Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-3 (2019) 670-678

Hermite-Hadamard Type Inequalities For Two Times Differentiable Arithmetic-Harmonically Convex Functions

Huriye KADAKAL

Ministry of Education, Hamdi Bozbağ Anatolian High School, Giresun, TURKEY

Received: 26.03.2019; Accepted: 11.09.2019 *http://dx.doi.org/10.17776/csj.544984*

Abstract. In this work, by using both an integral identity and the Hölder, the power-mean integral inequalities it is established several new inequalities for two times differentiable arithmetic-harmonically-convex function. Also, a few applications are given for some means of real numbers.

Keywords: Convex function, Arithmetic-harmonically-convex function, Hermite-Hadamard's inequality.

İki Kez Türevlenebilen Aritmetik-Harmonik Konveks Fonksiyonlar İçin Hermite-Hadamard Tip Eşitsizlikler

Özet. Bu çalışmada, hem bir integral özdeşlik hem de Hölder ve power-mean integral eşitsizlikleri kullanılarak iki kez türevlenebilen aritmetik-harmonik konveks fonksiyonlar için birkaç yeni eşitsizlik elde edilmiştir.

Anahtar Kelimeler: Konveks fonksiyon, Aritmetik-harmonik konveks fonksiyon, Hermite-Hadamard eşitsizliği.

1. INTRODUCTION

Theory of convex sets and convex functions play an important role in mathematics and the other pure and applied sciences. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences.

Definition 1.1 A function $f: I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if the inequality

$$f(tx+(1-t)y) \leq tf(x)+(1-t)f(y)$$

valid for all $x, y \in I$ and $t \in [0,1]$. If this inequality reverses, then f is said to be concave on the interval $I \neq \emptyset$. This definition is well known in the literature.

Theorem 1.2 Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval I of real numbers and $a, b \in I$ with $a < b$. The following inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1}$$

* Corresponding author. *Email address: huriyekadakal@hotmail.com*

This double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions. See [2, 4], for the results of the generalization, improvement and extension of the famous integral inequality (1).

Definition 1.3 [5] A function $f: I \subset \mathbb{R} \to (0,\infty)$ is said to be arithmetic-harmonically (AH) convex function if for all $x, y \in I$ and $t \in [0,1]$ the equality

$$f(tx+(1-t)y) \le \frac{f(x)f(y)}{tf(y)+(1-t)f(x)} \tag{2}$$

holds. If the inequality (2) is reversed then the function $f(x)$ is said to be arithmetic-harmonically (AH) concave function.

In this study, in order to establish some new inequalities of Hermite-Hadamard type inequalities for arithmetic harmonically convex functions, we will use the following lemma obtained in the specials case of identity given in [3].

Lemma 1.4 Let $f: I \subseteq \mathbb{R} \to \mathbb{R}$ be two times-differentiable mapping on I° and $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$, we have the identity

$$bf(b) - af(a) - \frac{b^2 f'(b) - a^2 f'(b)}{2} - \int_a^b f(x)dx = -\frac{1}{2}\int_a^b x^2 f''(x)dx. \tag{3}$$

For shortness, throughout this paper, we will use the following notations for special means of two nonnegative numbers a, b with $b > a$:

1. The arithmetic mean
$$A := A(a,b) = \frac{a+b}{2}, \quad a,b > 0,$$
2. The geometric mean
$$G := G(a,b) = \sqrt{ab}, \quad a,b \ge 0$$
3. The logarithmic mean
$$L := L(a,b) = \begin{cases} \dfrac{b-a}{\ln b - \ln a}, & a \ne b \\ a, & a = b \end{cases}; \quad a,b > 0$$
4. The p-logaritmic mean
$$L_p := L_p(a,b) = \begin{cases} \left(\dfrac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right)^{\frac{1}{p}}, & a \ne b, p \in \mathbb{R}\backslash\{-1,0\} \\ a, & a = b \end{cases}; \quad a,b > 0.$$

These means are often used in numerical approximation and in other areas. However, the following simple relationships are known in the literature:

$$H \le G \le L \le I \le A.$$

It is also known that L_p is monotonically increasing over $p \in \mathbb{R}$, denoting $L_0 = I$ and $L_{-1} = L$.

2. MAIN RESULTS

Theorem 2.1 Let $f: I \subset (0,\infty) \to (0,\infty)$ be a two-times differentiable mapping on I°, and $a, b \in I^\circ$ with $a < b$. If $|f''|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:

i) If $|f''(a)| - |f''(b)| \neq 0$, then

$$\left|\frac{bf(b)-af(a)}{b-a} - \frac{b^2 f'(b) - a^2 f'(b)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{(b-a)|f''(a)||f''(b)|}{2}\left[\frac{1}{B_f}\left(\frac{b^2-a^2}{2} - (b-a)C_f\right) + \frac{C_f^2}{L(|f''(a)|,|f''(b)|)}\right], \quad (4)$$

ii) If $|f''(a)| - |f''(b)| = 0$, then

$$\left|\frac{bf(b)-af(a)}{b-a} - \frac{b^2 f'(b) - a^2 f'(b)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{|f''(b)|}{6}[2A(a^2,b^2) + G^2(a,b)],$$

where

$$B_f = B_f(a,b;n) = |f''(a)| - |f''(b)|$$

$$C_f = C_f(a,b;n) = \frac{b|f''(b)| - a|f''(a)|}{B_f}.$$

Proof. i) Let $|f''(a)| - |f''(b)| \neq 0$. If $|f''|$ is an arithmetic-harmonically convex function on the interval $[a,b]$, using Lemma 1.4 and the following inequality

$$|f''(x)| = \left|f''\left(\frac{b-x}{b-a}a + \frac{x-a}{b-a}b\right)\right| \leq \frac{|f''(a)||f''(b)|}{\frac{b-x}{b-a}|f''(b)| + \frac{x-a}{b-a}|f''(a)|}$$
$$= \frac{(b-a)|f''(a)||f''(b)|}{(b-x)|f''(b)| + (x-a)|f''(a)|},$$

we get

$$\left|bf(b) - af(a) - \frac{b^2 f'(b) - a^2 f'(b)}{2} - \int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2}\int_a^b x^2|f''(x)|dx$$

$$\leq \frac{1}{2}\int_a^b \frac{x^2(b-a)|f''(a)||f''(b)|}{(b-x)|f''(b)|+(x-a)|f''(a)|}dx$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2}\int_a^b \frac{x^2}{(b-x)|f''(b)|+(x-a)|f''(a)|}dx. \qquad (5)$$

From here, we can write the following inequality

$$\left|bf(b)-af(a)-\frac{b^2f'(b)-a^2f'(b)}{2}-\int_a^b f(x)dx\right|$$

$$\leq \frac{(b-a)|f''(a)||f''(b)|}{2B_f}\int_a^b \frac{x^2}{x+C_f}dx$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2B_f}\int_a^b \left(x-C_f+\frac{C_f^2}{x+C_f}\right)dx$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2B_f}\left[\frac{x^2}{2}-xC_f+C_f^2\ln\left(x+C_f\right)\right]\Bigg|_a^b$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2B_f}\left[\frac{b^2-a^2}{2}-(b-a)C_f+C_f^2\ln\left(\frac{b+C_f}{a+C_f}\right)\right]$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2B_f}\left[\frac{b^2-a^2}{2}-(b-a)C_f+C_f^2\ln\left(\frac{|f''(a)|}{|f''(b)|}\right)\right]$$

$$= \frac{(b-a)|f''(a)||f''(b)|}{2}\left[\frac{1}{B_f}\left(\frac{b^2-a^2}{2}-(b-a)C_f\right)+\frac{C_f^2}{L(|f''(a)|,|f''(b)|)}\right].$$

So, we get the desired inequality.

ii) Let $|f''(a)|-|f''(b)|=0$. Then, substituting $|f''(a)|=|f''(b)|$ in (5), we obtain

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^2f'(b)-a^2f'(b)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{f''(b)}{2}\int_a^b x^2dx$$

$$= \frac{|f''(b)|}{6}\left[2A(a^2,b^2)+G^2(a,b)\right]. \qquad (6)$$

This completes the proof of theorem.

Theorem 2.2 Let $f: I \subset (0,\infty) \to (0,\infty)$ be two-times differentiable mapping on I°, and $a,b \in I^\circ$ with $a<b$. *If* $|f''|^q$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:

i) If $|f''(a)|^q - |f''(b)|^q \neq 0$, then

$$\left|\frac{bf(b)-af(a)}{b-a} - \frac{b^2f'(b)-a^2f'(b)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2}\frac{L_{2p}^2(a,b)G^2(|f''(a)|,|f''(b)|)}{\left[L(|f''(a)|,|f''(b)|)L_{q-1}^{q-1}(|f''(a)|,|f''(b)|)\right]^{\frac{1}{q}}}, \quad (7)$$

ii) If $|f''(a)|^q - |f''(b)|^q = 0$, then

$$\left|\frac{bf(b)-af(a)}{b-a} - \frac{b^2f'(b)-a^2f'(b)}{2(b-a)} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{1}{2}|f''(b)|L_{2p}^2(a,b),$$

where

$$\frac{1}{p}+\frac{1}{q}=1.$$

Proof. i) Let $|f''(a)|^q - |f''(b)|^q \neq 0$. If $|f''|^q$ for $q>1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then using Lemma 1.4, well known Hölder integral inequality and the following inequality

$$|f''(x)|^q = \left|f''\left(\frac{b-x}{b-a}a+\frac{x-a}{b-a}b\right)\right|^q \leq \frac{(b-a)|f''(a)|^q|f''(b)|^q}{(b-x)|f''(b)|^q+(x-a)|f''(a)|^q},$$

we can write,

$$\left|bf(b)-af(a) - \frac{b^2f'(b)-a^2f'(b)}{2} - \int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2}\left(\int_a^b x^{2p}dx\right)^{\frac{1}{p}}\left(\int_a^b |f''(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq \frac{1}{2}\left((b-a)L_{2p}^{2p}(a,b)\right)^{\frac{1}{p}}\left(\int_a^b \frac{(b-a)|f''(a)|^q|f''(b)|^q}{(b-x)|f''(b)|^q+(x-a)|f''(a)|^q}dx\right)^{\frac{1}{q}}. \quad (8)$$

From here, we get

$$\left|bf(b)-af(a) - \frac{b^2f'(b)-a^2f'(b)}{2} - \int_a^b f(x)dx\right|$$

$$\leq \frac{1}{2}(b-a)^{\frac{1}{p}}L_{2p}^2(a,b)(b-a)^{\frac{1}{q}}\frac{|f''(a)||f''(b)|}{\left[|f''(b)|^q-|f''(a)|^q\right]^{\frac{1}{q}}}\left(\int_{(b-a)|f''(a)|^q}^{(b-a)|f''(b)|^q}\frac{1}{u}du\right)^{\frac{1}{q}}$$

$$= \frac{1}{2}(b-a)L_{2p}^2(a,b)\frac{|f''(a)||f''(b)|}{\left[|f''(b)|^q-|f''(a)|^q\right]^{\frac{1}{q}}}\left(\ln u\Big|_{(b-a)|f''(a)|^q}^{(b-a)|f''(b)|^q}\right)^{\frac{1}{q}}$$

$$=\frac{1}{2}(b-a)L_{2p}^{2}(a,b)|f''(a)||f''(b)|\left(\frac{\ln(b-a)|f''(b)|^{q}-\ln(b-a)|f''(a)|^{q}}{|f''(b)|^{q}-|f''(a)|^{q}}\right)^{\frac{1}{q}}$$

$$=\frac{1}{2}(b-a)L_{2p}^{2}(a,b)|f''(a)||f''(b)|\left(\frac{\ln|f''(b)|^{q}-\ln|f''(a)|^{q}}{|f''(b)|^{q}-|f''(a)|^{q}}\right)^{\frac{1}{q}}$$

$$=\frac{1}{2}(b-a)\frac{L_{2p}^{2}(a,b)G^{2}(|f''(a)|,|f''(b)|)}{\left[L(|f''(a)|,|f''(b)|)L_{q-1}^{q-1}(|f''(a)|,|f''(b)|)\right]^{\frac{1}{q}}},$$

where

$$\int_{a}^{b}x^{2p}dx=(b-a)L_{2p}^{2p}(a,b),$$

$$\frac{ln|f''(b)|^{q}-ln|f''(a)|^{q}}{|f''(b)|^{q}-|f''(a)|^{q}}=\frac{ln|f''(b)|-ln|f''(a)|}{|f''(b)|-|f''(a)|}\frac{q[|f''(b)|-|f''(a)|]}{|f''(b)|^{q}-|f''(a)|^{q}}$$

$$=\left[L(|f''(a)|,|f''(b)|)L_{q-1}^{q-1}(|f''(a)|,|f''(b)|)\right]^{-1}.$$

Therefore, the required result is obtained.

ii) Let $|f''(a)|^{q}-|f''(b)|^{q}=0$. Then, from (8) we obtain the following:

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^{2}f'(b)-a^{2}f'(b)}{2(b-a)}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{1}{2}|f''(b)|L_{2p}^{2}(a,b). \qquad (9)$$

This completes the proof of the Theorem.

Theorem 2.3 *Let* $f:I\subset(0,\infty)\to(0,\infty)$ be two-times differentiable mapping on I°, and $a,b\in I^{\circ}$ *with* $a<b$. *If* $|f''|^{q},q\geq 1$ is an arithmetic-harmonically convex function on the interval $[a,b]$, then the following inequality holds:

i) If $|f''(a)|^{q}-|f''(b)|^{q}\neq 0$, then

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^{2}f'(b)-a^{2}f'(b)}{2(b-a)}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{b-a}{2}|f''(a)||f''(b)|L_{2}^{2\left(1-\frac{1}{q}\right)}(a,b)$$

$$\times\left[\frac{1}{B_{f}}\left(\frac{b^{2}-a^{2}}{2}-(b-a)C_{f}\right)+\frac{C_{f}^{2}}{L(|f''(a)|^{q},|f''(b)|^{q})}\right]^{\frac{1}{q}}, \qquad (10)$$

ii) If $|f''(a)|^{q}-|f''(b)|^{q}=0$, then

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^{2}f'(b)-a^{2}f'(b)}{2(b-a)}-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|$$

$$\leq\frac{1}{6}|f''(b)|[2A(a^{2},b^{2})+G^{2}(a,b)],$$

where

$$B_{q,f}=B_{q,f}(a,b;n)=|f''(a)|^q-|f''(b)|^q$$

$$C_{q,f}=C_{q,f}(a,b;n)=\frac{b|f''(b)|^q-a|f''(a)|^q}{B_{q,f}}.$$

Proof. i) Let $|f''(a)|^q-|f''(b)|^q\neq 0$. If $|f''|^q$ for $q\geq 1$ is an AH-convex function on $[a,b]$, then using Lemma 1.4 and well known power-mean integral inequality, we have

$$\left|bf(b)-af(a)-\frac{b^2f'(b)-a^2f'(b)}{2}-\int_a^b f(x)dx\right| \tag{11}$$

$$\leq\frac{1}{2}\left(\int_a^b x^2dx\right)^{1-\frac{1}{q}}\left(\int_a^b x^2|f''(x)|^q dx\right)^{\frac{1}{q}}$$

$$\leq\frac{1}{2}\left(\int_a^b x^2dx\right)^{1-\frac{1}{q}}\left(\int_a^b \frac{x^2(b-a)|f''(a)|^q|f''(b)|^q}{(b-x)|f''(b)|^q+(x-a)|f''(a)|^q}dx\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{2}|f''(a)||f''(b)|L_2^{2\left(1-\frac{1}{q}\right)}(a,b)\left(\int_a^b \frac{x^2}{(b-x)|f''(b)|^q+(x-a)|f''(a)|^q}dx\right)^{\frac{1}{q}}$$

$$=\frac{b-a}{2}|f''(a)||f''(b)|L_2^{2\left(1-\frac{1}{q}\right)}(a,b)\left[\frac{1}{B_{q,f}}\int_a^b\left(x-C_{q,f}+\frac{C_{q,f}^2}{x+C_{q,f}}\right)dx\right]^{\frac{1}{q}}$$

$$=\frac{b-a}{2}|f''(a)||f''(b)|L_2^{2\left(1-\frac{1}{q}\right)}(a,b)\left[\frac{1}{B_{q,f}}\left(\frac{b^2-a^2}{2}-(b-a)C_{q,f}+C_{q,f}^2\ln\left(\frac{b+C_{q,f}}{a+C_{q,f}}\right)\right)\right]^{\frac{1}{q}}$$

$$=\frac{b-a}{2}|f''(a)||f''(b)|L_2^{2\left(1-\frac{1}{q}\right)}(a,b)\left[\frac{1}{B_{q,f}}\left(\frac{b^2-a^2}{2}-(b-a)C_{q,f}\right)+\frac{C_{q,f}^2}{L(|f''(a)|^q,|f''(b)|^q)}\right]^{\frac{1}{q}}.$$

Therefore, we get the following inequality:

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^2f'(b)-a^2f'(b)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{b-a}{2}|f''(a)||f''(b)|L_2^{2\left(1-\frac{1}{q}\right)}(a,b)$$

$$\times\left[\frac{1}{B_f}\left(\frac{b^2-a^2}{2}-(b-a)C_f\right)+\frac{C_f^2}{L(|f''(a)|^q,|f''(b)|^q)}\right]^{\frac{1}{q}}$$

ii) Let $|f''(a)|^q-|f''(b)|^q=0$. By using the inequality (2.8), we have

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^2f'(b)-a^2f'(b)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$

$$\leq\frac{1}{6}|f''(b)|[2A(a^2,b^2)+G^2(a,b)]. \tag{12}$$

This completes the proof of the Theorem.

Corollary 2.4 If we take $q = 1$ in the inequality (10), we get the following inequality:

$$\left|\frac{bf(b)-af(a)}{b-a}-\frac{b^2f'(b)-a^2f'(b)}{2(b-a)}-\frac{1}{b-a}\int_a^b f(x)dx\right|$$
$$\leq \frac{1}{2}|f''(a)||f''(b)|\left[\frac{1}{B_f}\left(\frac{b^2-a^2}{2}-(b-a)C_f\right)+\frac{C_f^2}{L(|f''(a)|,|f''(b)|)}\right].$$

3. APPLICATIONS FOR SPECIAL MEANS

If $p \in (-1,0)$ then the function $f(x) = x^p, x > 0$ is an arithmetic harmonically-convex function [1]. Using this function we obtain following propositions related to means:

Proposition 3.1 Let $0 < a < b$ and $p \in (-1,0)$. Then we have the following inequalities:

$$L_{p+2}^{p+2}(a,b) \leq \left(\frac{G^p(a,b)}{pL_{p-1}^{p-1}(a,b)}\right)^2\left[\frac{(p+1)L_p^{2p}(a,b)}{L_{p-1}^{p-1}(a,b)L(a,b)}-pA(a,b)L_{p-1}^{p-1}(a,b)-(p+1)L_p^p(a,b)\right]$$

Proof. Let $p \in (-1,0)$. Then we consider the function $f(x) = \frac{x^{p+2}}{(p+1)(p+2)}, x > 0$. Under the assumption of the Proposition

$$|f''(x)| = x^p$$

is an arithmetic harmonically-convex function. Therefore, the assertion follows from the inequality (4) in the Theorem 2.1, for $f: (0,\infty) \to \mathbb{R}$, $f(x) = \frac{x^{p+2}}{(p+1)(p+2)}$.

Proposition 3.2 Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-1,0)$. Then, we have the following inequality:

$$L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b) \leq \frac{L_{2p}^2(a,b)G^{\frac{2m}{q}}(a,b)}{L^{\frac{1}{q}}(a,b)L_{m-1}^{\frac{m-1}{q}}(a,b)}.$$

Proof. The assertion follows from the inequality (6) in the Theorem 2.2. Let

$$f(x) = \frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)}x^{\frac{m}{q}+2}, \qquad x \in (0,\infty).$$

Then $|f''(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.2.

Proposition 3.3 Let $a, b \in (0,\infty)$ with $a < b$, $q > 1$ and $m \in (-1,0)$. Then, we have the following inequality:

$$L_{\frac{m}{q}+2}^{\frac{m}{q}+2}(a,b) \le \left[\frac{G^m(a,b)}{mL_{m-1}^{m-1}(a,b)}\right]^{\frac{2}{q}} L_2^{2\left(1-\frac{1}{q}\right)}(a,b)$$

$$\times \left\{\frac{(m+1)^2 L_m^{2m}(a,b)}{L(a,b)L_{m-1}^{m-1}(a,b)} - (m+1)L_m^m(a,b) - mA(a,b)L_{m-1}^{m-1}(a,b)\right\}^{\frac{1}{q}}. \qquad (13)$$

Proof. The assertion follows from the inequality (8) in the Theorem 2.3. Let

$$f(x) = \frac{1}{\left(\frac{m}{q}+1\right)\left(\frac{m}{q}+2\right)} x^{\frac{m}{q}+2}, \qquad x \in (0,\infty).$$

Then $|f''(x)|^q = x^m$ is an arithmetic harmonically-convex on $(0,\infty)$ and the result follows directly from Theorem 2.3.

Corollary 3.4 If we take $q = 1$ in the inequality (13), we get the following inequality:

$$L_{m+2}^{m+2}(a,b) \le \frac{G^{2m}(a,b)}{m^2 L_{m-1}^{2(m-1)}(a,b)} \left[\frac{(m+1)^2 L_m^{2m}(a,b)}{L(a,b)L_{m-1}^{m-1}(a,b)} - (m+1)L_m^m(a,b) - mA(a,b)L_{m-1}^{m-1}(a,b)\right].$$

REFERENCES

[1] Dragomir, S.S., Inequalities of Hermite-Hadamard type for AH-convex functions, Stud. Univ. Babeş-Bolyai Math., 61-4 (2016) 489-502.

[2] Dragomir S.S. and Pearce, C.E.M., Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.

[3] Maden, S., Kadakal, H., Kadakal M. and İşcan, İ., Some new integral inequalities for n-times differentiable convex and concave functions. J. Nonlinear Sci. Appl., 10 (2017) 6141-6148.

[4] Sarikaya, M.Z. and Aktan, N., On the generalization of some integral inequalities and their applications, Mathematical and Computer Modelling, 54 (2011) 2175-2182.

[5] Zhang, T.Y. and Qi, F., Integral Inequalities of Hermite-Hadamard Type for m-AH Convex Functions, Turkish Journal of Analysis and Number Theory, 2-3 (2014) 60-64.