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Some generalised integral inequalities for bidimensional preinvex stochastic processes

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Abstract

In this study, we generalize some integral inequalities for bidimensional preinvex stochastic processes. The main results consist of two parts. In the first part, we obtain a generalization of H-H type integral inequality for bidimensional preinvex stochastic processes. In the second part, we derive a generalization of Ostrowski type integral inequality for bidimensional preinvex stochastic processes. For this reason, we use mean-square integrable preinvex stochastic processes and verify generalization of H-H type integral inequality and Ostrowski type integral inequality for preinvex stochastic processes on the real, respectively.

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1. Introduction

In the literature, the following inequality is well-known as Hermite-Hadamard type integral inequality (H-H integral inequality) for convex functions [1]:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Alomari [2] generalized this classical H-H type integral inequality and Ostrowski type inequality for convex function on $[a,b]$. Dragomir [3] proved H-H type integral inequality for convex functions on the coordinates. Nwaeze [4] proved some generalizations of H-H type integral inequality and Ostrowski type inequality for coordinated convex functions.

It is necessary to know that preinvexity indicates a generalization of convexity. There are many results for preinvex functions in the literature [5-13].

In terms of probability theory, H-H type integral inequality gives minimum and maximum bounds for the expectation value of a random variable. Based on importance of convexity, many researchers studied on this issue, for examples Kumar [14], Gavrea [15]. In this sense, researchers investigated many problems related convexity and inequality for stochastic processes [16-22].

Akdemir et al. [23] defined the following preinvex stochastic process with respect to η *($P_\eta SP$)* on the real line:

Definition 1.1([23]**).** *A set* $I \subseteq \mathbb{R}$ *is called invex with respect to the continuous function* $\eta: I \times I \to \mathbb{R}$ *if* $\left(x+\lambda\eta(y,x)\right) \in I$ *for all* $x, y \in I, \lambda \in [0,1]$. *An invex function is also preinvex under following Condition C:*

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Condition C: *Let $I \subseteq \mathbb{R}$ be invex with respect to $\eta: I \times I \to \mathbb{R}$. It is told that the function η satisfies Condition C if*
$\eta(y, y + \lambda\eta(x, y)) = -\lambda\eta(x, y);$
$\eta(x, y + \lambda\eta(x, y)) = (1 - \lambda)\eta(x, y)$
for all $x, y \in I$ and $\lambda \in [0,1]$.

Definition 1.2([23]**)**. *The process $X: I \times \Omega \to \mathbb{R}$ is called $P_\eta SP$ if the following inequality holds almost everywhere:*
$X(t + \lambda\eta(s, t),\cdot) \leq (1 - \lambda)X(t,\cdot) + \lambda X(s,\cdot)$
for all $t, s \in I \subseteq \mathbb{R}$ and $\lambda \in [0,1]$.

Theorem 1.1([23]**)**. *If η satisfies Condition C, then almost everywhere*

$$X\left(\frac{2u + \eta(v, u)}{2},\cdot\right) \leq \frac{1}{\eta(v, u)} \int_u^{u+\eta(v,u)} X(t,\cdot)dt \leq \frac{X(u,\cdot) + X(v,\cdot)}{2}. \tag{1}$$

In the light of all the above mentioned information, our main aim is to generalize some integral inequalities for bidimensional preinvex stochastic processes such as H-H type integral inequality and Ostrowski type integral inequality.

2. Materials and Methods

2.1. Apparatus

In this section, we use as materials the following definitions for bidimensional preinvex stochastic process with respect to η_1 and η_2 ($P_\eta^2 SP$); and H-H type integral inequality for these processes:

Definition 2.1([25]**)**. *Let the continuous functions $\eta_1 : T \times T \to \mathbb{R}$ and $\eta_2 : S \times S \to \mathbb{R}$ be invex on the sets $T, S \subseteq \mathbb{R}$, respectively. Then $\Delta := T \times S$ is called invex set with respect to η_1 and η_2 if $(u + \lambda_1\eta_1(t, u), v + \lambda_2\eta_2(s, v)) \in \Delta$ for all $(t, s), (u, v) \in \Delta$, $\lambda_1, \lambda_2 \in [0,1]$.*

Definition 2.2([25]**)**. *Let Δ be called invex with respect to the η_1 and η_2. The process $X: \Delta \times \Omega \to \mathbb{R}$ is called a $P_\eta^2 SP$ on Δ if the following inequality holds almost everywhere*
$X\left((t_1 + \lambda\eta_1(t_2, t_1), s_1 + \lambda\eta_2(s_2, s_1)),\cdot\right) \leq (1 - \lambda)X((t_1, s_1),\cdot) + \lambda X((t_2, s_2),\cdot).$
for all (t_1, s_1) , $(t_2, s_2) \in \Delta$ and $\lambda \in [0,1]$.

Lemma 2.1([25]**).** *Let $(\Omega, \Im, P)$ be an arbitrary probability space and $X: \Delta \times \Omega \to \mathbb{R}$ be a $P_\eta^2 SP$ on Δ, then X can be known as*

(i) mean-square continuous (differentiable) on Δ,
(ii) monotonic if it is increasing or decreasing,
(iii) mean-square differentiable at a point $(t, s) \in \Delta$,
(iv) mean-square integrable on $\Lambda := [u_1, u_1 + \eta_1(u_2, u_1)] \times [v_1, v_1 + \eta_2(v_2, v_1)] \subseteq \Delta$.

From here on out, assume that
$\Lambda \coloneqq \Lambda_u \times \Lambda_v$ with $\Lambda_u := [u_1, u_1 + \eta_1(u_2, u_1)]$, $\Lambda_v \coloneqq [v_1, v_1 + \eta_2(v_2, v_1)]$
and
$\Lambda_u^+ := 2u_1 + \eta_1(u_2, u_1)$, $\Lambda_u^- := \eta_1(u_2, u_1) > 0$, $\Lambda_v^+ := 2v_1 + \eta_2(v_2, v_1)$, $\Lambda_v^- := \eta_2(v_2, v_1) > 0$.

Theorem 2.1([25]**)**. *Let $X: \Lambda \times \Omega \to \mathbb{R}_+$ be a $P_\eta^2 SP$ on Λ. If X is mean-square integrable on Λ, then almost everywhere*

$$X\left(\left(\frac{\Lambda_u^+}{2},\frac{\Lambda_v^+}{2}\right),\cdot\right)\leq\frac{1}{2\Lambda_u^-}\int_{u_1}^{u_1+\Lambda_u^-}X\left(\left(t,\frac{\Lambda_v^+}{2}\right),\cdot\right)dt+\frac{1}{2\Lambda_v^-}\int_{v_1}^{v_1+\Lambda_v^-}X\left(\left(\frac{\Lambda_u^+}{2},s\right),\cdot\right)ds$$
$$\leq\frac{1}{\Lambda_u^-\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-}X\big((t,s),\cdot\big)\,dsdt \tag{2}$$
$$\leq\frac{1}{4\Lambda_u^-}\int_{u_1}^{u_1+\Lambda_u^-}\Big(X\big((t,v_1),\cdot\big)+X\big((t,v_2),\cdot\big)\Big)dt+\frac{1}{4\Lambda_v^-}\int_{v_1}^{v_1+\Lambda_v^-}\Big(X\big((u_1,s),\cdot\big)+X\big((u_2,s),\cdot\big)\Big)ds$$
$$\leq\frac{X\big((u_1,v_1),\cdot\big)+X\big((u_1,v_2),\cdot\big)}{4}+\frac{X\big((u_2,v_1),\cdot\big)+X\big((u_2,v_2),\cdot\big)}{4}.$$

3. Results and Discussion

This section consists of two parts. In the first subsection, we obtain a generalization of H-H type integral inequality for bidimensional preinvex stochastic processes. In the second subsection, we derive a generalization of Ostrowski type integral inequality for bidimensional preinvex stochastic processes.

3.1. Generalization of H-H type integral inequality for bidimensional preinvex stochastic processes

In this subsection, we generalize H-H type integral inequality for bidimensional preinvex stochastic processes. For this reason, we need the following generalization of H-H type integral inequality for preinvex stochastic processes on the real line:

Lemma 3.1. *Let* $X: [u,u+\eta(v,u)]\times\Omega\to\mathbb{R}_+$ *be a* $P_\eta SP$ *and mean-square integrable on* $[u,u+\eta(v,u)]$. *Then almost everywhere*

$$\frac{\eta(v,u)}{n}\sum_{k=1}^{n}X\left(\left(\frac{2t_{k-1}+\eta(t_k,t_{k-1})}{2}\right),\cdot\right)\leq\int_{u}^{u+\eta(v,u)}X\,(t,\cdot)dt$$
$$\leq\frac{\eta(v,u)}{2n}\left(X(u,\cdot)+2\sum_{k=1}^{n-1}X(t_k,\cdot)+X(v,\cdot)\right), \tag{3}$$

where $t_k=u+k\frac{\eta(v,u)}{n}$, $k=0,1,2,\dots,n$; $u<v$, $n\in N$.

Proof. Using preinvexity of X on each sub-interval $[t_{k-1},t_{k-1}+\eta(t_k,t_{k-1})]\subseteq[u,u+\eta(v,u)]$, $k=1,2,\dots,n$, then for all $\lambda\in[0,1]$

$$X(t_{k-1}+\lambda\eta(t_k,t_{k-1}),\cdot)\leq\lambda X(t_k,\cdot)+(1-\lambda)\,X(t_{k-1},\cdot). \tag{4}$$

Integrating (5) with respect to λ on $[0,1]$

$$\int_0^1X(t_{k-1}+\lambda\eta(t_k,t_{k-1}),\cdot)d\lambda\leq\frac{X(t_{k-1},\cdot)+X(t_k,\cdot)}{2}. \tag{5}$$

Changing of variable $t=t_{k-1}+\lambda\eta(t_k,t_{k-1})$ in (4)

$$\int_{t_{k-1}}^{t_{k-1}+\eta(t_k,t_{k-1})}X(t,\cdot)dt\leq\frac{\eta(t_k,t_{k-1})}{2}\big(X(t_{k-1},\cdot)+X(t_k,\cdot)\big). \tag{6}$$

Taking the sum over k from 1 to n on (5), we get

$$\sum_{k=1}^{n}\int_{t_{k-1}}^{t_{k-1}+\eta(t_k,t_{k-1})} X(t,\cdot)dt = \int_{u}^{u+\eta(v,u)} X(t,\cdot)dt \le \sum_{k=1}^{n}\frac{\eta(t_k,t_{k-1})}{2}\big(X(t_{k-1},\cdot)+X(t_k,\cdot)\big)$$
$$\le \frac{1}{2}\max_k\{\eta(t_k,t_{k-1})\}\sum_{k=1}^{n}\big(X(t_{k-1},\cdot)+X(t_k,\cdot)\big)$$
$$= \frac{\eta(v,u)}{2n}\left(X(t_0,\cdot)+X(t_1,\cdot)+\sum_{k=2}^{n-1}\big(X(t_{k-1},\cdot)+X(t_k,\cdot)\big)+X(t_{n-1},\cdot)+X(t_n,\cdot)\right) \tag{7}$$
$$= \frac{\eta(v,u)}{2n}\left(X(u,\cdot)+2\sum_{k=1}^{n-1}X(t_k,\cdot)+X(v,\cdot)\right).$$

Because of preinvexity of X on $[t_{k-1}, t_{k-1}+\eta(t_k,t_{k-1})]$, then for $\lambda \in [0,1]$

$$X\left(\frac{2t_{k-1}+\eta(t_k,t_{k-1})}{2},\cdot\right) = X\left(\frac{t_{k-1}+\lambda\eta(t_k,t_{k-1})}{2}+\frac{t_{k-1}+\lambda\eta(t_k,t_{k-1})}{2},\cdot\right)$$
$$\le \frac{1}{2}[X(t_{k-1}+\lambda\eta(t_k,t_{k-1}),\cdot)+X(t_{k-1}+\lambda\eta(t_k,t_{k-1}),\cdot)]. \tag{8}$$

Applying on (7) by using similar way in (4)-(6)

$$\frac{\eta(v,u)}{n}\sum_{k=1}^{n}X\left(\left(\frac{2t_{k-1}+\eta(t_k,t_{k-1})}{2}\right),\cdot\right) \le \int_{u}^{u+\eta(v,u)} X(t,\cdot)dt. \tag{9}$$

From (6) and (8), we obtain (3).

Remark 3.1. *In Lemma 3.1 for $n=1$, then we obtain (1).*

Now, we can give a generalization of H-H type integral inequality for bidimensional preinvex stochastic processesesas follows:

Theorem 3.1. *Let $X: \Lambda \times \Omega \to \mathbb{R}_+$ be a P_η^2SP on Λ. If X is mean-square integrable on Λ, then almost everywhere*

$$\frac{\Lambda_v^-}{2n}\sum_{k=1}^{n}\int_{u_1}^{u_1+\Lambda_u^-} X\left(\left(t,\frac{\Lambda_s^+}{2}\right),\cdot\right)dt + \frac{\Lambda_u^-}{2n}\sum_{k=1}^{n}\int_{v_1}^{v_1+\Lambda_v^-} X\left(\left(\frac{\Lambda_t^+}{2},s\right),\cdot\right)ds$$
$$\le \frac{1}{\Lambda_u^-\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t_1,s_1),\cdot\big)dtds$$
$$\le \frac{\Lambda_v^-}{4n}\int_{v_1}^{v_1+\Lambda_v^-}\Big(X\big((t,v_1),\cdot\big)+X\big((t,v_2),\cdot\big)\Big)dt + \frac{\Lambda_u^-}{4n}\int_{v_1}^{v_1+\Lambda_v^-}\Big(X\big((u_1,s),\cdot\big)+X\big((u_2,s),\cdot\big)\Big)ds \tag{10}$$
$$+\frac{\Lambda_v^-}{2n}\sum_{k=1}^{n-1}\int_{u_1}^{u_1+\Lambda_u^-} X\big((t,s_k),\cdot\big)dt + \frac{\Lambda_u^-}{2n}\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t_k,s),\cdot\big)ds,$$

where $t_k = u_1 + k\frac{\Lambda_u^-}{n}$, $s_k = v_1 + k\frac{\Lambda_v^-}{n}$, $k = 0,1,2,\ldots,n$; $n \in N$.

Proof. By using Lemma 3.1, we get

$$\frac{\Lambda_v^-}{n}\sum_{k=1}^{n} X_t\left(\frac{\Lambda_s^+}{2},\cdot\right) \le \int_{v_1}^{v_1+\Lambda_v^-} X_t(s,\cdot)ds \le \frac{\Lambda_v^-}{2n}\left(X_t(v_1,\cdot)+2\sum_{k=1}^{n-1} X_t(s_k,\cdot)+X_t(v_2,\cdot)\right).$$

Thus

$$\begin{aligned}&\frac{\Lambda_v^-}{n}\sum_{k=1}^{n} X\left(\left(t,\frac{\Lambda_s^+}{2}\right),\cdot\right) \le \int_{v_1}^{v_1+\Lambda_v^-} X((t,s),\cdot)ds \\ &\le \frac{\Lambda_v^-}{2n}\left(X\big((t,v_1),\cdot\big)+X\big((t,v_2),\cdot\big)+2\sum_{k=1}^{n-1} X\big((t,s_k),\cdot\big)\right).\end{aligned} \tag{11}$$

Integrating all sides of (11) on Λ_u, we have

$$\begin{aligned}&\frac{\Lambda_v^-}{2n}\sum_{k=1}^{n}\int_{u_1}^{u_1+\Lambda_u^-} X\left(\left(t,\frac{\Lambda_s^+}{2}\right),\cdot\right)dt \le \frac{1}{\Lambda_u^-\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t_1,s_1),\cdot\big)dtds \\ &\le \frac{\Lambda_v^-}{2n}\left(\int_{u_1}^{u_1+\Lambda_u^-} X\big((t,s),\cdot\big)dt+\int_{u_1}^{v_1+\Lambda_v^-} X\big((t,v_2),\cdot\big)dt+2\sum_{k=1}^{n-1}\int_{u_1}^{u_1+\Lambda_u^-} X\big((t,s_k),\cdot\big)dt\right),\end{aligned} \tag{12}$$

and

$$\begin{aligned}&\frac{\Lambda_u^-}{n}\sum_{k=1}^{n} X\left(\left(\frac{\Lambda_t^+}{2},s\right),\cdot\right)ds \le \int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t,s),\cdot\big)dtds \\ &\le \frac{\Lambda_u^-}{2n}\left(\int_{v_1}^{v_1+\Lambda_v^-} X\big((u_1,s),\cdot\big)ds+\int_{v_1}^{v_1+\Lambda_v^-} X\big((u_2,s),\cdot\big)ds+2\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t_k,s),\cdot\big)ds\right).\end{aligned} \tag{13}$$

Adding (12) and (13), we obtain (10). That completes the proof of Theorem 3.1.

Remark 3.2. *In Theorem 3.1 for* $n = 1$, *then we obtain (2).*

Corollary 3.1. *Using Theorem 3.1, we have almost everywhere*

i)
$$\begin{aligned}&\sum_{k=1}^{n} X\left(\left(\frac{\Lambda_u^+}{2},\frac{\Lambda_s^+}{2}\right),\cdot\right)+\sum_{k=1}^{n} X\left(\left(\frac{\Lambda_t^+}{2},\frac{\Lambda_v^+}{2}\right),\cdot\right) \\ &\le \frac{n}{\Lambda_v^-}\int_{v_1}^{v_1+\Lambda_v^-} X\left(\left(\frac{\Lambda_u^+}{2},s\right),\cdot\right)ds+\frac{n}{\Lambda_u^-}\int_{u_1}^{u_1+\Lambda_u^-} X\left(\left(t,\frac{\Lambda_v^+}{2}\right),\cdot\right)dt;\end{aligned}$$

ii)
$$\begin{aligned}&\frac{n}{\Lambda_v^-}\int_{v_1}^{v_1+\Lambda_v^-}\Big(X\big((u_1,s),\cdot\big)+X\big((u_2,s),\cdot\big)\Big)ds+\frac{n}{\Lambda_u^-}\int_{u_1}^{u_1+\Lambda_u^-}\Big(X\big((t,v_1),\cdot\big)+X\big((t,v_2),\cdot\big)\Big)dt \\ &\le X\big((u_1,v_1),\cdot\big)+X\big((u_1,v_2),\cdot\big)+X\big((u_2,v_1),\cdot\big)+X\big((u_2,v_2),\cdot\big) \\ &+\sum_{k=1}^{n-1}\Big(X\big((u_1,s_k),\cdot\big)+X\big((u_2,s_k),\cdot\big)+X\big((t_k,v_1),\cdot\big)+X\big((t_k,v_2),\cdot\big)\Big).\end{aligned}$$

3.2. Generalization of Ostrowski type integral inequality for bidimensional preinvex stochastic processes

In this subsection, we generalize Ostrowski type integral inequality for bidimensional preinvex stochastic processes. For this reason, we need the following generalization of Ostrowski type integral inequality for preinvex stochastic processes on the real line:

Lemma 3.2. *Let* $X: [u, u+\eta(v,u)] \times \Omega \to \mathbb{R}_+$ *be a* $P_\eta SP$ *and mean-square integrable on* $[u, u+\eta(v,u)]$. *Then almost everywhere*

$$\int_u^{u+\eta(v,u)} X(t,\cdot)dt - \eta(v,u)X(s,\cdot) \leq \frac{\eta(v,u)}{2n}\left(X(u,\cdot) + 2\sum_{k=1}^{n-1} X(t_k,\cdot) + X(v,\cdot)\right). \tag{14}$$

for all $s \in [u, u+\eta(v,u)],\ t_k = u + k\frac{\eta(v,u)}{n}, k = 0,1,2,\ldots,n;\ u_1 < u_2,\ n \in N.$

Proof. Fix $s \in [t_{k-1}, t_{k-1} + \eta(t_k, t_{k-1})]$, $k = 1,2,\ldots,n$. Because of preinvexity of X on $[t_{k-1}, t_{k-1} + \eta(s, t_{k-1})], k = 1,2,\ldots,n$, we obtain

$$\int_{t_{k-1}}^{t_{k-1}+\eta(s,t_{k-1})} X(t,\cdot)dt \leq \frac{\eta(s,t_{k-1})}{2}\big(X(t_{k-1},\cdot) + X(s,\cdot)\big). \tag{15}$$

Using preinvexity of X on $[s, s+\eta(t_k,s)], k = 1,2,\ldots,n$, we get

$$\int_s^{s+\eta(t_k,s)} X(t,\cdot)dt \leq \frac{\eta(t_k,s)}{2}\big(X(s,\cdot) + X(t_k,\cdot)\big). \tag{16}$$

Adding the inequalities (15) and (16), we get

$$\begin{aligned}\int_{t_{k-1}}^{t_{k-1}+\eta(s,t_{k-1})} X(t,\cdot)dt + \int_s^{s+\eta(t_k,s)} X(t,\cdot)dt &= \int_{t_{k-1}}^{t_{k-1}+\eta(t_k,t_{k-1})} X(t,\cdot)dt \\ &\leq \frac{\eta(s,t_{k-1})}{2}\big(X(t_{k-1},\cdot) + X(s,\cdot)\big) + \frac{\eta(t_k,s)}{2}\big(X(s,\cdot) + X(t_k,\cdot)\big) \\ &\leq \frac{\eta(t_k,t_{k-1})}{2}\{X(t_{k-1},\cdot) + X(t_k,\cdot)\} + \frac{\eta(v,u)}{n}X(s,\cdot).\end{aligned} \tag{17}$$

Taking the sum over k from 1 to n on (17), we get (14).

Now, we can give a generalization of Ostrowski type integral inequality for bidimensional preinvex stochastic processes as follows:

Theorem 3.2. *Let* $X: \Lambda \times \Omega \to \mathbb{R}_+$ *be a* $P_\eta^2 SP$ *on* Λ. *If* X *is mean-square integrable on* Λ, *then almost everywhere*

$$\begin{aligned}&\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t,s),\cdot\big)dtds \\ &\leq \frac{\Lambda_u^-(n+1)}{4n}\left(\int_{v_1}^{v_1+\Lambda_v^-} X\big((u_1,s),\cdot\big)ds + \int_{v_1}^{v_1+\Lambda_v^-} X\big((u_2,s),\cdot\big)ds + \frac{2}{n+1}\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\big((t_k,s),\cdot\big)ds\right) \\ &+ \frac{\Lambda_v^-(n+1)}{4n}\left(\int_{u_1}^{u_1+\Lambda_u^-} X\big((t,v_1),\cdot\big)dt + \int_{u_1}^{u_1+\Lambda_u^-} X\big((t,v_2),\cdot\big)dt + \frac{2}{n+1}\sum_{k=1}^{n-1}\int_{u_1}^{u_1+\Lambda_u^-} X\big((t,s_k),\cdot\big)dt\right),\end{aligned} \tag{18}$$

where t_k *and* s_k *are defined as in Theorem 3.1.*

Proof. By using Lemma 3.2 for $X_s(t,\cdot) = X((t,s),\cdot)$ at $t = u_2$, we obtain

$$\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,s),\cdot\right)dt - \eta_1(u_2,u_1)X\left((u_2,s),\cdot\right) \le \frac{\Lambda_u^-}{2n}\left(X\left((u_1,s),\cdot\right) + 2\sum_{k=1}^{n-1} X\left((t_k,s),\cdot\right) + X\left((u_2,s),\cdot\right)\right). \quad (19)$$

Integrating all of (19) with respect to s on Λ_v, we get

$$\int_{v_1}^{v_1+\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,s),\cdot\right)dsdt \le \frac{\Lambda_u^-}{2n}\left(\begin{array}{c}\displaystyle\int_{v_1}^{v_1+\Lambda_v^-} X\left((u_1,s),\cdot\right)ds + (1+2n)\int_{v_1}^{v_1+\Lambda_v^-} X\left((u_2,s),\cdot\right)ds \\ \displaystyle +2\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\left((t_k,s),\cdot\right)ds\end{array}\right), \quad (20)$$

and for $X_s(t,\cdot) = X((t,s),\cdot)$ at $t = u_1$,

$$\int_{v_1}^{v_1+\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,s),\cdot\right)dsdt \le \frac{\Lambda_u^-}{2n}\left(\begin{array}{c}\displaystyle(1+2n)\int_{v_1}^{v_1+\Lambda_v^-} X\left((u_1,s),\cdot\right)ds + \int_{v_1}^{v_1+\Lambda_v^-} X\left((u_2,s),\cdot\right)ds \\ \displaystyle +2\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\left((t_k,s),\cdot\right)ds\end{array}\right). \quad (21)$$

Adding (20) and (21),

$$\int_{v_1}^{v_1+\Lambda_v^-}\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,s),\cdot\right)dsdt \le \frac{\Lambda_u^-}{2n}\left(\begin{array}{c}\displaystyle(n+1)\int_{v_1}^{v_1+\Lambda_v^-} X\left((u_1,s),\cdot\right)ds + (n+1)\int_{v_1}^{v_1+\Lambda_v^-} X\left((u_2,s),\cdot\right)ds \\ \displaystyle +2\sum_{k=1}^{n-1}\int_{v_1}^{v_1+\Lambda_v^-} X\left((s_k,s),\cdot\right)ds\end{array}\right). \quad (22)$$

By using the similar way for $X_t(s,\cdot) = X((t,s),\cdot)$ at $s = v_1$ and $s = v_2$ with Lemma 3.2, respectively,

$$\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\left((t,s),\cdot\right)dtds \le \frac{\Lambda_v^-}{2n}\left(\begin{array}{c}\displaystyle(n+1)\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,v_1),\cdot\right)dt + (n+1)\int_{u_1}^{u_1+\Lambda_u^-} X\left((t,v_2),\cdot\right)dt \\ \displaystyle +2\sum_{k=1}^{n-1}\int_{u_1}^{u_1+\Lambda_u^-} X\left((t_k,s),\cdot\right)ds\end{array}\right). \quad (23)$$

The desired inequality (18) is obtained by adding (22) and (23).

Corollary 3.2. *According to Theorem 3.2 for* $n = 1, 2$, *respectively, we get almost everywhere*

i)
$$\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\left((t,s),\cdot\right)dtds$$
$$\le \frac{\Lambda_v^-}{2}\int_{u_1}^{u_1+\Lambda_u^-}\left(X\left((t,v_1),\cdot\right) + X\left((t,v_2),\cdot\right)\right)dt + \frac{\Lambda_u^-}{2}\int_{v_1}^{v_1+\eta_2(v_2,v_1)}\left(X\left((u_1,s),\cdot\right) + X\left((u_2,s),\cdot\right)\right)ds\,;$$

ii)
$$\int_{u_1}^{u_1+\Lambda_u^-}\int_{v_1}^{v_1+\Lambda_v^-} X\left((t,s),\cdot\right)dtds$$

$$\leq \frac{\Lambda_v^-}{8} \int_{u_1}^{u_1+\Lambda_u^-} \left(3X\big((t, v_1), \cdot\big) + 2\left(X\left(t, \frac{\Lambda_v^+}{2}\right), \cdot \right) + 3X((t, v_2), \cdot) \right) dt$$

$$+ \frac{\Lambda_u^-}{8} \int_{v_1}^{v_1+\Lambda_v^-} \left(3X\big((u_1, s), \cdot\big) + 2X\left(\left(\frac{\Lambda_u^+}{2}, s \right), \cdot \right) + 3X((u_2, s), \cdot) \right) ds.$$

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Conflicts of interest

The outhors state that did not have conflict of interests.

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