Araştırma Makalesi / Research Article

(p,q)-Baskakov Operators

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Abstract
In the present paper, we give a new analogue of Baskakov operators and we call them (p,q)-Baskakov operators which are a generalization of q-Baskakov operators. We obtain their respective formulae for central moments. Also, we study the rate of convergence and approximation properties for these operators using the modulus of smoothness.

Keywords: (p,q)-calculus, Korovkin's theorem, Baskakov operators, Modulus of continuity.

(p,q)-Baskakov Operatörleri

Öz
Bu çalışmada q-Baskakov operatörünün bir genelleşmesi olan (p,q)-Baskakov operatörü olarak adlandırılan Baskakov operatörünün yeni bir türü tanıtılmıştır. Merkezi momentler için formüller elde edilmiştir. Aynı zamanda süreklilik modülü kullanılarak bu operatörlerin yaklaşım özellikleri ve yakınsama oranı çalışılmıştır.

Anahtar kelimeler: (p,q)-analiz, Korovkin teoremi, Baskakov operatörleri, Süreklilik modülü.

1. Introduction

The quantum calculus (q-calculus) which is an ordinary calculus without taking limits has a lot of applications in various areas such as quantum theory, mathematics, theory of relativity and mechanics. Jackson [1] introduced q-functions in the beginning of the twentieth century and developed q-calculus. In the approximation theory, q-Bernstein polynomials were first presented as the applications of q-calculus by Lupas [2]. q-analogues of the well known operators have been proposed and their approximation behavior has been discussed (see [3-19]). After that various generalization of the q-Baskakov operators were intoroduced and their approximation properties have been studied in [4-7].

Recently, the (p,q)-calculus which the extension of the q-calculus to post-quantum calculus has been discussed by researchers in approximation theory. The (p,q)-calculus has been used efficiently not only in the approximation theory but also in different sciences areas such as differential equations, hypergeometric function, lie group, field theory and physical sciences. Many researchers have transferred well-known positive linear operators to the (p,q)- calculus and advantages of (p,q) analogue of them have intensively investigated [20-30]. Some basic definition and fundamental theorems in (p,q)-calculus was presented by Sadjang [20]. Mursaleen et. al. introduced (p,q)-Bernstein polynomials and studied their approximation properties [21]. Aral and Gupta [22] proposed the (p,q)-analog of the Baskakov operators $0<q<p\leq 1$ by

$$B_{n,p,q}(f;x)=\sum_{k=0}^{n} b_{n,k}^{p,q}(x) f\left(\frac{p^{n-1}[k]_{p,q}}{q^{k-1}[n]_{p,q}}\right), \text{ for } x\in(0,\infty]$$

where

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$$b_{n,k}^{p,q}(x) = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_{p,q} p^{k+n(n-1)/2} q^{k(k-1)/2} \frac{x^k}{(1\oplus x)_{p,q}^{n+k}}.$$

We recall significiant some basic notations, definitions on the concept of (p,q)-calculus for $0<q<p\leq 1$. The (p,q)- integers of the number η are given as

$$[\eta]_{p,q} := \frac{p^\eta - q^\eta}{p-q}, \quad p\neq 1 \ \text{ and } \ [\eta]_{1,q} := [\eta]_q$$

where $[\eta]_q$ means q-integers for $\eta = 0,1,2,....$ Also we have $[0]_{p,q} := 0$. The (p,q)-factorial is given by:

$$[\eta]_{p,q}! := [\eta]_{p,q}[\eta-1]_{p,q} \dots [1]_{p,q}, \quad \eta = 1,2,3,\cdots; \quad [0]_{p,q}! := 1.$$

The (p,q)-analogues of $(a+b)^\eta$ are expressed as

$$(a\oplus b)_{p,q}^\eta := \sum_{\nu=0}^{\eta} \begin{bmatrix} \eta \\ \nu \end{bmatrix}_{p,q} p^{(\eta-\nu)(\eta-\nu-1)/2} q^{\nu(\nu-1)/2} b^\nu a^{\eta-\nu}; \qquad (a\oplus b)_{p,q}^0 := 1.$$

where

$$\begin{bmatrix} \eta \\ \nu \end{bmatrix}_{p,q} := \frac{[\eta]_{p,q}!}{[\nu]_{p,q}![\eta-\nu]_{p,q}!} \qquad 0\leq \nu \leq \eta.$$

are the (p,q)-binomial coefficients.
Here we note that if p tends to 1, all the concepts defined above reduce to the q-analogs.
We take the (p,q)-derivative of a function f, denoted by $D_{p,q}f$, that is,

$$\left(D_{p,q}f\right)(x) = \frac{f(px)-f(qx)}{(p-q)x}, \qquad x\neq 0$$

and $\left(D_{p,q}f\right)(0) = f'(0)$ provided $f'(0)$ exists.

The (p,q)-partial derivative of a function $f(x,y)$ of two variables is given as

$$\frac{\partial_{p,q} f(x,y)}{\partial_{p,q} x} = \frac{f(px,y)-f(qx,y)}{(p-q)x}, \quad x\neq 0$$

with respect to x. The (p,q)-partial derivative of $f(x,y)$ with respect to y is given similarly.

In this paper, we define a new and different from ones exists in literature (p,q)-Baskakov operators and discussed approximation results and compute rate of convergence of the sequences of this operators.

2. Material and Method

For $f\in C[0,\infty)$, $0<q<p\leq 1$ and each positive integer m, Şimşek and Tunç [31] introduced the following operators:

$$L_{m,p,q}(f,x) = \sum_{\nu=0}^{\infty} \frac{1}{[\nu]_{p,q}!} \frac{\partial_{p,q}^\nu \varphi_{m,p,q}(x,u)}{\partial_{p,q} u^\nu}\bigg|_{u=0} f\left(\frac{[\nu]_{p,q}}{\alpha_{m,\nu,p,q}}\right), \tag{1}$$

where $\alpha_{m,\nu,p,q}$ are positive numbers and $\{\varphi_{m,p,q}\}$ generating real functions defined on $I\times[0,\infty)$ where $I\subset[0,\infty)$ is an interval, have this conditions:

(i) For all $m\in\mathbb{N}$ and $x\in I$, $\varphi_{m,p,q}(x,0)\neq 0$ and $\varphi_{m,p,q}(x,1) = 1$.
(ii) $\frac{\partial_{p,q}^\nu \varphi_{m,p,q}(x,u)}{\partial_{p,q} u^\nu}\Big|_{u=0}$ exist and are continuous functions of x for all $\nu\in\mathbb{N}_0$ and $m\in\mathbb{N}$.

(iii) $\frac{\partial_{p,q}^{\nu}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u^{\nu}} \geq 0,\, m \in \mathbb{N}$ and for all $\nu \in \mathbb{N}_0,\, x,u \geq 0,$

As a result of (iii), we can see clearly that $L_{m,p,q}$ are positive linear operators on $B(I)$ which is denoted the space of bounded functions on I.
In order to use the Korovkin's theorem, we need to give the test functions

$$e_{r,i}(k) = \left(\frac{k}{1+(1-i)k}\right)^r,\, r \in \mathbb{N}_0,\, i = 0,1,2\,.$$

The test function is used by many operators. For example, $e_{r,1}$ are used for (p,q)-Bernstein, (p,q)-Szasz-Mirakyan, (p,q)-Lupas and (p,q)-Baskakov operators.
For the numbers $\alpha_{m,\nu,p,q}$ indicated in (1), we assume the following:

$$e_{r,i}\left(\frac{[\nu]_{p,q}}{\alpha_{m,\nu,p,q}}\right) = \frac{[\nu]_{p,q}^r}{\alpha_{m,p,q}^r},\, r \in \mathbb{N}_0,\, m,\nu \in \mathbb{N},$$

where $\alpha_{m,p,q}$ are positive numbers independent of ν.

Theorem 2.1. [31]

Let $L_{m,p,q}(f;x)$ is given by (1). We have the following equalities for any $x \geq 0$ and $0 < q < p \leq 1$,
$L_{m,p,q}(e_{0,i};x) = 1;$
$L_{m,p,q}(e_{1,i};x) = \frac{1}{\alpha_{m,p,q}}\frac{\partial_{p,q}\varphi_{m,p,q}\,(x,u)}{\partial_{p,q}u}\bigg|_{u=1};$
$L_{m,p,q}(e_{2,i};x) = \frac{1}{\alpha_{m,p,q}^2}\left\{q\,\frac{\partial_{p,q}^2}{\partial_{p,q}u^2}\bigg|_{u=1} + \frac{\partial_{p,q}\varphi_{m,p,q}\,(x,u)}{\partial_{p,q}u}\bigg|_{u=p}\right\}.$

Now, we construct the new type of (p,q)-analogue of Baskakov operators. For $m \in \mathbb{N}$ and $0 < q < p \leq 1$, we take the function

$$\varphi_{m,p,q}(x,u) = \frac{((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus ux)_{p,q}^m},\; x \in [0,\infty) \tag{2}$$

for every $m \in \mathbb{N}$ and $x \in [0,\infty)$, $\varphi_{m,p,q}(x,0) \neq 0$ and $\varphi_{m,p,q}(x,1) = 1$. By the definition of (p,q)-partial derivatives, we have

$$\frac{\partial_{p,q}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u} = \frac{\varphi_{m,p,q}(x,pu) - \varphi_{m,p,q}(x,qu)}{(p-q)u}$$
$$= \frac{\frac{((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus pux)_{p,q}^m} - \frac{((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus qux)_{p,q}^m}}{(p-q)u}$$
$$= \frac{[m]_{p,q}xp((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus pux)_{p,q}^{m+1}}.$$

Therefore, we obtain

$$\frac{\partial_{p,q}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u} = \frac{[m]_{p,q}xp((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus pux)_{p,q}^{m+1}}.$$
By induction, we obtain that
$$\frac{\partial_{p,q}^{\nu}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u^{\nu}} = \frac{[m]_{p,q,\nu}x^{\nu}p^{\frac{\nu(\nu+1)}{2}}((1+x)\ominus x)_{p,q}^m}{((1+x)\ominus p^{\nu}ux)_{p,q}^{m+\nu}}\;,\; \nu \in \mathbb{N}$$

where $[m]_{p,q,\nu} = [m]_{p,q}[m+1]_{p,q}[m+2]_{p,q}\cdots[m+\nu-1]_{p,q}$.

If we write $u = 0$ in the last equality, then we get

$$\left.\frac{\partial_{p,q}^{\nu}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u^{\nu}}\right|_{u=0} = \frac{[m]_{p,q}x^{\nu}p^{\frac{\nu(\nu+1)}{2}}((1+x)\ominus x)_{p,q}^{m}}{(1+x)_{p,q}^{m+\nu}}$$

and

$$\frac{1}{[\nu]_{p,q}!}\left.\frac{\partial_{p,q}^{\nu}\varphi_{m,p,q}(x,u)}{\partial_{p,q}u^{\nu}}\right|_{u=0} = ((1+x)\ominus x)_{p,q}^{m}\begin{bmatrix} m+\nu-1 \\ k \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}}. \tag{3}$$

The right hand of equality (3) is a rational function of x because of it does not have any singular points in $[0,\infty)$. So the conditions (ii) and (iii) hold since $0 < q < p \leq 1$ and $x \in [0,\infty)$. So, the functions $\varphi_{m,p,q}(x,u)$ given by (2) generate some positive and linear operators.
If we take $\alpha_{m,\nu,p,q} = [m]_{p,q}$ and using (3) in the operators $L_{m,p,q}$ given by (1), then we can obtain

$$L_{m,p,q}(f;x) = ((1+x)\ominus x)_{p,q}^{m}\sum_{\nu=0}^{\infty}\begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}} f\left(\frac{[\nu]_{p,q}}{[m]_{p,q}}\right).$$

3. Results

In this part, we give some classical approximation results of the operators $L_{m,p,q}$. Let $0 < q_m < p_m \leq 1$ and $1 - q_m = o\left(\frac{1}{m}\right)$, $1 - p_m = o\left(\frac{1}{m}\right)$ when $m \to \infty$. In the sequel for $j \in \mathbb{N}_0$, $m \in \mathbb{N}$, we use notations:

$$\mu_{m,j}(x,p,q) := L_{m,p,q}\left(\left(e_1(.) - e_1(x)\right)^{j}; x\right),$$
$$I_A := [0,A], \quad A > 0.$$

We obtain the following lemmas by simple calculations from Theorem 2.1.

Lemma 3.1. For $x \in [0,\infty)$, the following identities hold for $0 < q < p \leq 1$, $n \in \mathbb{N}$

$$L_{m,p,q}(e_0;x) = 1\,;$$
$$L_{m,p,q}(e_1;x) = \frac{xp((1+x)\ominus x)_{p,q}^{m}}{((1+x)\ominus px)_{p,q}^{m+1}}\ ;$$
$$L_{m,p,q}(e_2;x) = \frac{qp^3[m+1]_{p,q}}{[m]_{p,q}}\frac{x^2((1+x)\ominus x)_{p,q}^{m}}{((1+x)\ominus p^2x)_{p,q}^{m+2}} + \frac{p}{[m]_{p,q}}\frac{x((1+x)\ominus x)_{p,q}^{m}}{((1+x)\ominus p^2x)_{p,q}^{m+1}}$$

where $e_r(t) := t^r$, $r = 0,1,2$.

Lemma 3.2. Using Lemma 3.1., we have

$$\mu_{m,1}(x,p,q) = \frac{xp\left((1+x)\ominus x\right)_{p,q}^{m}}{\left((1+x)\ominus px\right)_{p,q}^{m+1}} - x\,;$$
$$\mu_{m,2}(x,p,q) = \frac{qp^3[m+1]_{p,q}}{[m]_{p,q}}\frac{x^2((1+x)\ominus x)_{p,q}^{m}}{((1+x)\ominus p^2x)_{p,q}^{m+2}} + \frac{p}{[m]_{p,q}}\frac{x((1+x)\ominus x)_{p,q}^{m}}{((1+x)\ominus p^2x)_{p,q}^{m+1}}$$

$$-\frac{2x^2p\big((1+x)\ominus x\big)_{p,q}^m}{((1+x)\ominus px)_{p,q}^{m+1}}+x^2 \ .$$

Lemma 3.3. For all $t,x\in I_A$, we have

$$\left|L_{m,p,q}(e_r;x)-e_r(x)\right|\le rA^{r-1}\sqrt{\mu_{m,2}(x,p,q)},\ r\in\mathbb{N}_0.$$

Proof. The assertion is obvious for $r=0$. We assume that $r\in\mathbb{N}$. For $t,x\in I_A$ with $A>0$,

$$\begin{aligned}&|e_r(t)-e_r(x)|=|t-x||t^{r-1}+\cdots+x^{r-1}|\\&\le|e_1(t)-e_1(x)||A^{r-1}+\cdots+A^{r-1}|\\&=rA^{r-1}|e_1(t)-e_1(x)|.\end{aligned}$$

Taking into account that $L_{m,p,q}$ is monoton and by the Cauchy -Schwarz inequality, the following holds

$$\left|L_{m,p,q}(e_r;x)-e_r(x)\right|\le rA^{r-1}L_{m,p,q}(|e_1(.)-e_1(x)|;x)\le rA^{r-1}\sqrt{L_{m,p,q}\left(\big(e_1(.)-e_1(x)\big)^2;x\right)}$$

for all $m\in\mathbb{N}$. Thus, we obtain

$$\left|L_{m,p,q}(e_r;x)-e_r(x)\right|\le rA^{r-1}\sqrt{\mu_{m,2}(x,p,q)}$$

Lemma 3.4. For each $x\in I_A$, we have

$$\lim_{m\to\infty}L_{m,p_m,q_m}(e_r,x)=e_r(x)\ ,\quad r=0,1,2.$$

Proof.

$$\mu_{m,2}(x,p_m,q_m)=\frac{q_mp_m^3[m+1]_{p_m,q_m}}{[m]_{p_m,q_m}}\frac{x^2\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_m^2x\big)_{p_m,q_m}^{m+2}}+\frac{p_m}{[m]_{p_m,q_m}}\frac{x\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_m^2x\big)_{p_m,q_m}^{m+1}}$$
$$-\frac{2x^2p_m\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_mx\big)_{p_m,q_m}^{m+1}}+x^2\ .$$

So, we have

$$\begin{aligned}\mu_{m,2}(x,p_m,q_m)&=\frac{q_mp_m^3\big(q_m^m+p_m[m]_{p_m,q_m}\big)}{[m]_{p_m,q_m}}\frac{x^2((1+x)\ominus x)_{p_m,q_m}^m}{((1+x)\ominus p_m^2x)_{p_m,q_m}^{m+2}}\\&\quad+\frac{p_m}{[m]_{p_m,q_m}}\frac{x\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus x\big)_{p_m,q_m}^{m+1}}-\frac{2x^2p_m\big((1+x)\ominus x\big)_{p_m,q_m}^m}{((1+x)\ominus p_mx)_{p_m,q_m}^{m+1}}+x^2\end{aligned}$$

$$\begin{aligned}&=\frac{q_mp_m^4x^2((1+x)\ominus x)_{p_m,q_m}^m}{((1+x)\ominus p_m^2x)_{p_m,q_m}^{m+2}}+x^2-\frac{2x^2p_m\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_mx\big)_{p_m,q_m}^{m+1}}\\&\quad+\frac{q_m^{m+1}p_m^3}{[m]_{p_m,q_m}}\frac{x^2\big((1+x)\ominus x\big)_{p_m,q_m}^m}{((1+x)\ominus p_m^2x)_{p_m,q_m}^{m+2}}+\frac{p_m}{[m]_{p_m,q_m}}\frac{x\big((1+x)\ominus x\big)_{p_m,q_m}^m}{((1+x)\ominus p_m^2x)_{p_m,q_m}^{m+1}}\\&\le\frac{x^2\big((1+x)\ominus p_mx\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_mx\big)_{p_m,q_m}^{m+2}}+x^2-\frac{2x^2p_m\big((1+x)\ominus x\big)_{p_m,q_m}^m}{\big((1+x)\ominus p_mx\big)_{p_m,q_m}^{m+1}}+\frac{1}{[m]_{p_m,q_m}}\frac{x^2\big((1+x)\ominus x\big)_{p_m,q_m}^m}{((1+x)\ominus x)_{p_m,q_m}^{m+2}}\\&\quad+\frac{1}{[m]_{p_m,q_m}}\frac{x\big((1+x)\ominus x\big)_{p_m,q_m}^m}{((1+x)\ominus x)_{p_m,q_m}^{m+1}}\end{aligned}$$

$$\leq \frac{x^2}{p_m^{2m+3}} + x^2 - 2x^2 p_m q_m^{\frac{m(m+1)}{2}} + \frac{1}{[m]_{p_m,q_m}} \frac{x^2(2-2q_m^{m+2}) + x}{q_m^{m+2}}$$

Consequently, we obtain

$$\lim_{m\to\infty} \mu_{m,2}(x, p_m, q_m) = 0.$$

Finally from Lemma 3.3, we arrive at the required result.

Corollary 3.5. Let $f \in C(I_A)$. Then we have

$$\lim_{m\to\infty} L_{m,p_m,q_m}(f, x) = f(x), \quad x \in I_A.$$

Now, we recall the usual modulus of continuity for function $f \in C(I_A)$. The first modulus of continuity is defined by

$$\omega(f;\delta) \coloneqq \sup_{u,s\in I_A, |u-s|\leq\delta} |f(u) - f(s)|, \ \delta > 0.$$

Thus, it implies for any $\delta > 0$

$$|f(u) - f(s)| \leq \left(1 + \tfrac{|u-s|}{\delta}\right) \omega(f, \delta) \tag{4}$$

is satisfied.

Theorem 3.6. Let $f \in C(I_A)$, then the inequality

$$\left|L_{m,p,q}(f;x) - f(x)\right| \leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\mu_{m,2}(x,p,q)}\right)$$

holds for any $\delta > 0$.

Proof. Since $L_{m,p,q}(1;x) = 1$, then we obtain

$$\left|L_{m,p,q}(f;x) - f(x)\right| \leq L_{m,p,q}(|f(.) - f(x)|; x) \tag{5}$$

for all $m \in \mathbb{N}$. Now using (4) in inequality (5) we obtain

$$|f(t) - f(x)| \leq \omega(|t-x|) \leq \left(1 + \tfrac{|t-x|}{\delta}\right)\omega(f,\delta), \tag{6}$$

For all $\delta > 0$. Using the Cauchy-Schwartz Inequality and (6) it follows that

$$\begin{aligned}
&\left|L_{m,p,q}(f;x) - f(x)\right| \leq \omega(f,\delta)\left(L_{m,p,q}(1;x) + \frac{1}{\delta} L_{m,p,q}(|t-x|;x)\right) \leq \\
&\leq \omega(f,\delta)\left(1 + \frac{1}{\delta}\left[L_{m,p,q}((t-x)^2;x)\right]^{1/2}\left[L_{m,p,q}(1;x)\right]^{1/2}\right) = \\
&= \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{L_{m,p,q}(t^2,x) - 2xL_{m,p,q}(t,x) + x^2 L_{m,p,q}(1,x)}\right) \\
&= \omega(f,\delta)\left(1 + \frac{1}{\delta}\sqrt{\mu_{m,2}(x,p,q)}\right)
\end{aligned}$$

Thus, for any $\delta > 0$

$$\left|L_{m,p,q}(f;x)-f(x)\right| \leq \omega(f,\delta)\left(1+\frac{1}{\delta}\sqrt{\mu_{m,2}(x,p,q)}\right).$$

Let us obtain the rate of convergence in terms of the Lipschitz class $Lip_M(\alpha)$ $(M>0,\ \ 0<\alpha\leq 1)$. A function f belongs to $Lip_M\alpha$ if

$$|f(u)-f(s)| \leq M|u-s|^{\alpha}$$

is satisfied for all $u,s \in I_A$.

Theorem 3.7. Let $f \in Lip_M(\alpha)$ and $x \in I_A$, we have

$$\left|L_{m,p,q}(f;x)-f(x)\right| \leq M\big(\mu_{m,2}(x,p,q)\big)^{\alpha/2}.$$

Proof. Since $f \in Lip_M(\alpha)$ and the operator $L_{m,p,q}$ is linear and monotone, we have

$$\left|L_{m,p,q}(f;x)-f(x)\right| \leq L_{m,p,q}(|f(t)-f(x)|;x)$$
$$\leq L_{m,p,q}(f;x) = ((1+x)\ominus x)_{p,q}^{m}\sum_{\nu=0}^{\infty}\begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}}\left|f\left(\frac{[\nu]_{p,q}}{[m]_{p,q}}\right)-f(x)\right| \leq$$
$$\leq M((1+x)\ominus x)_{p,q}^{m}\sum_{\nu=0}^{\infty}\begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}}\left|\frac{[\nu]_{p,q}}{[m]_{p,q}}-x\right|^{\alpha}$$

If we consider the Hölder's inequality for $p=\frac{2}{\alpha}$ and $q=\frac{2}{2-\alpha}$, the following inequality holds,

$$\left|L_{m,p,q}(f;x)-f(x)\right|$$
$$\leq M\left(((1+x)\ominus x)_{p,q}^{m}\ \sum_{\nu=0}^{\infty}\begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}}\left(\frac{[\nu]_{p,q}}{[m]_{p,q}}-x\right)^{2}\right)^{\alpha/2}$$
$$\left(((1+x)\ominus x)_{p,q}^{m}\ \sum_{\nu=0}^{\infty}\begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_{p,q} x^{\nu}(1+x)^{-m-\nu}p^{\frac{\nu(\nu+1)}{2}}\right)^{\frac{2-\alpha}{2}} = M\big(\mu_{m,2}(x,p,q)\big)^{\alpha/2}$$

This ends the proof.

4. Graphical Analysis

In this part, with the help of MATLAB, we illustrate several graphs and make comparisons in terms of the convergence by allocating different values of parameters of operators for the function of $f(x)=sinx$.

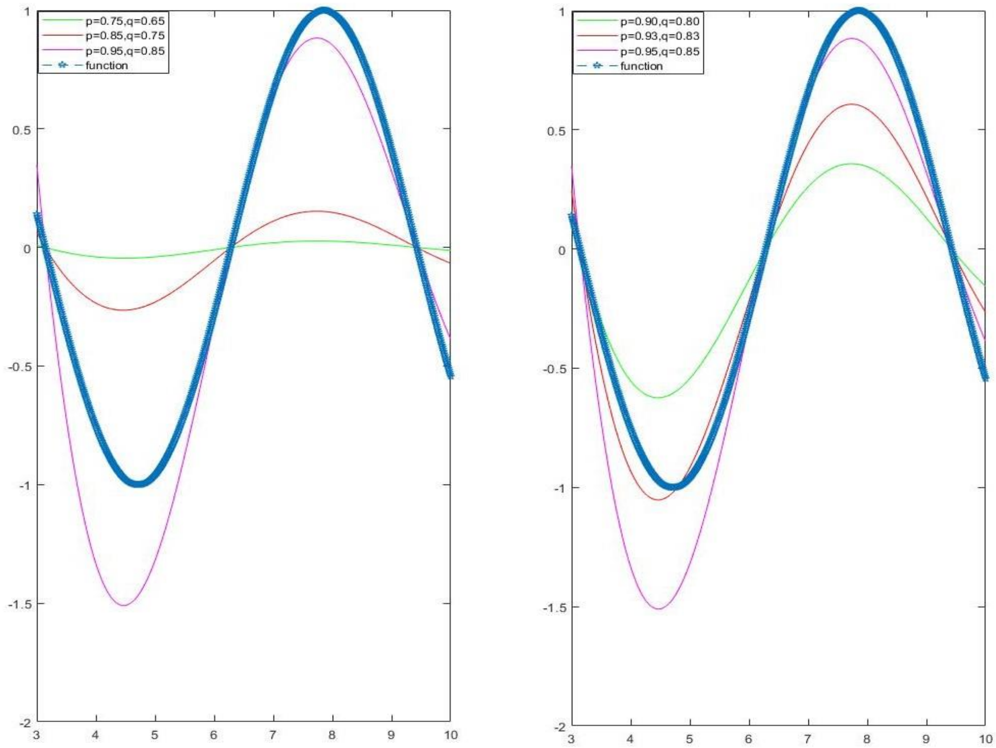

Figure 1. Approximation by (p,q) Baskakov operators to $f(x)$ for $x \in [3,10]$ and from $\nu = 0$ to $\nu = 30$ and $m = 5$.

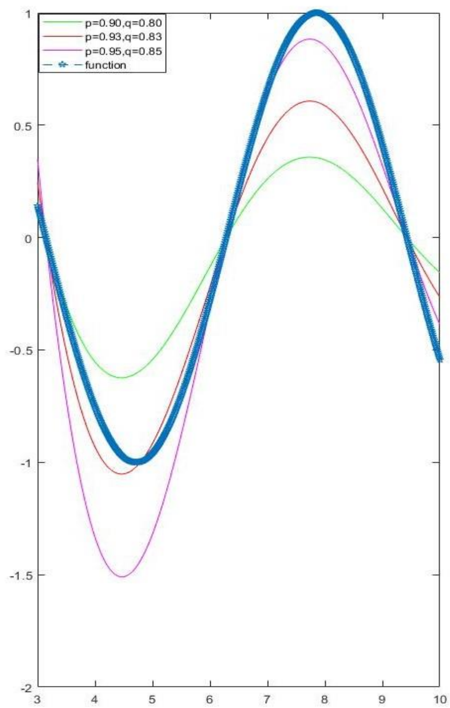

Figure 2. Approximation by (p,q) Baskakov operators to $f(x)$ for $x \in [3,10]$ and from $\nu = 0$ to $\nu = 50$ and $m = 5$.

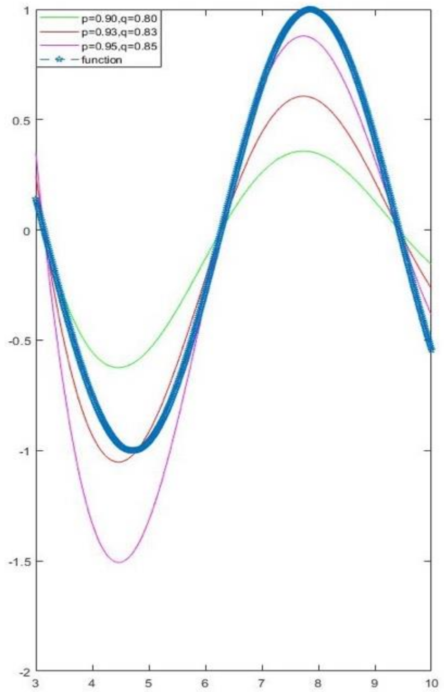

Figure 3. Approximation by (p, q) Baskakov operators to $f(x)$ for $x \in [3,10]$ and from $\nu = 0$ to $\nu = 10$ and $m = 5$.

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