x \circ y = xy \vee a$ where L is multiplicative lattice and $a \in L$. Note that $0_{L/z} = z$. D.D. Anderson and the current authors have studied on multiplicative lattices in a series of articles [1-4]. An element $a \in L$ is said to be proper if $a < 1_L$. A proper element $p \in L$ is called a prime element if $ab \leq p$ implies $a \leq p$ or $b \leq p$, where $a, b \in L$. Then p is called 2-absorbing element of L if $x_1x_2x_3 \leq p$ for some x_1, x_2 and x_3 in L, then $x_1x_2 \leq p$ or $x_1x_3 \leq p$ or $x_2x_3 \leq p$.

Let a finite set V be a vertex set and $E(V) = \{(u,v) | u, v \in V, u \neq v\}$. A pairwise $G = (V, E)$ is called a graph on V where $E \subseteq E(V)$. The elements of V are the vertices of G, and those of E the

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edges of G. Consider that the edges (x,y) and (y,x) denote the same edge (For more information, see [3-8].

A hypergraph H is a pair (V,E) of disjoint sets, where the elements of E are nonempty subsets of V. The elements of V are called the vertices of H and the elements of E are called the hyperedges of H. If the size of any hyperedge e in the hypergraph H is n, then H is called n-uniform hypergraph. Let H be an n-uniform hypergraph. An alternating sequence of distinct vertices and hyperedges is called a path with the form $v_1, e_1, v_2, e_2, \dots, v_m$ such that v_i, v_{i+1} are in e_i for all $1 \leq i \leq m-1$. The length of a path is the number of hyperedges of it. The distance $d(x,y)$ between two vertices x and y of H is the length of the shortest path from x to y. If no such path between x and y exists, then $d(x,y) = \infty$. The diameter $diam(H)$ of H is the greatest distance between any two vertices. The hypergraph H is said to be connected if $diam(H) < \infty$. A cycle in a hypergraph H is an alternating sequence of distinct vertices and hyperedges of the form $v_1, e_1, v_2, e_2, \dots, v_m, e_m, v_1$ such that $v_i, v_{i+1} \in e_i$ and $v_m, v_1 \in e_m$ for all $1 \leq i \leq m$. The girth $gr(H)$ of a hypergraph H containing a cycle is the smallest size of the length of cycles of H. (For more information, see [5]). A hypergraph H is called trivial if it has a single vertex and also it is called empty if it has no hyperedges.

The concept of a zero-divisor graph of a commutative ring was first introduced in [6]. Let R be a commutative ring and $k \geq 2$ be an integer. A nonzero nonunit element x_1 in R is said to be a k-zero-divisor in R if there are $k-1$ distinct nonunit elements $x_2, x_3, \dots, x_k$ in R different from x_1 such that $x_1 x_2 x_3 \dots x_k = 0$ and the product of no elements of any proper subset of $A = \{x_1, x_{2,} x_3, \dots, x_k\}$ is zero. The set of k-zero divisor elements of R is denoted by $Z_k(R)$. Let I be a proper ideal of R. The 3-zero-divisor hypergraph of R with respect to I, denoted by $H_3(R,I)$, is the hypergraph whose vertices are the set $\{x_1 \in R\backslash I | x_1 x_2 x_3 \in I \ for\ some\ x_2, x_3 \in R\backslash I\ such\ that\ x_1 x_2 \notin I, x_2 x_3 \notin I\ and\ x_1 x_3 \notin I\}$ where distinct vertices x_1, x_2 and x_3 are adjacent if and only if $x_1 x_2 x_3 \in I, x_1 x_2 \notin I, x_2 x_3 \notin I\ and\ x_1 x_3 \notin I$ (See [9]). Let I be a proper ideal of R. Recall that I is called a 2-absorbing ideal of R if $x_1 x_2 x_3 \in I$ for some x_1, x_2 and x_3 in R, then $x_1 x_2 \in I$ or $x_2 x_3 \in I$ or $x_1 x_3 \in I$ (For more information, see [10]). Hence $H_3(R,I)$ is not empty if and only if I is not a 2-absorbing ideal of R (see Proposition 1 in [9]).

Let z be a proper element of L. A proper element a_1 of L is called n-zero divisor element with respect to z in L if there are $n-1$ distinct elements $a_2, a_3, \dots, a_n$ in L different from a_1 such that $a_2 a_3 \dots a_n \leq z$ and the product of no elements of any proper subset of $A = \{a_1, a_{2,}, \dots, a_n\}$ is less than or equals to z. The set of all n-zero divisor element with respect to z in L is denoted by $Z_n(L,z)$. For example, consider the lattice of ideals of $\mathbb{Z}$, $L = \mathrm{I}(\mathbb{Z})$ the set of all ideals of $\mathbb{Z}$. The ideal (2) is a 3-zero-divisor with respect to (8) in L since $(2)(3)(6) \subseteq (8)$, and the product of no elements of any proper subset of $\{(2),(3),(6)\}$ is contained by (8).

Throughout this paper, we assume that a lattice L is a multiplicative lattice. Let z be a proper element of L. The 3-zero-divisor hyper-graph of L with respect to z, denoted by $H_3(L,z)$, is a hypergraph whose vertices are elements of the set $\left\{x_1 \in L - \{z\} \middle| \begin{array}{c} x_1 x_2 x_3 \leq z \Rightarrow x_1 x_2 \nleq z, x_2 x_3 \nleq z\ and\ x_1 x_3 \nleq z \\ for\ some\ x_2, x_3 \in L - \{z\} \end{array} \right\}$ such that distinct vertices x_1, x_2 and x_3 are adjacent, that is, $\{x_1, x_2, x_3\}$ is a hyperedge if and only if $x_1 x_2 x_3 \leq z \Rightarrow x_1 x_2 \nleq z$, $x_2 x_3 \nleq z\ and\ x_1 x_3 \nleq z$. It can be seen that $H_3(L,z)$ is a 3-uniform hypergraph. In this paper, we show that $H_3(L,z)$ is empty if and only if z is a 2-absorbing element of L and also, $H_3(L/z)$ is empty hypergraph if and only if $H_3(L,z)$ is empty hypergraph. Then we give that $H_3(L,z)$ is connected and $diam(H_3(L,z)) \leq 4$. Additionally, we show that $H_3(L,z)$ is a complete 3-partite hypergraph if p_1, p_2 and p_3 are prime elements of L and $z = p_1 \wedge p_2 \wedge p_3 \neq 0_L$ and the converse is true if L is reduced lattice. Finally, we see that $H_3(L,z)$ has no cut-point.

2. ZERO DIVISOR HYPERGRAPH H_3 (L,z) WITH RESPECT TO z

Definition 1. Let z be a proper element of L. The 3-zero-divisor hypergraph of L with respect to z is a hypergraph whose vertices are elements of the set $\left\{x_1 \in L-\{z\} \middle| \begin{matrix} x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z, x_2x_3 \nleq z \text{ and } x_1x_3 \nleq z \\ \text{for some } x_2, x_3 \in L-\{z\} \end{matrix} \right\}$. Also, distinct vertices x_1, x_2 and x_3 are adjacent, that is, $\{x_1,x_2,x_3\}$ is a hyperedge if and only if $x_1x_2x_3 \leq z \Rightarrow x_1x_2 \nleq z,\ x_2x_3 \nleq z \text{ and } x_1x_3 \nleq z$. Throughout this paper, the hypergraph is denoted by $H_3(L,z)$.

Let $z = 0_L$. Then it is clear that $H_3(L) = H_3(L,0_L)$ is the hypergraph whose vertices are elements of the set $\left\{x_1 \in Z_3(L) \middle| \begin{matrix} x_1x_2x_3 = 0_L \Rightarrow x_1x_2 \neq 0_L, x_2x_3 \neq 0_L \text{ and } x_1x_3 \neq 0_L \\ \text{for some } x_2, x_3 \in Z_3(L) \end{matrix} \right\}$ where distinct vertices x_1, x_2 and x_3 are adjacent if and only if $x_1x_2x_3 = 0_L \Rightarrow x_1x_2 \neq 0_L, x_2x_3 \neq 0_L \text{ and } x_1x_3 \neq 0_L$.

The hypergraphs $H_3(R)$ in [5] and $H_3(R,I)$ in [10], which are defined on a commutative ring R and a proper ideal I of R, are examples for the hypergraph $H_3(L,z)$.

We obtain the following results with the above definition and the definition of 2-absorbing element in L.

Proposition 1. Let z be a proper element of L. Then the following statements hold:

1) $H_3(L,z)$ is empty hypergraph if and only if z is a 2-absorbing element of L.
2) $H_3(L/z)$ is empty hypergraph if and only if $H_3(L,z)$ is empty hypergraph.

Proof. 1). $(\Rightarrow)$: Let $H_3(L,z)$ be empty hypergraph. Suppose that z is not a 2-absorbing element of L. Take $x_1x_2x_3 \leq z$ for some $x_1, x_2, x_3 \in L$. Then we get $x_1x_2 \nleq z,\ x_2x_3 \nleq z$ and $x_1x_3 \nleq z$. Hence $e = \{x_1,x_2,x_3\}$ is a hyperedge of $H_3(L,z)$, a contradiction.

$(\Leftarrow)$: It is obvious.

2). $(\Rightarrow)$: Assume that $H_3(L,z)$ is not an empty hypergraph. Then it has a hyperedge $e = \{x_1,x_2,x_3\}$. Consider $x_1 \vee z, x_2 \vee z, x_3 \vee z \in L/z$. It is clear that $x_1 \vee z, x_2 \vee z, x_3 \vee z$ are different from z. Then we have that $(x_1 \vee z)(x_2 \vee z)(x_3 \vee z) = 0_{L/z}$, $(x_1 \vee z)(x_2 \vee z) \neq 0_{L/z}$, $(x_2 \vee z)(x_3 \vee z) \neq 0_{L/z}$ and $(x_1 \vee z)(x_3 \vee z) \neq 0_{L/z}$. Thus $e' = \{x_1 \vee z, x_2 \vee z, x_3 \vee z\}$ is a hyperedge of $H_3(L/z)$, a contradiction.

$(\Leftarrow)$: Let $H_3(L/z)$ be not an empty hypergraph. Then it has a hyperedge $e = \{y_1,y_2,y_3\}$ for some $y_1, y_2, y_3 \in V(H_3(L/z))$. Then $y_1 \circ y_2 \circ y_3 = 0_{L/z}$, that is, $y_1y_2y_3 \leq z$ and since $y_1 \circ y_2$, $y_2 \circ y_3$ and $y_1 \circ y_3$ are different from $0_{L/z}$, then $y_1y_2, y_2y_3, y_1y_3 \nleq z$. Therefore, $e = \{y_1,y_2,y_3\}$ is a hyperedge of $H_3(L,z)$, a contradiction.

Theorem 1. Let $H_3(L,z)$ be a 3-zero-divisor hypergraph of L with respect to z. If $x^2 \nleq z$ for each 3-zero-divisor $x \in L$ with respect to z, then $H_3(L,z)$ is connected and $diam(H_3(L,z)) \leq 4$. Furthermore, if $H_3(L,z)$ has a cycle, then $gr(H_3(L,z)) \leq 9$.

Proof. Let $e_1 = \{x_1,x_2,x_3\}$ and $e_2 = \{y_1,y_2,y_3\}$ be hyperedges of $H_3(L,z)$. If $e_1 \cap e_2 \neq \emptyset$, the proof is completed. Assume that $e_1 \cap e_2 = \emptyset$. We show that there are hyperedges e_3, e_4 such that they satisfy one of the followings:

(1) $\boldsymbol{e_3 \cap e_1 \neq \emptyset, e_3 \cap e_2 \neq \emptyset}$

(2) $e_3 \cap e_1 \neq \emptyset, e_4 \cap e_2 \neq \emptyset, e_4 \cap e_3 \neq \emptyset$

Assume that G is the partite graph such that $V(G) = e_1 \cup e_2$ and $x_iy_j \in E(G)$ if and only if $x_iy_j \leq z$.

Assume that G has two isolated vertices such that one is in e_1 and the other is in e_2. Let $deg_G(x_3) = deg_G(y_3) = 0$. Suppose that there is $a \in \{x_1,x_2,y_1,y_2\}$ where $x_3y_3a \leq z$. Then $e_3 = \{x_3,y_3,a\}$ is a hyperedge which holds the condition (1). Let the case not satisfy. If $x_3y_3 \notin \{x_1,x_2,y_1,y_2\}$, then $e_3 = \{x_1,x_2,x_3y_3\}$ and $e_4 = \{y_1,y_2,x_3y_3\}$ are two hyperedges which satisfy the condition (2). In the contrary case, without loss of generality (wlog.), suppose that $x_3y_3 = x_1$. Hence $e_3 = \{x_1,y_1,y_2\}$ is a hyperedge satisfying the condition (1). Consequently, $H_3(L,z)$ is connected. Now, we show that $diam(H_3(L,z)) \leq 4$. We consider the number of edges G for the rest of the proof.

Case 1. Assume that $|E(G)| \leq 2$. Then G has two isolated vertices such that one is in e_1 and the other is in e_2.

Case 2. Let $|E(G)| = 3$. Take account of the next four different subcases for this case:

Case 2.1: Let $deg_G(a) = 1$ for each vertex a of G. Assume that $E(G) = \{x_1y_1,x_2y_2,x_3y_3\}$. We consider $\{x_1,x_2y_3,y_1 \vee y_2\}$. If $x_1 = x_2y_3$, then $x_1y_2 = x_2y_3y_2 \leq z$, a contradiction. If $x_1 = y_1 \vee y_2$, then $y_1x_2x_3 \leq z$. Thus $e_3 = \{y_1,x_2,x_3\}$ satisfies the condition (1). If $y_1 \vee y_2 = x_2y_3$, then $x_1y_2x_3 \leq z$ and so the condition (1) is satisfied for $e_3 = \{x_1,y_2,x_3\}$. On the contrary, reconsider $e_3 = \{x_1,x_2y_3,y_1 \vee y_2\}$. If e_3 is not a hyperedge, then $x_1x_2y_3 \leq z$ or $x_2y_3(y_1 \vee y_2) \leq z$, that is, $x_2y_3y_1 \leq z$. Then $e'_3 = \{x_1,x_2,y_3\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_2,y_3,x_1\}$ is a hyperedge satisfying the condition (1). Let $e_3 = \{x_1,x_2y_3,y_1 \vee y_2\}$ be a hyperedge. In a similar way, we consider $\{y_1,x_2y_3,x_1 \vee x_3\}$. If e_4 is not a hyperedge, then $y_1x_2y_3 \leq z$ or $x_2y_3(x_1 \vee x_3) \leq z$, that is, $x_2y_3x_1 \leq z$. Then $e''_3 = \{y_1,x_2,y_3\}$ is a hyperedge satisfying the condition (1) or $e''_4 = \{x_2,y_3,x_1\}$ is a hyperedge satisfying the condition (1). Assume that $e_4 = \{y_1,x_2y_3,x_1 \vee x_3\}$ is a hyperedge. Then we have two hyperedges $e_3 = \{x_1,x_2y_3,y_1 \vee y_2\}$ and $e_4 = \{y_1,x_2y_3,x_1 \vee x_3\}$ with e_3 and e_4 satisfying the condition (2).

Case 2.2. Let $deg_G(a) = 1$ for only an element a of G. Wlog., suppose that $E(G) = \{x_1y_1,x_1y_2,x_2y_3\}$. We consider $\{x_2,x_3y_1,x_1 \vee y_3\}$. If $x_2 = x_3y_1$, then $x_1x_2 \leq z$, is a contradiction. If $x_2 = x_1 \vee y_3$, then $x_2y_2y_1 \leq z$ and so the condition (1) is satisfied for $e_3 = \{x_2,y_2,y_1\}$. If $x_1 \vee y_3 = x_3y_1$, then $x_3y_1y_2y_1 \leq z$. In the circumstances, if $x_3 = y_1y_2$, then $x_1x_3 \leq z$, a contradiction. If $y_1 = y_1y_2$, then $y_1y_3 \leq z$, a contradiction. Hence, the condition (1) holds for $e_3 = \{x_3,y_1y_2,y_1\}$. Let the above conditions not hold. If $e_3 = \{x_2,x_3y_1,x_1 \vee y_3\}$ is not a hyperedge, then $x_2x_3y_1 \leq z$ or $x_3y_1(x_1 \vee y_3) \leq z$, that is, $x_3y_1y_3 \leq z$. Then $e'_3 = \{x_2,x_3,y_1\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_3,y_1,y_3\}$ is a hyperedge satisfying the condition (1). Suppose that $e_3 =$

$\{x_2, x_3y_1, x_1 \vee y_3\}$ is a hyperedge. Now, similarly we consider $\{y_2, x_3y_1, y_3\}$. If $e_4 = \{y_2, x_3y_1, y_3\}$ is not a hyperedge, then $y_2x_3y_1 \leq z$ or $x_3y_1y_3 \leq z$. Then $e''_3 = \{y_2, x_3, y_1\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_3, y_1, y_3\}$ is a hyperedge satisfying the condition (1). Let $\{y_2, x_3y_1, x_1 \vee y_3\}$ be a hyperedge. Then we obtain two hyperedges $e_3 = \{x_2, x_3y_1, x_1 \vee y_3\}$ and $e_4 = \{y_2, x_3y_1, y_3\}$ with e_3 and e_4 satisfying the condition.

Case 2.3. Let $deg_G(a) = deg_G(b) = 2$ for $a, b \in V(G)$. Wlog., suppose that $E(G) = \{x_1y_1, x_1y_2, x_2y_2\}$. Then $deg_G(x_3) = deg_G(y_3) = 0$ and so the proof is completed.

Case 2.4. Let $deg_G(a) = 3$ for only one element a of G. Wlog., suppose that $E(G) = \{x_1y_1, x_1y_2, x_1y_3\}$. Let ${x_1}^2x_2 \nleq z$. Consider $\{x_1x_2 \vee y_1, x_1, x_3\}$. If $x_1x_2 \vee y_1 = x_1$, then $y_2y_1 \leq z$, a contradiction. If $x_1x_2 \vee y_1 = x_3$, then $x_3y_3y_2 \leq z$, a contradiction. Hence $e_3 = \{x_3, y_2, y_3\}$ is a hyperedge satisfying the condition (1). In the other case, $e_3 = \{x_1x_2 \vee y_1, x_1, x_3\}$ is a hyperedge. In a similar way, we consider $\{x_1x_2 \vee y_1, y_2, y_3\}$. Then we have a hyperedge e_3 which satisfies the condition (1) or $e_4 = \{x_1x_2 \vee y_1, y_2, y_3\}$ is a hyperedge with e_3 and e_4 satisfying the condition (2). Let ${x_1}^2x_2 \leq z$. We consider $\{x_1 \vee y_1, x_1, x_2\}$. If $x_1 \vee y_1 = x_2$, then $x_2y_3y_2 \leq z$, a contradiction. Thus $e_3 = \{x_1 \vee y_1, x_1, x_2\}$ is a hyperedge. In a similar way, we consider $\{x_1 \vee y_1, y_2, y_3\}$. Then we have a hyperedge e_3 which satisfies the condition (1) or $e_4 = \{x_1x_2 \vee y_1, y_2, y_3\}$ is a hyperedge with e_3 and e_4 satisfying the condition (2).

Case 3. Assume that $|E(G)| = 4$. Consider four different subcases for this case:

Case 3.1. Let $deg_G(a) = 3$ for only one element a of G. Wlog., suppose that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_3\}$. We consider $\{x_3y_1, x_2, x_1 \vee y_3\}$. If $x_3y_1 = x_2$, then $x_3y_3y_1 \leq z$, a contradiction. Thus $e_3 = \{x_3, y_1, y_3\}$ is a hyperedge which holds (1). If $x_3y_1 = x_1 \vee y_3$, then ${x_1}^2 \leq z$, is a contradiction. If $x_2 = x_1 \vee y_3$, then ${y_3}^2 \leq z$, a contradiction. In the other condition, consider again $e_3 = \{x_3y_1, x_2, x_1 \vee y_3\}$. If $e_3 = \{x_3y_1, x_2, x_1 \vee y_3\}$ is not a hyperedge, then $x_2x_3y_1 \leq z$ or $x_3y_1(x_1 \vee y_3) \leq z$, that is, $x_3y_1y_3 \leq z$. Then $e'_3 = \{x_2, x_3, y_1\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_3, y_1, y_3\}$ is a hyperedge satisfying the condition (1). Assume that $e_3 = \{x_3y_1, x_2, x_1 \vee y_3\}$ is a hyperedge. In a similar way, we consider $\{x_3y_1, y_2, y_3\}$. If $e_4 = \{x_3y_1, y_2, y_3\}$ is not a hyperedge, then $x_3y_1y_2 \leq z$ or $x_3y_1y_3 \leq z$. Then $e''_3 = \{y_2, x_3, y_1\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_3, y_1, y_3\}$ is a hyperedge satisfying the condition (1). Suppose that $e_4 = \{x_3y_1, y_2, y_3\}$ is a hyperedge. Then we get two hyperedges $e_3 = \{x_3y_1, x_2, x_1 \vee y_3\}$. and $e_4 = \{x_3y_1, y_2, y_3\}$ with e_3 and e_4 satisfying the condition (2).

Case 3.2. Assume that the degree of four vertices of G equals to two. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$. Then $deg_G(x_3) = deg_G(y_3) = 0$ and so the proof is completed.

Case 3.3. Suppose that the degree of three vertices of G is two. Wlog. assume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3\}$. We consider $\{x_3y_3, x_1, x_2\}$. If $x_3y_3 = x_1$ or $x_3y_3 = x_2$, then $x_3y_3y_2 \leq z$ and so (1) is satisfied for a hyperedge $e_3 = \{x_3, y_2, y_3\}$. In the other case, let us view $e_3 = \{x_3y_3, x_1, x_2\}$. If $e_3 = \{x_3y_3, x_1, x_2\}$ is not a hyperedge, then $x_3y_3x_1 \leq z$ or $x_3y_3x_2 \leq z$. Then $e'_3 = \{x_3, y_3, x_1\}$ is a hyperedge satisfying the condition (1) or $e'_4 = \{x_3, y_3, x_2\}$ is a hyperedge satisfying the condition (1). Let $e_3 = \{x_3y_3, x_1, x_2\}$ be a hyperedge. In a similar way, we consider $\{x_3y_3, y_1, y_2\}$. If $e_4 = \{x_3y_3, y_1, y_2\}$ is not a hyperedge, then $x_3y_3y_1 \leq z$ or $x_3y_3y_2 \leq z$. Then $e''_3 = \{x_3, y_3, y_1\}$ is a hyperedge satisfying the condition (1) or $e''_4 = \{x_3, y_3, y_2\}$ is a hyperedge satisfying the condition (1). Let $e_4 = \{x_3y_3, y_1, y_2\}$ be a hyperedge. Then we get two hyperedges $e_3 = \{x_3y_3, x_1, x_2\}$ and $e_4 = \{x_3y_3, y_1, y_2\}$ with e_3 and e_4 satisfying the condition (2).

Case 3.4. Let $deg_G(a) = deg_G(b) = 2$ for $a, b \in V(G)$. Then, we have two different cases and we can choose one of these sets $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_3y_3\}$ and $E(G) = \{x_1y_1, x_1y_2, x_2y_3, x_3y_3\}$. In the first choice, we consider $\{x_3y_1, x_2, x_1 \vee y_2\}$. If $x_3y_1 = x_2$, then $x_3y_1y_2 \leq z$ and so $e_3 = \{x_3, y_1, y_2\}$ is an edge satisfying (1). If $x_3y_1 = x_1 \vee y_2$, then ${x_1}^2 \leq z$, a contradiction. If $x_2 = x_1 \vee y_2$, then ${y_2}^2 \leq z$, is a contradiction. In the other case, consider $e_3 = \{x_3y_1, x_2, x_1 \vee y_2\}$. If $e_3 = \{x_3y_1, x_2, x_1 \vee y_2\}$ is not a hyperedge, then $x_3y_1x_2 \leq z$ or $x_3y_1(x_1 \vee y_2) \leq z$, that is, $x_3y_1y_2 \leq z$. Then $e''_3 = \{x_3, y_1, y_2\}$ is a hyperedge satisfying the condition (1) or $e''_4 = \{x_3, y_1, y_3\}$ is a hyperedge satisfying the condition (1). Let $e_4 = \{x_3y_1, y_2, y_3\}$ be a hyperedge. Then we get two hyperedges $e_3 = \{x_3y_1, x_2, x_1 \vee y_2\}$ and $e_4 = \{x_3y_1, y_2, y_3\}$ with e_3 and e_4 satisfying the condition (2).

In a similar manner, we consider $\{x_1 \vee y_1, x_2, x_3\}$ and $\{x_1 \vee y_1, y_2, y_3\}$ for the second choice. Hence, we have a hyperedge e_3 which holds (1) or two hyperedges e_3 and e_4 which hold the condition (2).

Case 4. Assume that $|E(G)| = 5$. Consider four different subcases for this case:

Case 4.1. Wlog. assume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2\}$. We consider $\{x_3y_3, x_2, x_1 \vee y_2\}$. If $x_3y_3 = x_2$, then $x_3y_3x_2 \leq z$, and so the condition (1) is satisfied for a hyperedge $e_3 = \{x_2, x_3, y_3\}$. If $x_3y_3 = x_1 \vee y_2$, then ${x_1}^2 \leq z$, a contradiction. If $x_2 = x_1 \vee y_2$, then $y_1y_2 \leq z$, yielding a contradiction. On the other hand, $e_3 = \{x_3y_3, x_2, x_1 \vee y_2\}$ is a edge in G. In a similar way, we consider $\{x_3y_3, y_1, y_2\}$. If $e_4 = \{x_3y_3, y_1, y_2\}$ is not a hyperedge, then $x_3y_3y_1 \leq z$ or $x_3y_3y_2 \leq z$. Then $e''_3 = \{x_3, y_3, y_1\}$ is a hyperedge satisfying the condition (1) or $e''_4 = \{x_3, y_3, y_2\}$ is a hyperedge satisfying the condition (1). Let $e_4 = \{x_3y_3, y_1, y_2\}$ be a hyperedge. Then we get two hyperedges $e_3 = \{x_3y_3, x_2, x_1 \vee y_2\}$ and $e_4 = \{x_3y_3, y_1, y_2\}$ with e_3 and e_4 satisfying the condition (2).

Case 4.2. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_3y_2\}$. We consider $\{x_1 \vee y_1, x_2, y_2\}$. If $x_1 \vee y_1 = x_2$, then ${y_1}^2 \leq z$, is a contradiction. If $x_1 \vee y_1 = y_2$, then ${x_1}^2 \leq z$, is a contradiction. In the following situations, $e_3 = \{x_1 \vee y_1, x_2, x_3y_3\}$ is a hyperedge of G satisfying (1).

Case 4.3. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_3y_2\}$. We consider $\{x_1 \vee y_1, x_2, y_2\}$. If $x_1 \vee y_1 = x_2$ then ${y_1}^2 \leq z$, is a contradiction. If $x_1 \vee y_1 = y_2$ then $x_2x_3y_2 \leq z$. Thus $e_3 = \{x_2, x_3, y_2\}$ is a hyperedge satisfying (1). In the other case, $e_3 = \{x_1 \vee y_1, x_2, y_2\}$ is a hyperedge satisfying (1).

Case 4.4. Wlog., let $E(G) = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2, x_3y_3\}$. We consider $\{x_3 \vee y_1, x_1, y_3\}$. If $x_3 \vee y_1 = x_1$ or $x_3 \vee y_1 = y_3$, then $x_1x_2y_3 \leq z$. Then $e_3 = \{x_1, x_2, y_3\}$ is a hyperedge satisfying the condition (1). In the other case, $e_3 = \{x_3 \vee y_1, x_1, y_3\}$ is a hyperedge satisfying the condition (1).

Case 4.5. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_2y_2, x_2y_3, x_3y_3\}$. We consider $\{x_1 \vee y_2, x_2, y_1\}$. If $x_1 \vee y_2 = x_2$, then ${y_2}^2 \leq z$, is a contradiction. If $x_1 \vee y_2 = y_1$, then ${x_1}^2 \leq z$, is a contradiction. Then $e_3 = \{x_1 \vee y_2, x_2, y_1\}$ is a hyperedge satisfying the condition (1).

Case 5. Let $|E(G)| = 6$. Consider three different subcases for this case:

Case 5.1. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_3y_1\}$.

We consider $\{x_1 \vee y_1, x_2, x_3\}$ and $\{x_1 \vee y_1, y_2, y_3\}$. If $x_1 \vee y_1 = x_2$, then $y_1y_2 \leq z$, a contradiction. If $x_1 \vee y_1 = x_3$, then ${y_1}^2 \leq z$, is a contradiction. If $x_1 \vee y_1 = y_2$ or $x_1 \vee y_1 = y_3$, then ${x_1}^2 \leq z$, is a contradiction. Thus $e_3 = \{x_1 \vee y_1, x_2, x_3\}$ and $e_4 = \{x_1 \vee y_1, y_2, y_3\}$ are hyperedges satisfying the condition (2).

Case 5.2. Wlog., presume that $E(G) = \{x_1y_1, x_1y_2, x_1y_3, x_2y_1, x_2y_2, x_3y_3\}$.

We consider $\{x_1 \vee y_3, x_3, y_1\}$. If $x_1 \vee y_3 = x_3$, then ${y_3}^2 \leq z$, is a contradiction. If $x_1 \vee y_3 = y_1$, then ${x_1}^2 \leq z$, is a contradiction. Thus $e_3 = \{x_1 \vee y_3, x_3, y_1\}$ is a hyperedge satisfying the condition (1).

Case 5.3. Wlog., presume that $E(G) = \{x_1y_1, x_1y_3, x_2y_1, x_2y_2, x_3y_2, x_3y_3\}$. We consider $\{x_1 \vee y_3, x_3, y_1\}$. If $x_1 \vee y_3 = x_3$, then ${y_3}^2 \leq z$, is a contradiction. If $x_1 \vee y_3 = y_1$, then ${x_1}^2 \leq z$, is a contradiction. Thus $e_3 = \{x_1 \vee y_3, x_3, y_1\}$ is a hyperedge satisfying the condition (1).

Case 6. If $7 \leq |E(G)| \leq 9$, then we have two vertices which are degree three in e_1 and the other in e_2. We suppose that $deg_G(x_1) = deg_G(y_1) = 3$. We consider $\{x_1 \vee y_1, x_2, x_3\}$ and $\{x_1 \vee y_1, y_2, y_3\}$. If $x_1 \vee y_1 = x_2$ or $x_1 \vee y_1 = x_3$, then ${y_1}^2 \leq z$, is a contradiction. If $x_1 \vee y_1 = y_2$ or $x_1 \vee y_1 = y_3$, then ${x_1}^2 \leq z$, is a contradiction. Hence $e_3 = \{x_1 \vee y_1, x_2, x_3\}$ and $e_4 = \{x_1 \vee y_1, y_2, y_3\}$ are hyperedges satisfying the condition (2).

By the fact that $gr(H_3(L,z)) \leq 2diam(H_3(L,z)) + 1$, we have that $gr(H_3(L,z)) \leq 9$.

2.1. Complete 3-Partıte Hypergraph

Definition 2. [10] A hypergraph H is called an n-partite if the vertex set V can be partitioned into disjoint subsets $V_1, V_2, \ldots, V_n$ of V such that a hyperedge in the hyperedge set E composes of a choice of completely one vertex from each subset of V. Also, a hypergraph H is called a complete n-partite hypergraph if the vertex set V can be partitioned into disjoint subsets $V_1, V_2, \ldots, V_n$ of V and each element of V_i for each $1 \leq i \leq n$ creates a hyperedge of H.

Proposition 2. Let $H_3(L,z)$ be a complete 3-partite hypergraph.

If $xy \leq z$, then x and y are contained by same subset V_i for some $i \in \{1,2,3\}$.

Proof. Let $H_3(L,z)$ has disjoint subsets V_1, V_2, V_3 which are partitions of the vertex set V. Let a be a vertex with $xya \leq z$. Without loss of generality, assume that $x \in V_1$ and $a \in V_2$. Then $e = \{x, y, a\}$ is not a hyperedge in $H_3(L,z)$ by our assumption. If $y \in V_3$, then e is a hyperedge since $H_3(L,z)$ is a complete 3-partite hypergraph, a contradiction. If $y \in V_2$, then there is a vertex $b \in V_3$ such that $e' = \{x, y, b\}$. But this contradicts the fact that $xy \leq z$. Therefore, y must be in V_1.

Theorem 2. Let z be a proper element of L. Then the following statements hold:

(1) If p_1, p_2 and p_3 are prime elements of L and $z = p_1 \wedge p_2 \wedge p_3 \neq 0_L$, then $H_3(L,z)$ is a complete 3-partite hypergraph.

(2) Let $a^2 \leq z$ for every 3-zero-divisor $a \in L$ with respect to z and $H_3(L,z)$ be a complete 3-partite hypergraph over the reduced lattice L. Then there exist prime elements p_1, p_2 and p_3 of L such that $p_1 \wedge p_2 \wedge p_3 \leq z$.

Proof. (1). Let $e = \{a,b,c\}$ be a hyperedge of $H_3(L,z)$. Then $abc \leq z = p_1 \wedge p_2 \wedge p_3$, that is, $abc \leq p_1, p_2, p_3$. Since p_i is a prime element for any $i \in \{1,2,3\}$, then $a \leq p_1$ or $b \leq p_1$ or $c \leq p_1$ and $a \leq p_2$ or $b \leq p_2$ or $c \leq p_2$ and $a \leq p_3$ or $b \leq p_3$ or $c \leq p_3$. Additionally, $ab \nleq p_i$ and $bc \nleq p_j$ and $ac \nleq p_k$ for some $i,j,k \in \{1,2,3\}$ since $ab, bc, ac \nleq z = p_1 \wedge p_2 \wedge p_3$. Wlog., we assume $ab \nleq p_1$. Then $a \nleq p_1$ and $b \nleq p_1$. Thus, we have $c \leq p_1$. Indeed, if $ac \nleq p_1$, then $b \leq p_1$, a contradiction. In a similar manner, suppose that $ac \nleq p_2$. Then $a \nleq p_2$ and $c \nleq p_2$. Thus, this yields $b \leq p_2$. Indeed, if $bc \nleq p_1$, then $a \leq p_1$, a contradiction and if $bc \nleq p_2$, then $a \leq p_2$, a contradiction. Thus, it must be $bc \nleq p_3$. Then, we get $a \leq p_3$. We assume that $a \leq p_3$ and $a \nleq p_1, p_2$, $b \leq p_2$ and $b \nleq p_1, p_3$ and $c \leq p_1$ and $c \nleq p_2, p_3$. Consequently, $H_3(L,z)$ is a complete 3-partite hypergraph with parts V_i for any $i \in \{1,2,3\}$ whose vertices must be only less than or equal to p_i.

(2). Let $H_3(L,z)$ be a complete 3-partite hypergraph and it has parts V_1, V_2 and V_3. Set $p_1 = V_1 \vee z$, $p_2 = V_2 \vee z$ and $p_3 = V_3 \vee z$. Then $x_1x_2x_3 \leq z$ for every $x_i \leq p_i$ for any $i \in \{1,2,3\}$. It is clear that $\left(\bigvee_{x_1 \in V_1} x_1\right)\left(\bigvee_{x_2 \in V_2} x_2\right)\left(\bigvee_{x_3 \in V_3} x_3\right) \vee z \leq z$, that is, $p_1p_2p_3 \leq z$ since L is a multiplicative lattice. As L is reduced, then $p_1 \wedge p_2 \wedge p_3 \leq z$. We assume that p_1 is not a prime element of L, that is, $ab \leq p_1$ and $a, b \nleq p_1$ for some $a, b \in L$. Since $ab \leq p_1 = V_1 \vee z$ then $ab \leq z$ or $ab \in V_1$. We have three cases for this assumption.

Case 1. Let $ab \in V_1$ and $ab \leq z$. This contradicts the definition of vertex set of $H_3(L,z)$.

Case 2. Let $ab \in V_1$ and $ab \nleq z$. Since $ab \in V_1$ and $a \notin V_1$, then $a \in V_2$ or $a \in V_3$. Wlog., assume that $a \in V_2$. So, $\{ab, a, c\}$ must be a hyperedge of $H_3(L,z)$ for any $c \in V_3$. However, since $a^2 \leq z$ for every 3-zero-divisor $a \in L$, then $a^2b \leq z$, contradiction.

Case 3. Let $ab \notin V_1$ and $ab \leq z$. By Proposition 2, a and b must be in the same V_i for any $i = \{2,3\}$. Wlog., let $a, b \in V_2$. Then, $xay \leq z, xa \nleq z, xy \nleq z, ay \nleq z$ and $xby \leq z, xb \nleq z, xy \nleq z, by \nleq z$ for some $x \in V_1$ and $y \in V_3$. By Proposition 2, we obtain that $xa \in V_3$, $xb \in V_3$, $ay \in V_1$, $by \in V_1$. Therefore, $\{ay, b, xa\}$ must be a hyperedge, since $H_3(L,z)$ is a complete 3-partite hypergraph. However, $a^2yx \leq z$ for $a^2 \leq z$, contradiction. We have a contradiction for each cases. Therefore, a or b must be less than or equal to p_1. Similarly, it can be seen that p_2 and p_3 are prime elements in L.

2.2. Cut Points and Bridge of $H_3(L,z)$

Definition 3. [6] A vertex a of a connected graph G is called a cut-point of G if there are vertices x and y of G with $a \neq x$ and $a \neq y$ such that a is in every path which is from x to y.

Theorem 3. Let $z \in L$ and $S = \{u \in L | u \leq z \text{ and } u \nleq a\}$. If $S \neq \emptyset$, then a is not a cut-point in $H_3(L,z)$.

Proof. Let a be in every path which is from x to y with $a \neq x$ and $a \neq y$. We know that $d(x,y) = 2, 3$ or 4 by Theorem 1. Consider $a \vee u$. Note that it is a vertex in $H_3(L,z)$ which is different from a. We consider the following cases:

Case 1. Let $d(x,y) = 2$. Then there are two hyperedges $e_1 = \{x, a, c_1\}$ and $e_2 = \{a, y, c_2\}$ for some vertices c_1, c_2 in $H_3(L,z)$ such that $x -_{e_1} a -_{e_2} y$ is a path. Consider $e'_1 = \{x, a \vee u, c_1\}$ and $e'_2 = \{a \vee u, y, c_2\}$.

Let $a \vee u \neq x$, $a \vee u \neq y$ and $a \vee u \neq c_i$ for $i \in \{1,2\}$. It is easily seen that e'_1 and e'_2 are two hyperedges such that $x -_{e'_1} a \vee u -_{e'_2} y$ is a path.

i. If $a \vee u = x$ or $a \vee u = y$, then x and y are adjacent.
ii. Consider $a \vee u = c_1$ or $a \vee u = c_2$. Wlog., assume that $a \vee u = c_1$. Then $e''_1 = \{x, a \vee u, a\}$ and $e'_2 = \{a \vee u, y, c_2\}$ are two hyperedges such that $x -_{e''_1} a \vee u -_{e'_2} y$ is a path.
Thus a is not a cut point.

Case 2. Let $d(x,y) = 3$. Then there are three hyperedges $e_1 = \{x, a, c_1\}$ and $e_2 = \{a, b, c_2\}$ and $e_3 = \{b, y, c_3\}$ for some vertices b, c_1, c_2, c_3 in $H_3(L,z)$ such that $x -_{e_1} a -_{e_2} b -_{e_3} y$ is a path. If $a \vee u$ is different from each of x, b and c_i for $i \in \{1,2,3\}$, then there is a path from x to y which does not contain a. Now, we consider other situations.

- **i.** Let $a \vee u = x$. Then consider $e'_2 = \{a \vee u, b, c_2\}$ and e_3. Note that there is a path $a \vee u -_{e'_2} b -_{e_3} y$. Thus a is not a cut point.
- **ii.** Let $a \vee u = b$. Consider $e'_1 = \{x, a \vee u, c_1\}$ and e_3. Clearly, there is a path $x -_{e'_1} a \vee u -_{e_3} y$. Hence a is not a cut point.
- **iii.** Let $a \vee u = y$. Consider $e'_1 = \{x, a \vee u, c_1\}$. Thus x and y are adjacent. Hence a is not a cut point.
- **iv.** $a \vee u = c_i$ for $i \in \{1,2\}$. It can be seen in a similar way in Case 1 (ii).
- **v.** Let $a \vee u = c_3$. Consider $e'_3 = \{b, y, a \vee u\}$ and $e'_1 = \{x, a \vee u, c_1\}$. Then there is a path such that $x -_{e'_1} a -_{e'_3} y$.

Case 3. Let $d(x,y) = 4$. Then there are four hyperedges $e_1 = \{x, a, c_1\}$ and $e_2 = \{a, b, c_2\}$, $e_3 = \{b, c, c_3\}$ and $e_4 = \{c, y, c_4\}$ for some vertices b, c, c_1, c_2, c_3, c_4 in $H_3(L,z)$ such that $x -_{e_1} a -_{e_2} b -_{e_3} y -_{e_4} c$ is a path. If $a \vee u$ is different from each of x, b, c, y and c_i for $i \in \{1,2,3,4\}$, then there is a path from x to y which does not contain a. Now, we consider other situations.

- **i.** Let $a \vee u = x$. Now, consider $e'_2 = \{a \vee u, b, c_2\}$. Then note that e'_2 is a hyperedge and there is a path $a \vee u -_{e'_2} b -_{e_3} c -_{e_4} y$.
- **ii.** Let $a \vee u = b$ Consider $e'_1 = \{x, a \vee u, c_1\}$ and $e'_3 = \{a \vee u, c, c_3\}$. Then note that e'_1 and e'_3 are two hyperedges and there is a path x$-_{e'_1} a \vee u -_{e'_3} c -_{e_4} y$.
- **iii.** Let $a \vee u = c$. Consider $e'_1 = \{x, a \vee u, c_1\}$ and $e'_4 = \{a \vee u, y, c_4\}$. Then note that e'_4 is a hyperedge and there is a path x$-_{e'_1} a \vee u -_{e'_4} y$.
- **iv.** Let $a \vee u = y$. Consider $e'_1 = \{x, a \vee u, c_1\}$. Note that e'_1 is a hyperedge and x and y are adjacent.
- **iv.** Let $a \vee u = c_i$ for $i \in \{1,2\}$. It can be seen in a similar way in Case 1 (ii).

v. Let $a \vee u = c_i$ for $i \in \{3,4\}$. It can be seen in a similar way in Case 2 (v).

We obtain the following result by the previous theorem.

Corollary 1. Let a be a vertex in $H_3(L,z)$ and $z \nleq a$. Then a is not a cut-point of $H_3(L,z)$.

Proposition 3. If $H_3(L,z)$ is connected, then $H_3(L,z)$ has not any bridge.

Proof. Let $e = \{a,b,c\}$ be a bridge of $H_3(L,z)$. Then $H_3(L,z)$ is disconnected if e is omitted in hypergraph. Take an element y with $0_L \neq y \nleq z$. Then $a \vee y, b \vee y, c \vee y \nleq z$. Also each of $e_1 = \{a \vee y, b, c\}$, $e_2 = \{a, b \vee y, c\}$ and $e_3 = \{a, b, c \vee y\}$ is a hyperedge. Thus, there is a cycle $a -_{e_3} b -_{e_1} c -_{e_2} a$. Indeed if e is omitted in hypergraph, $H_3(L,z)$ is connected. Thus, $H_3(L,z)$ has not any bridge.

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