Adıyaman University Journal of Science

https://dergipark.org.tr/en/pub/adyujsci
DOI: 10.37094/adyujsci.508986

ADYUJSCI

9 (2) (2019)
314-328

Chebyshev Series Solutions for a Class of System of Linear Integro-Differential Equations with Weakly Singular Kernel

Yalçın ÖZTÜRK*

Muğla Sıtkı Koçman University, Ula Ali Koçman Vocational School, Muğla, Turkey

yozturk@mu.edu.tr, ORCID: 0000-0002-4142-5633

Abstract

In this study, a numerical algorithm for solving a class of system of linear integro differential equations with weakly singular kernel is presented. This algorithm is based on polynomial approximation and collocation method, using the first kind Chebyshev polynomial basis. This method transforms the equations and the given conditions into matrix equation which corresponds to a system of linear algebraic equation. To show the validity and applicability of the numerical method some experiments are examined. Present method is compared some numerical methods.

Keywords: Singular system of integro-differential equations, Weakly singular kernel, Abel's equation, Collocation method, Chebyshev polynomials

Zayıf Tekil Çekirdekli Lineer İntegro Diferansiyel Denklemlerin Bir Sınıfının Chebyshev Seri Çözümleri

Öz

Bu çalışmada, zayıf tekil çekirdekli lineer integro diferansiyel denklemlerin bir sınıfı için bir nümerik algoritma sunulacaktır. Bu algoritma birinci tip Chebyshev polinom bazı yardımıyla polinom yaklaşımı ve sıralama metodunu temel almaktadır. Bu metot verilen denklem ve koşulları bir matris denklemine dönüştürür. Nümerik metodun uygulanabilirliğini ve doğruluğunu göstermek amacıyla bazı örnekler incelenecektir. Sunulan metot diğer metotlar ile kıyaslanmıştır.

* Corresponding Author

Received: 06 January 2019 Accepted: 26 November 2019

Anahtar Kelimeler: İntegro-diferansiyel denklemlerin singular sistemleri, Zayıf tekil çekirdek, Abel denklemi, Sıralama metodu, Chebyshev polinomları

1. Introduction

Applications in many important fields, like fracture mechanics, elastic contact problems, the theory of porous filtering and combined infrared radiation and molecular conduction [1-4], contain integral and integro-differential equations with singular kernel. Singular integral and integro-differential equations are usually difficult to solve analytically so it is required to obtain the approximate solution and the aim in the present research is to develop an accurate as well as easy to implement numerical solution scheme to treat such equations.

These type equations have been solved numerically by some authours. We can give some studies in literature [5-17].

We consider the class of system of linear integro-differential eqeuations with weakly singular kernel

$$\sum_{p=0}^{n}\sum_{q=0}^{m} P_{pq}^{j}(t)y_q^{(p)}(t)=\lambda_j\int_0^t \frac{y_j(x)}{\sqrt{t-x}}dx+f_j(t)\,,\ j=0,1,\ldots m \tag{1}$$

with conditions

$$\sum_{r=0}^{n-1}\sum_{p=0}^{m} c_{rp}^{j}y_p^{(r)}(t)=\lambda_j \tag{2}$$

where $f_j(t)$ and $P_{pq}^{j}(t)$ are analytic functions, c_{rp}^{j} and λ_j are constant. For $q=0$, Eq. (1) is a nonsystem integro-differential equation and is Abel's equation with convenient coefficients. We construct to the shifted Chebyshev series solutions that is;

$$y_j^N(t)=\sum_{r=0}^{N} a_r^j T_r(t) \tag{3}$$

where $T_r(t)$ denotes the Chebyshev polynomials of the first kind, a_r^j $(0\le r\le N)$ are unknown Chebyshev coefficients, and N is chosen any positive integer.

The Chebyshev polynomials $T_r(t)$ of the first kind are the polynomials in t of degree r, defined by relation [18, 19]

$$T_r(t) = \cos n\theta \text{, when } t = \cos\theta .$$

If the range of the variable t is the interval $[-1,1]$, the range the corresponding variables θ can be taken $[0,\pi]$. These polynomials have the following properties [18, 19]:

i) The Chebyshev polynomials with degree $r+1$ has absolutly complete $r+1$ real zeroes which is called The Chebyshev-Gaus nodes on the interval $[-1,1]$ and these roots compute as

$$t_i = \cos\frac{(2(r-i)+1)\pi}{2(r+1)},\ i = 0,1,\ldots,N . \tag{4}$$

ii) The Chebyshev polynomials with degree r is ortogonal on $[-1,1]$ together with the weight function $w(t) = (1-t^2)^{-\frac{1}{2}}$.

iii) The passing equation between the powers t^n and the Chebyshev polynomials $T_r(t)$ is

$$t^{2r} = 2^{-2r+1}\sum_{s=0}^{r}\binom{r}{r-s}T_{2s}(t), \tag{5}$$

$$t^{2r+1} = 2^{-2r}\sum_{s=0}^{r}\binom{2r+1}{r-s}T_{2s+1}(t). \tag{6}$$

2. Fundamental Matrix Relations

In this section, we convert the part of Eq. (1) and conditions Eq. (2) into the matrix form. Using the Eq. (3), we have the matrix relation of solutions and its derivatives as:

$$y_j^N(t) = \mathbf{T}(t)\mathbf{A}^j,\ \left(y_j^N\right)^{(s)}(t) = \mathbf{T}^{(s)}(t)\mathbf{A}^j \quad j = 0,1,\ldots m, \tag{7}$$

where

$$\mathbf{T}(t) = [T_0\ (t)\ T_1\ (t) \ldots T_N\,(t)] \qquad \mathbf{A}^j = [a_0^j\ a_1^j \ldots a_N^j]^T$$

By using the expression (5) and (6) and taking $r = 0,1,\ldots,N$ we find the corresponding matrix relation as follows

$$(\mathbf{Y}(t))^T = \mathbf{D}(\mathbf{T}(t))^T \quad and \quad \mathbf{Y}(t) = \mathbf{T}(t)\mathbf{D}^T\,, \tag{8}$$

where

$$\mathbf{Y}(t) = [1\ \ t \ldots t^N\,],$$

and for odd N,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \binom{N}{(N-1)/2} & 0 & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix},$$

for even N,

$$\mathbf{D} = \begin{bmatrix} \frac{1}{2}\binom{0}{0}2^1 & 0 & 0 & \cdots & 0 \\ 0 & \binom{1}{0}2^0 & 0 & \cdots & 0 \\ \frac{1}{2}\binom{2}{1}2^{-1} & 0 & \binom{2}{0}2^{-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{1}{2}\binom{N}{N/2}2^{1-N} & 0 & \binom{N}{(N-2)/2}2^{1-N} & \cdots & \binom{N}{0}2^{1-N} \end{bmatrix}.$$

Then, by taking into account (8) we obtain

$$\mathbf{T}(t) = \mathbf{Y}(t)(\mathbf{D}^{-1})^T$$

and

$$(\mathbf{T}(t))^{(q)} = \mathbf{Y}^{(q)}(t)(\mathbf{D}^{-1})^T \ , \ \ q = 0,1,\ldots,n\,. \tag{9}$$

To obtain the matrix $\mathbf{Y}^{(s)}(t)$ in terms of the matrix $\mathbf{Y}(t)$, we can use the following relation:

$$\mathbf{Y}^{(q)}(t)=\mathbf{Y}(t)(\mathbf{B}^T)^q, \tag{10}$$

where $(\mathbf{B}^T)^0=I_{(n+1)\times(n+1)}$ and

$$\mathbf{B}=\begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 2 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & N & 0 \end{bmatrix}.$$

Consequently, by substituting the matrix form (9) and (10) into (7), we get the matrix representation of approximate solution and its derivatives as follows:

$$\left(y_j^N\right)^{(q)}(t)=\mathbf{Y}(t)\left(\mathbf{B}^T\right)^q(\mathbf{D}^T)^{-1}\mathbf{A}^j. \tag{11}$$

On the other hand, the matrix representation of the conditions Eq. (2) are given by matrix relation as:

$$\sum_{r=0}^{n-1}\sum_{p=0}^{m}c_{rp}^j\mathbf{Y}(t)\left(\mathbf{B}^T\right)^r(\mathbf{D}^T)^{-1}\mathbf{A}^p=\lambda_j. \tag{12}$$

For the part $\lambda_j\int_0^t\frac{y_j(x)}{\sqrt{t-x}}dx$, we have the following equation [7]

$$\int_0^t\frac{x^N}{\sqrt{t-x}}dx=\frac{\sqrt{\pi}\Gamma(N+1)x^{(\frac{1}{2}+N)}}{\Gamma(N+\frac{3}{2})}$$

and so, we get

$$\lambda_j\int_0^t\frac{y_j(x)}{\sqrt{t-x}}dx=\lambda_j\left(\int_0^t\frac{\mathbf{Y}(t)}{\sqrt{t-x}}dt\right)(\mathbf{D}^{-1})^T\mathbf{A}^j=\lambda_j\mathbf{Q}(t)(\mathbf{D}^{-1})^T\mathbf{A}^j, \tag{13}$$

where

$$\mathbf{Q}(t)=\begin{bmatrix}\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} & \frac{\sqrt{\pi}\Gamma(2)}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} & \cdots & \frac{\sqrt{\pi}\Gamma(N+1)}{\Gamma(N+\frac{3}{2})}x^{\frac{1}{2}+N}\end{bmatrix}.$$

3. Description Method

In this chapter, we present the fundamental matrix equation which give us the matrix form of Eq. (1). To obtain the matrix form of Eq. (1), If Eq. (11) is put into Eq. (1), we have

$$\sum_{p=0}^{n}\sum_{q=0}^{m} P_{pq}^{j}(t)\mathbf{Y}(t)\left(\mathbf{B}^{T}\right)^{q}(\mathbf{D}^{T})^{-1}\mathbf{A}^{q}=\lambda_{j}\int_{0}^{t}\frac{y_{j}(x)}{\sqrt{t-x}}dx+f_{j}(t),\ j=0,1,\ldots m \tag{14}$$

then, it can be written as:

$$\left(\sum_{q=0}^{m}\mathbf{P}_{q}\ \overline{\mathbf{Y}(t)}\,\mathbf{B}_{q}\mathbf{D}-\lambda_{j}\ \overline{\mathbf{Q}(t)}\,\mathbf{D}\right)\mathbf{A}=\mathbf{F}, \tag{15}$$

where

$$\overline{\mathbf{Y}(t)}=\begin{bmatrix}\mathbf{Y}(t) & 0 & \cdots & 0\\ 0 & \mathbf{Y}(t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Y}(t)\end{bmatrix},\ \mathbf{B}_{q}=\begin{bmatrix}\left(\mathbf{B}^{T}\right)^{q} & 0 & \cdots & 0\\ 0 & \left(\mathbf{B}^{T}\right)^{q} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \left(\mathbf{B}^{T}\right)^{q}\end{bmatrix},$$

$$\mathbf{D}=\begin{bmatrix}(\mathbf{D}^{T})^{-1} & 0 & \cdots & 0\\ 0 & (\mathbf{D}^{T})^{-1} & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & (\mathbf{D}^{T})^{-1}\end{bmatrix},\ \mathbf{P}_{q}(t)=\begin{bmatrix}P_{q0}^{0}(t) & P_{q1}^{0}(t) & \cdots & P_{qn}^{0}(t)\\ P_{q0}^{1}(t) & P_{q1}^{1}(t) & \cdots & P_{qn}^{1}(t)\\ \vdots & \vdots & \ddots & \vdots\\ P_{q0}^{m}(t) & P_{q1}^{m}(t) & \cdots & P_{qn}^{m}(t)\end{bmatrix},$$

$$\overline{\mathbf{Q}(t)}=\begin{bmatrix}\mathbf{Q}(t) & 0 & \cdots & 0\\ 0 & \mathbf{Q}(t) & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & \mathbf{Q}(t)\end{bmatrix},\ \mathbf{F}=\begin{bmatrix}f_{0}(t)\\ f_{1}(t)\\ \vdots\\ f_{m}(t)\end{bmatrix},\ \mathbf{A}=\begin{bmatrix}\mathbf{A}^{0}\\ \mathbf{A}^{1}\\ \vdots\\ \mathbf{A}^{m}\end{bmatrix},$$

If the Chebyshev-Gauss grid nodes are put into Eq. (14), the matrix form of the Eq. (1) is obtained

$$\left(\sum_{q=0}^{m}\overline{\mathbf{P}_{q}}\ \overline{\mathbf{Y}}\,\mathbf{B}_{q}\mathbf{D}-\lambda_{j}\ \overline{\mathbf{Q}}\,\mathbf{D}\right)\mathbf{A}=\overline{\mathbf{F}}, \tag{16}$$

$$\overline{\mathbf{Y}} = \begin{bmatrix} \overline{\mathbf{Y}(t_0)} & 0 & \cdots & 0 \\ 0 & \overline{\mathbf{Y}(t_1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\mathbf{Y}(t_N)} \end{bmatrix}, \overline{\mathbf{P}_q} = \begin{bmatrix} \mathbf{P}_{q0} & 0 & \cdots & 0 \\ 0 & \mathbf{P}_{q1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_{qN} \end{bmatrix},$$

$$\overline{\mathbf{Q}} = \begin{bmatrix} \overline{\mathbf{Q}(t_0)} & 0 & \cdots & 0 \\ 0 & \overline{\mathbf{Q}(t_1)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \overline{\mathbf{Q}(t_N)} \end{bmatrix}, \ \overline{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_0 \\ \mathbf{F}_1 \\ \vdots \\ \mathbf{F}_N \end{bmatrix},$$

where

$$\overline{\mathbf{Y}(t_i)} = \begin{bmatrix} \mathbf{Y}(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{Y}(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Y}(t_i) \end{bmatrix}, \ \mathbf{P}_{qi} = \begin{bmatrix} \mathbf{P}_q(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{P}_q(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{P}_q(t_i) \end{bmatrix},$$

$$\overline{\mathbf{Q}(t_i)} = \begin{bmatrix} \mathbf{Q}(t_i) & 0 & \cdots & 0 \\ 0 & \mathbf{Q}(t_i) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathbf{Q}(t_i) \end{bmatrix}, \ \mathbf{F}_i = \begin{bmatrix} f_0(t_i) \\ f_1(t_i) \\ \vdots \\ f_m(t_i) \end{bmatrix},$$

where the dimension of matrices $\overline{\mathbf{Y}}$, $\mathbf{B}_q$, $\mathbf{D}$, $\overline{\mathbf{P}_q}$, $\overline{\mathbf{Q}}$ diagonal matrices and the dimension of these matrices are $m(N+1)\times m(N+1)$, $\mathbf{P}_k(t)$, $\mathbf{B}_k$ are $(m+1)\times(s+1)$ and $\overline{\mathbf{F}}$ is $m(N+1)\times 1$.

Hence, the matrix equation (16) corresponding to Eq. (1) can be written in the form

$$\mathbf{WA} = \overline{\mathbf{F}} \ \text{ or } \left[\mathbf{W}; \overline{\mathbf{F}}\right], \tag{17}$$

where

$$\mathbf{W} = \sum_{q=0}^{m} \overline{\mathbf{P}_q}\, \overline{\mathbf{Y}}\, \mathbf{B}_q \mathbf{D}.$$

Moreover, the matrix form for conditions can be written as

$$\mathbf{UA} = \mathbf{G}, \tag{18}$$

where for $k = 0,1,\ldots,m$, $l = 0,1,\ldots,m(N+1)$,

$$\mathbf{U} = [u_{kl}] = \begin{bmatrix} \mathbf{U}_0 & 0 & 0 & \cdots & 0 \\ 0 & \mathbf{U}_1 & 0 & \cdots & 0 \\ 0 & 0 & \mathbf{U}_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{U}_m \end{bmatrix}, \ \mathbf{G} = \begin{bmatrix} \lambda_0 \\ \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_m \end{bmatrix},$$

and

$$\mathbf{U}_j = c_{rp}^j \mathbf{Y}(t)\left(\mathbf{B}^T\right)^r (\mathbf{D}^T)^{-1}, \ j = 0,1,\ldots,m.$$

If the condition matrix Eq. (18) relocate by the last n rows of the matrix Eq. (17) to get the approximate solution of system of linear integro-differential equations with weakly singular kernel Eq. (1) with the conditions Eq. (2) by Chebyshev polynomials, we obtain the augmented matrix:

$$\overline{\mathbf{W}}\,\mathbf{A} = \dot{\mathbf{F}}, \tag{19}$$

where

$$\overline{\mathbf{W}} = \begin{bmatrix} w_{00} & w_{01} & \cdots & w_{0\times m(N+1)} \\ w_{10} & w_{11} & \cdots & w_{1\times m(N+1)} \\ \vdots & \vdots & \ddots & \vdots \\ w_{m(N-1)\times 0} & w_{m(N-1\times)1} & \cdots & w_{m(N-1)\times m(N+1)} \\ u_{00} & u_{01} & \cdots & u_{0\times m(N+1)} \\ \vdots & \vdots & \cdots & \vdots \\ u_{m0} & u_{m1} & \cdots & u_{m\times m(N+1)} \end{bmatrix}$$

and

$$\dot{\mathbf{F}} = \begin{bmatrix} \mathbf{F}_0 \\ \vdots \\ \mathbf{F}_{N-1} \\ \mathbf{G} \end{bmatrix}.$$

Therefore, Eq. (19) give us a algebraic systems which include $m(N+1)\times m(N+1)$ linear algebraic equations with $m(N+1)$ unknown Chebyshev coefficients. If $\overline{\mathrm{W}}$ is invertible, then it can be written $\mathbf{A} = (\overline{\mathbf{W}})^{-1}\dot{\mathbf{F}}$. Thus, the matrix $\mathbf{A}$ (thereby the coefficients matrix $\mathbf{A}^j$, $j = 0,1,\ldots,m$) is uniquely determined.

4. Test Problems

In this section, we give five test problem corresponding to equation (1) to demonstrate the efficiency of proposed method. Examples 1 and 2 is are given systems of integro-differential equations, 3 and 4 are Abel's equations. All numerical scheme is calculated by using Maple 15. The absolute errors in Tables are the values of $N_e = \left|y_j(t) - y_j^N(t)\right|$, those at selected points.

Example 1. Let us consider the following singular integro-differential equation

$$y_0^{'}(t) + y_0(t) - ty_1(t) = \int_0^t \frac{y_0(x)}{\sqrt{t-x}}\,dx + 1 + t - t^3 - \frac{4}{3}t^{4/3},$$

$$y_1^{'}(t) + ty_0(t) + y_1(t) = \int_0^t \frac{y_1(x)}{\sqrt{t-x}}\,dx + 2t + 2t^2 - \frac{16}{15}t^{5/2},$$

with subject to conditions

$$y_0(0) = y_1(0) = 0,$$

with eaxact solutions $y_0(t) = t$ ve $y_1(t) = t^2$. For $N = 5$, we seek the approximate solutions, we have the fundamental matrix equation from Eq.(16)

$$\left(\overline{\mathbf{P}_0}\;\overline{\mathbf{Y}}\,\mathbf{D} + \overline{\mathbf{P}_1}\;\overline{\mathbf{Y}}\,\mathbf{B}_1\mathbf{D} - \overline{\mathbf{Q}}\,\mathbf{D}\right)\mathbf{A} = \overline{\mathbf{F}}, \tag{20}$$

with

$$\mathbf{P}_1(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{P}_0(t)=\begin{bmatrix}1 & -t\\ t & 1\end{bmatrix},$$

$$\mathbf{Q}(t)=\begin{bmatrix}\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(\frac{3}{2})}x^{\frac{1}{2}} & \frac{\sqrt{\pi}\Gamma(2)}{\Gamma(\frac{5}{2})}x^{\frac{3}{2}} & \frac{\sqrt{\pi}\Gamma(3)}{\Gamma(\frac{7}{2})}x^{\frac{5}{2}} & \frac{\sqrt{\pi}\Gamma(4)}{\Gamma(\frac{9}{2})}x^{\frac{7}{2}} & \frac{\sqrt{\pi}\Gamma(5)}{\Gamma(\frac{11}{2})}x^{\frac{9}{2}}\end{bmatrix},$$

and with collocation points, we have

$$\overline{\mathbf{P}_0}=\begin{bmatrix}\mathbf{P}_{00} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{P}_{01} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{P}_{02} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{P}_{03} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbf{P}_{04} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{05}\end{bmatrix},\ \overline{\mathbf{P}_1}=\begin{bmatrix}\mathbf{P}_{10} & 0 & 0 & 0 & 0 & 0\\ 0 & \mathbf{P}_{11} & 0 & 0 & 0 & 0\\ 0 & 0 & \mathbf{P}_{12} & 0 & 0 & 0\\ 0 & 0 & 0 & \mathbf{P}_{13} & 0 & 0\\ 0 & 0 & 0 & 0 & \mathbf{P}_{14} & 0\\ 0 & 0 & 0 & 0 & 0 & \mathbf{P}_{15}\end{bmatrix},\ \overline{\mathbf{F}}=\begin{bmatrix}\mathbf{F}_0\\ \mathbf{F}_1\\ \mathbf{F}_2\\ \mathbf{F}_3\\ \mathbf{F}_4\\ \mathbf{F}_5\end{bmatrix},$$

$$\overline{\mathbf{B}_1}=\begin{bmatrix}\mathbf{B} & 0\\ 0 & \mathbf{B}\end{bmatrix},\ \mathbf{D}=\begin{bmatrix}(\mathbf{D}^T)^{-1} & 0\\ 0 & (\mathbf{D}^T)^{-1}\end{bmatrix},\ ,\ \mathbf{A}=\begin{bmatrix}\mathbf{A}^0\\ \mathbf{A}^1\end{bmatrix}$$

and where

$$\mathbf{P}_{1i}(t)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},\ \mathbf{P}_{0i}(t_i)=\begin{bmatrix}1 & -t_i\\ t_i & 1\end{bmatrix},\ \mathbf{F}_i=\begin{bmatrix}1+t_i-t_i^{\ 3}-\frac{4}{3}t_i^{\ 4/3}\\ 2t_i+2t_i^{\ 2}-\frac{16}{15}t_i^{\ 5/2}\end{bmatrix},$$

$$\mathbf{Q}(t_i)=\begin{bmatrix}\frac{\sqrt{\pi}\Gamma(1)}{\Gamma(\frac{3}{2})}t_i^{\frac{1}{2}} & \frac{\sqrt{\pi}\Gamma(2)}{\Gamma(\frac{5}{2})}t_i^{\frac{3}{2}} & \frac{\sqrt{\pi}\Gamma(3)}{\Gamma(\frac{7}{2})}t_i^{\frac{5}{2}} & \frac{\sqrt{\pi}\Gamma(4)}{\Gamma(\frac{9}{2})}t_i^{\frac{7}{2}} & \frac{\sqrt{\pi}\Gamma(5)}{\Gamma(\frac{11}{2})}t_i^{\frac{9}{2}}\end{bmatrix},$$

$$(\mathbf{D}^T)^{-1}=\begin{bmatrix}1 & 0 & -1 & 0 & 1 & 0\\ 0 & 1 & 0 & -3 & 0 & 5\\ 0 & 0 & 2 & 0 & -8 & 0\\ 0 & 0 & 0 & 4 & 0 & -20\\ 0 & 0 & 0 & 0 & 8 & 0\\ 0 & 0 & 0 & 0 & 0 & 16\end{bmatrix},\ \mathbf{B}^T=\begin{bmatrix}0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 2 & 0 & 0 & 0\\ 0 & 0 & 0 & 3 & 0 & 0\\ 0 & 0 & 0 & 0 & 4 & 0\\ 0 & 0 & 0 & 0 & 0 & 5\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}\cdot$$

Moreover, the matrix form for conditions can be written as:

$$\begin{bmatrix}y_0(0)\\ y_1(0)\end{bmatrix}\equiv\begin{bmatrix}\mathbf{U}_0 & 0\\ 0 & \mathbf{U}_0\end{bmatrix}\begin{bmatrix}\mathbf{A}^0\\ \mathbf{A}^1\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}.$$

Solving the new augmented matrix based on conditions, truncated Chebyshev coefficients matrix are obtained as:

$$\mathbf{A}^0 = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ and } \mathbf{A}^1 = \begin{bmatrix} 0.375 \\ 0.5 \\ 0.125 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, the solutions of the problem for $N = 5$ become $y_0^5(t) = t$ and $y_1^5(t) = t^2$ which are the exact solution of problem.

Example 2. Let us consider the following the systems of Volterra integro-differential equation with weakly singular kernel

$$y_0'(t) + y_1'(t) + e^{-t} y_0(t) + y_1(t) = \int_0^t \frac{y_0(x)}{\sqrt{t-x}} dx + f_0(t),$$

$$y_0'(t) - y_1'(t) + y_0(t) - e^{-t} y_1(t) = \int_0^t \frac{y_1(x)}{\sqrt{t-x}} dx + f_1(t),$$

where

$$f_0(t) = 2e^t \cos(t) + e^t \sin(t) + \cos(t) - \int_0^t \frac{e^x \cos(x)}{\sqrt{t-x}} dx,$$

$$f_1(t) = e^t \cos(t) - 2e^t \sin(t) - \sin(t) - \int_0^t \frac{e^x \sin(x)}{\sqrt{t-x}} dx,$$

with $y_0(0) = 1$ and $y_1(0) = 0$ and exact solutions are $y_0(t) = e^t \cos(t)$ and $y_1(t) = e^t \sin(t)$. Approximately solving these systems by present method for $N = 5,6,9$ we obtain the numerical results in Table 1. Moreover, we display the absolute errors for $N = 5,6,9,10$ in Fig. 1 and Fig. 2.

Table 1. Absolute errors for various N

t	$N=5$		$N=6$		$N=9$	
	N_e^0	N_e^1	N_e^0	N_e^1	N_e^0	N_e^1
0.2	0.3994E-5	0.1026E-4	0.6405E-6	0.2181E-6	0.2772E-10	0.6070E-10
0.4	0.178E-5	0.2896E-5	0.4452E-6	0.5959E-7	0.2240E-10	0.4397E-10
0.6	0.2981E-5	0.6477E-5	0.3803E-6	0.5870E-7	0.1753E-10	0.2801E-10
0.8	0.4341E-5	0.8624E-5	0.1015E-5	0.6360E-7	0.1839E-10	0.2368E-10

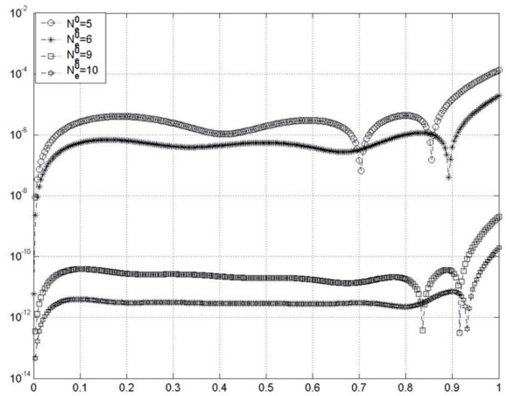

Figure 1. Comparison of the absolute errors for $y_0(t)$ in Ex. 2

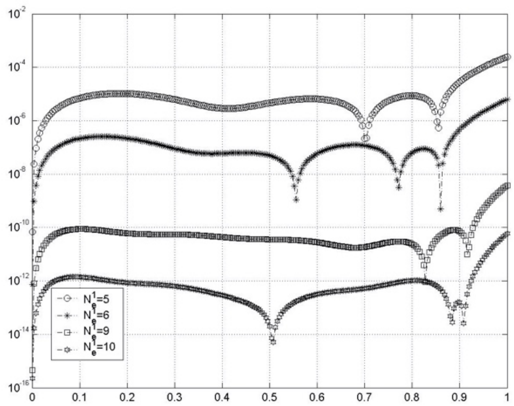

Figure 2. Comparison of the absolute errors for $y_1(t)$ in Ex. 2

Example 3. We consider the Volterra integro-differential equation with weakly singular kernel [7]

$$y''(x) + y(x) + \frac{1}{\sqrt{\pi}} \int_0^x \frac{y''(t)}{\sqrt{x-t}} = f(x)$$

with $y(0) = y'(0) = 1$. Here, if $f(x)$ is chosen as $3 + x + x^2 + \frac{4\sqrt{x}}{\sqrt{\pi}}$, the exact solution is $1 + x + x^2$. For $N = 4$, we obtain the approximate solution for digits 20

$$y_4(x) = 1 + t + t^2 + 0.13521E - 19t^3 - 0.31881E - 20t^4 .$$

In [12], it is obtained the approximate solution for $N = 5$ in [14]

$$y(x) = 1 + x + x^2 + 0.1E - 18x^5$$

and we compare the absolute errors present method and Berstein series solution [12] in Table 2.

Table 2. Comparision of Berstein method and present method

x	Exact solution	Işik [14] ($N = 5$)	Present met. ($N = 4$)
0.0	1.00	0.000E-0	0.200E-19
0.2	1.24	0.355E-18	0.200E-19
0.4	1.56	0.532E-18	0.000E-00
0.6	1.96	0.700E-18	0.000E-00
0.8	2.44	0.856E-18	0.000E-00
1.0	3.00	0.100E-18	0.000E-00

Example 4. Finally, we consider the following singular Volterra integral equation [8]:

$$y(t) = e^t\left(1 + \sqrt{\pi} erf\left(\sqrt{t}\right)\right) - \int_0^t \frac{y(x)}{\sqrt{t - x}} dx$$

which has the exact solution $y(t) = e^t$. We compare the maximal errors of Tau method and present method in Table 3.

Table 3: Comparison of maximal error of Tau method and present method

N	Tau method [18] Standart base	method	Legendre base	Present method
5	0.673E-3	0.321E-3	0.313E-3	0.100E-5
10	0.381E-6	0.209E-6	0.222E-6	0.100E-12
15	0.541E-9	0.354E-9	0.459E-6	0.548E-16
20	0.648E-11	0.411E-11	0.459E-11	0.628E-20
25	0.714E-13	0.601E-13	0.586E-13	0.746E-24

5. Conclusion

We have introduced a numerical scheme for solving systems of integro-differential equations with weakly singular kernel. The numerical scheme is based on collocation method with using the Chebyshev polynomials whose are orthogonal polynomial bases. Some examples have been solved to illustrate the validity and efficiency of the proposed technique. The examples show that the proposed numerical scheme produces the good results and produce the desired accuracy only in a few terms with high accuracy. The method is also quite straightforward to write computer code to construct a low-cost scheme. Increasing the degree of approximation also causes the convergency of the method. These facts motivate us to state the proposed method as a fast, reliable, valid and powerful tool for solving weakly singular Volterra integral equations. The proposed method can be transformed others singular equations by using some beneficial theorems in [19].

References

[1] Frankel, J., *A Galerkin solution to a regularized Cauchy singular integro-differential equation*, Quarterly of Applied Mathematics, L11 (2), 245–258, 1995.

[2] Green, C.D., Integral Equation Methods, Nelsson, New York, 1969.

[3] Hori, M., Nasser, N., *Asymptotic solution of a class of strongly singular integral equations*, SIAM Journal on Applied Mathematics (SIAP), 50 (3), 716–725, 1990.

[4] Muskhelishvili, N.I., Singular Integral Equations, Noordhoff, Leiden, 1953.

[5] Brunner, H., Pedas, A., Vainikko, G., *Piecewise polynomial collocation methods for linear Volterra integrodifferential equations with weakly singular kernels*, SIAM Journal on Numerical Analysis, 39, 957–982, 2001.

[6] Gülsu, M., Öztürk, Y., *Numerical approach for the solution of hypersingular integro differential equations*, Applied Mathematics and Computation, 230, 701-710, 2014.

[7] Işık, O.R., Sezer, M., Güney, Z., *Bernstein series solution of a class of linear integro-differential equations with weakly singular kernel*, Applied Mathematics and Computation, 217, 7009-7020, 2011.

[8] Karimi, S., Soleymani, F., *Tau approximate solution of weakly singular Volterra integral equations*, Mathematical and Computer Modelling, 57(3-4), 494-502, 2013.

[9] Koya, A.C., Erdoğan, F., *On the solution of integral equations with strongly singular kernels*, Quarterly of Applied Mathematics, 45, 105–122, 1987.

[10] Lakestani, M., Saray, B.N., Dehghan, M., *Numerical solution for the weakly singular Fredholm integro-differential equations using Legendre multiwavelets*, Journal of Computational and Applied Mathematics, 235, 3291-3303, 2011.

[11] Maleknejad, K., Arzhang, A., *Numerical solution of the Fredholm singular integro differential equation with Cauchy kernel by using Taylor-series expansion and Galerkin method*, Applied Mathematics and Computation, 182, 888-897, 2006.

[12] Öztürk, Y., Gülsu, M., *A collocation method for solving system of Volterra-differential-difference equations with terms of Chebyshev polynomials,* British Journal of Applied Science & Technology, 14(4), 1-20, 2016.

[13] Parts, I., Pedas, A., *Collocation approximations for weakly singular Volterra integro-differential equations*, Mathematical Modelling and Analysis, 8, 315–328, 2003.

[14] Pedas, A., Tamme E., *Spline collocation method for integro-differential equations with weakly singular kernels*, Journal of Computational and Applied Mathematics, 197, 253-269, 2006.

[15] Razlighi, B.B., Soltanalizadeh, B., *Numerical solution for system of singular nonlinear Volterra integro-differential equations by Newton-Product method*, Applied Mathematics and Computation, 219, 8375-8383, 2013.

[16] Singh, V.K., Postnikov, E.B., *Operational matrix approach for solution of integro-differential equations arising in theory ofanomalous relaxation processes in vicinity of singular point*, Applied Mathematical Modelling, 37, 6609-6616, 2013.

[17] Turhan, İ., Oğuz, H., Yusufoğlu, E., *Chebyshev polynomial solution of the system of Cauchy type singular integral equations of first kind*, Journal of Computational Mathematics, 90(5), 944-954, 2013.

[18] Fox, L., Parker, I.B., Chebyshev Polynomials in Numerical Analysis, Oxford University Press, London, 1968.

[19] Mason, J.C., Handscomb, D.C., Chebyshev Polynomials, Chapman and Hall/CRC, New York, 2003.