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A Theorem on Absolute Summability of Infinite Series

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Abstract. In this paper, a theorem on absolute summability of infinite series is obtained by taking almost increasing sequence instead of positive non-decreasing sequence. Also, some results of absolute summability are given.

***Keywords*:** Riesz mean, absolute summability, almost increasing sequence, Hölder inequality, Minkowski inequality.

Sonsuz Serilerin Mutlak Toplanabilmesi Üzerine Bir Teorem

Özet. Bu makalede, pozitif azalmayan dizi yerine hemen hemen artan dizi alınarak, sonsuz serilerin mutlak toplanabilmesi üzerine bir teorem elde edildi. Ayrıca, mutlak toplanabilme ile ilgili bazı sonuçlar verildi.

***Anahtar Kelimeler*:** Riesz ortalaması, mutlak toplanabilme, hemen hemen artan dizi, Hölder eşitsizliği, Minkowski eşitsizliği.

1. INTRODUCTION

A positive sequence (b_n) is said to be almost increasing if there exist a positive increasing sequence (c_n) and two positive constants K and L such that $Kc_n \le b_n \le Lc_n$ [1]. Obviously, every increasing sequence is almost increasing. However, the converse need not be true as can be seen by taking the example, say $b_n = ne^{(-1)^n}$. Let $\sum a_n$ be an infinite series with its partial sums (s_n). Let (φ_n) be a sequence of positive real numbers. The series $\sum a_n$ is said to be summable $\varphi - \left|\bar{N}, p_n; \delta\right|_k$, $k \ge 1$ and $\delta \ge 0$, if [2]

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k+k-1} \left|\theta_n - \theta_{n-1}\right|^k < \infty, \tag{1}$$

where (p_n) is a sequence of positive numbers such that

$$P_n = \sum_{v=0}^{n} p_v \to \infty \quad \text{as} \quad n \to \infty \quad (P_{-i} = p_{-i} = 0, \; i \ge 1) \tag{2}$$

and

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$$\theta_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v s_v \tag{3}$$

defines the sequence (θ_n) of the $(\bar{N}, p_n)$ mean of the sequence (s_n) generated by the sequence of coefficients (p_n) [3].

In the special case, if we take $\varphi_n = P_n / p_n$, then $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability reduces to $\left|\bar{N}, p_n; \delta\right|_k$ summability [4]. Also, if we take $\varphi_n = P_n / p_n$ and $\delta = 0$, $\varphi - \left|\bar{N}, p_n; \delta\right|_k$ summability reduces to $\left|\bar{N}, p_n\right|_k$ summability [5]. Finally, if we take $\varphi_n = n$, $\delta = 0$ and $p_n = 1$ for all values of n, then we get $\left|C,1\right|_k$ summability [6].

2. KNOWN RESULTS

Absolute summability methods are generally used to summability of an infinite series. There is an important application area of these methods. Especially, they have applications on different sequences such as positive non-decreasing, almost increasing and quasi power increasing sequences.

There are many different studies on absolute summability methods (see [2, 7-21]). Among them, in [7], the following theorem was proved.

Theorem 1 *Let* (X_n) *be a positive non-decreasing sequence and* (β_n), (λ_n) *be sequences such that*

$$\left|\Delta\lambda_n\right| \le \beta_n, \tag{4}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{5}$$

$$\sum_{n=1}^{\infty} n \left|\Delta\beta_n\right| X_n < \infty, \tag{6}$$

$$\left|\lambda_n\right| X_n = O(1) \quad as \quad n \to \infty \tag{7}$$

hold where $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$. *If* (p_n) *is a sequence of positive numbers such that*

$$P_n = O(np_n) \quad as \quad n \to \infty, \tag{8}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} \left|s_n\right|^k = O(X_m) \quad as \quad m \to \infty, \tag{9}$$

then the series $\sum a_n \lambda_n$ *is summable* $\left|\bar{N}, p_n\right|_k$, $k \ge 1$.

3. MAIN RESULT

Theorem 1 is generalized as in the following form under weaker conditions by using an almost increasing sequence instead of a positive non-decreasing sequence.

Theorem 2 *Let* (X_n) *be an almost increasing sequence and* $\varphi_n p_n = O(P_n)$. *If conditions (4)-(8) of Theorem 1 and*

$$\sum_{n=1}^{m} \varphi_n^{\delta k-1} |s_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{10}$$

$$\sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} = O\left(\varphi_v^{\delta k} \frac{1}{P_v} \right) \quad as \quad m \to \infty \tag{11}$$

are satisfied, then the series $\sum a_n \lambda_n$ *is summable* $\varphi - \left| \bar{N}, p_n; \delta \right|_k$, $k \geq 1$ *and* $0 \leq \delta < 1/k$.

Lemma 3 [22]. Under the conditions of Theorem 2, we have

$$nX_n \beta_n = O(1) \quad as \quad n \to \infty, \tag{12}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{13}$$

4. PROOF OF THEOREM 2

Let (M_n) be the sequence of $(\bar{N}, p_n)$ mean of the series $\sum a_n \lambda_n$. Then, we get

$$M_n = \frac{1}{P_n} \sum_{v=0}^{n} p_v \sum_{r=0}^{v} a_r \lambda_r = \frac{1}{P_n} \sum_{v=0}^{n} (P_n - P_{v-1})\, a_v \lambda_v.$$

Now, for $n \geq 1$

$$M_n - M_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n} P_{v-1} \lambda_v a_v.$$

From Abel's transformation, we obtain

$$M_n - M_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} P_v \Delta\lambda_v s_v - \frac{p_n}{P_n P_{n-1}} \sum_{v=1}^{n-1} p_v \lambda_v s_v + \frac{p_n s_n \lambda_n}{P_n}$$
$$= M_{n,1} + M_{n,2} + M_{n,3}$$

To prove that $\sum a_n \lambda_n$ is summable $\varphi - \left| \bar{N}, p_n; \delta \right|_k$, it is sufficient to show that

$$\sum_{n=1}^{\infty} \varphi_n^{\delta k + k - 1} \left| M_{n,r} \right|^k < \infty \text{ for } r = 1, 2, 3.$$

First, using the fact that $\varphi_n p_n = O\left(P_n\right)$ and the condition (4), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}^k} \left\{ \sum_{v=1}^{n-1} P_v \beta_v \left| s_v \right| \right\}^k .$$

Now, using Hölder's inequality and the condition (8),

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{n=2}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} \left\{ \sum_{v=1}^{n-1} p_v \left(v\beta_v \right)^k \left| s_v \right|^k \right\} \left\{ \frac{1}{P_{n-1}} \sum_{v=1}^{n-1} p_v \right\}^{k-1} .$$

Then, we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{v=1}^{m} p_v \left(v\beta_v \right)^k \left| s_v \right|^k \sum_{n=v+1}^{m+1} \varphi_n^{\delta k-1} \frac{1}{P_{n-1}} .$$

Here, using the condition (11), we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k} \frac{p_v}{P_v} \left(v\beta_v \right)^k \left| s_v \right|^k$$

$$= O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} \left(\frac{\varphi_v p_v}{P_v} \right) \left(v\beta_v \right)^k \left| s_v \right|^k ,$$

then using the fact that $\varphi_v p_v = O\left(P_v\right)$, we obtain

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} \left(v\beta_v \right) \left(v\beta_v \right)^{k-1} \left| s_v \right|^k .$$

Now using the fact that the sequence $\left(X_n\right)$ is almost increasing and the condition (12), we obtain $\left(v\beta_v\right)^{k-1} = O(1)$. Thus, we have

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{v=1}^{m} \varphi_v^{\delta k-1} v\beta_v \left| s_v \right|^k .$$

Then, using Abel's transformation, we get

$$\sum_{n=2}^{m+1} \varphi_n^{\delta k+k-1} \left| M_{n,1} \right|^k = O(1) \sum_{v=1}^{m-1} \Delta\left(v\beta_v \right) \sum_{i=1}^{v} \varphi_i^{\delta k-1} \left| s_i \right|^k + O(1)\, m\beta_m \sum_{v=1}^{m} \varphi_v^{\delta k-1} \left| s_v \right|^k .$$

Here considering the fact that

$$\Delta(v\beta_v) = v\beta_v - (v+1)\beta_{v+1} = v\Delta\beta_v - \beta_{v+1} ,$$

and using the condition (10), we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,1}\right|^k=O(1)\sum_{v=1}^{m-1}v\left|\Delta\beta_v\right|X_v+O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1}+O(1)\,m\,\beta_m X_m$$

$$=O(1)\quad\text{as}\quad m\to\infty,$$

by (6), (13) and (12), respectively.

Now, again using the fact that $\varphi_n p_n=O(P_n)$ and Hölder's inequality, we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k=O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}^k}\left\{\sum_{v=1}^{n-1}p_v\left|\lambda_v\right|\left|s_v\right|\right\}^k$$

$$=O(1)\sum_{n=2}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}\left\{\sum_{v=1}^{n-1}p_v\left|\lambda_v\right|^k\left|s_v\right|^k\right\}\left\{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1}p_v\right\}^{k-1}.$$

Then

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k=O(1)\sum_{v=1}^{m}p_v\left|\lambda_v\right|\left|\lambda_v\right|^{k-1}\left|s_v\right|^k\sum_{n=v+1}^{m+1}\varphi_n^{\delta k-1}\frac{1}{P_{n-1}}.$$

Here, using the fact that the sequence (X_n) is almost increasing and considering the condition (7), it is clear that $\left|\lambda_v\right|^{k-1}=O(1)$. Additionally, using the condition (11), we have

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k=O(1)\sum_{v=1}^{m}\varphi_v^{\delta k-1}\left|\lambda_v\right|\left|s_v\right|^k.$$

Hence, we get

$$\sum_{n=2}^{m+1}\varphi_n^{\delta k+k-1}\left|M_{n,2}\right|^k=O(1)\sum_{v=1}^{m-1}\Delta\left|\lambda_v\right|\sum_{i=1}^{v}\varphi_i^{\delta k-1}\left|s_i\right|^k+O(1)\left|\lambda_m\right|\sum_{v=1}^{m}\varphi_v^{\delta k-1}\left|s_v\right|^k$$

$$=O(1)\sum_{v=1}^{m-1}\beta_v X_v+O(1)\left|\lambda_m\right|X_m$$

$$=O(1)\quad\text{as}\quad m\to\infty,$$

by virtue of Abel's transformation, (4), (10), (13) and (7).

Finally, again using the fact that $\varphi_n p_n=O(P_n)$, we have

$$\sum_{n=1}^{m}\varphi_n^{\delta k+k-1}\left|M_{n,3}\right|^k=O(1)\sum_{n=1}^{m}\varphi_n^{\delta k-1}\left|\lambda_n\right|\left|s_n\right|^k$$

$$=O(1)\quad\text{as}\quad m\to\infty,$$

as in $M_{n,2}$. Thence, the proof of Theorem 2 is completed.

5. CONCLUSION

In this paper, generalized absolute summability of an infinite series is studied. A general theorem dealing with absolute summability is obtained. For the special cases of (X_n), (φ_n), (p_n) and δ, some results can be obtained. If we take (X_n) as a positive non-decreasing sequence in Theorem 2, we get an another theorem dealing with $\varphi-\left|\bar{N},p_n;\delta\right|_k$ summability of an infinite series. If we take $\varphi_n = P_n/p_n$ in Theorem 2, then we get a known theorem on $\left|\bar{N},p_n;\delta\right|_k$ summability of an infinite series [23]. Also, if we take (X_n) as a positive non-decreasing sequence, $\varphi_n = P_n/p_n$ and $\delta=0$ in Theorem 2, then the condition (10) reduces to the condition (9) and also the conditions $\varphi_n p_n = O(P_n)$ and (11) are automatically satisfied. Thus, Theorem 2 reduces to Theorem 1. Finally, if we take (X_n) as a positive non-decreasing sequence, $\varphi_n = n$, $\delta=0$ and $p_n=1$ for all values of n, then we get a known result of $|C,1|_k$ summability [24].

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