Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-2(2019) 505-517

Triangles In The De-Sitter Plane

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Received: 26.11.2018; Accepted: 20.03.2019 *http://dx.doi.org/10.17776/csj.487548*

Abstract. The triangular varieties in the de-Sitter plane were investigated and the formulas of triangles of non-degenerate and type of triangles were obtained in terms of dihedral angles.

Keywords: Triangle, Dihedral angles, de-Sitter plane

De-Sitter Düzleminde Üçgenler

Özet. Sitter düzlemindeki üçgen çeşitleri incelenip, bu üçgenlerin dejenere olmayanlarından olan ve tipindeki üçgenlerin alan formülleri dihedral açılar cinsinden elde edilmiştir.

Anahtar Kelimeler: Üçgen, Dihedral açı, de-Sitter düzlemi

1. INTRODUCTION

The hypothesis that the shortest distance between two points is the correct part between these points is also used by Archimedes . At the end of the nineteenth century, the concept of geodesy emerged from the problem of finding the shortest path connecting two points on a surface. In 1732 Euler published a differential equation of geodesics on a surface. Thus, it was shown that the geodesics given depending on the two points can be given only depending on the type of surface. Latin translations of Archimedes and Apollonius in the Middle Ages and the introduction of Fermat and Dekart in analytical geometry in 1637 led to the development of geometric techniques used to find the tangents of plane curves in the first half of the 19th century. The algebraic formula $y^2 = x^3$ obtained by using analytical geometry, which gives the length of a nonlinear curve, was found separately by Neil van Heuraet and Fermat around 1658. In the fourth quarter of the 19th century, the Euclidean arc length element was found independently by Newton and Leibnitz and calculated the arc length of these two geometric plane curves using integral. The concept of arc length in metric spaces was entered by Menger in 1930.

The use of other metric spaces used to concretize abstract concepts (such as relativity) in which the Euclidean metric cannot be a model is inevitable. Today we see these spaces as Lorentzian, global, hyperbolic and de-Sitter. Since the curvature at one point of a curve in these spaces measures the amount of deviation at this point and the curvature of the geodesics is zero, we can correctly consider the geodesy through the two given points of the space. In the space we consider, if we solve the differential equation of geodesics with respect to two points, then we see that solution will be unique. This coincides with a single true hypothesis from two points in the Euclidean space. In this case, the correct part of the Euclidean

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space is limited to two points and in this case it is a geodesic part limited to two points. The triangular region in the Euclidean space is the intersection of the semi-space determined by the external unit normals of the lines, and in this case it is the intersection of the semi-spaces determined by the geodesics. Furthermore, Asmus [1] obtain the type of triangles in de-Sitter plane by using geodesic segments.

Theorem 1

$p,q \in S_1^n$ and $V = Sp\{p,q\}$ is taken.

(i) If V spacelike, the parametrical equation of the line passing through p,q is

$$\alpha(t) = (\cos t)\, p + (\sin t)\left(\frac{q - \langle p,q\rangle p}{\left\| q - \langle p,q\rangle p \right\|}\right) \ , \ t \in IR$$

(ii) If V timelike, the parametrical equation of the line passing through p,q is

$$\beta(s) = (\cosh s)\, p + (\sinh s)\left(\frac{q - \langle p,q\rangle p}{\left\| q - \langle p,q\rangle p \right\|}\right) \ , \ s \in IR$$

(iii) If V null, the parametrical equation of the line passing through p,q is

$$\gamma(\lambda) = p + \lambda(q - p) \ , \ \lambda \in IR$$

Proof

It can be seen in [2] (in Proposition 28).

Theorem 2

$p,q \in S_1^n$ and $V = Sp\{p,q\}$ is taken.

(i) If V spacelike, $\langle p,q\rangle = \cos t_0$ respectively, the length of the line segment t_0 and the parametric equation limited with p,q are

$$\alpha(t) = (\cos t)\, p + (\sin t)\left(\frac{q - \cos t_0\, p}{\sin t_0}\right) \ , \ 0 \le t \le t_0 .$$

(ii) If V timelike, $\langle p,q\rangle = \cosh s_0$ respectively, the length of the line segment s_0 and the parametrical equation limited with p,q are

$$\beta(s) = (\cosh s)\, p + (\sinh s)\left(\frac{q - \cosh s_0\, p}{\sinh s_0}\right) \ , \ 0 \le s \le s_0 .$$

(iii) If V null

$$\gamma(\lambda) = p + \lambda(q - p)\ ,\ 0 \le \lambda \le 1$$

Proof

As Theorem 1.1 (i) is $\alpha(0) = p$, $\alpha(t_0) = q$ and α is also continuous $\alpha(t) = (\cos t)\,p + (\sin t)\left(\dfrac{q - \cos t_0\, p}{\sin t_0}\right)$, $0 \le t \le t_0$ point is on line segment limited with p, q.

As Theorem 1.1 (ii) is $\beta(0) = p$, $\beta(s_0) = q$ and β is also continuous $\beta(s) = (\cosh s)\,p + (\sinh s)\left(\dfrac{q - \cosh s_0\, p}{\sinh s_0}\right)$, $0 \le s \le s_0$ point is on line segment limited with p, q.

As from Theorem 1.1 $\gamma(0) = p$ and $\gamma(1) = q$ and $\gamma(1) = q$ and γ are continuous $\gamma(\lambda) = p + \lambda(q - p)$, $0 \le \lambda \le 1$ is parametrical equation of the line segment limited with p, q.

1.1. Triangle Types in de-Sitter Space

a. ${}_0\Delta_0^3$ Lightlike edged triangle

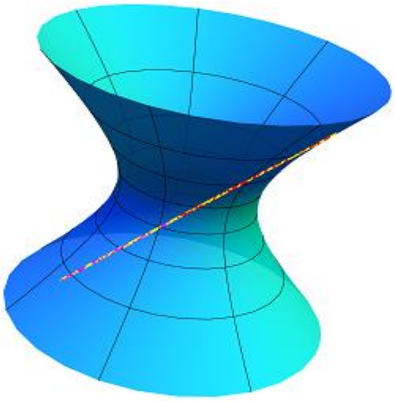

Figure 1. Lightlike edged triangle

b. ${}_1\Delta_0^2$ Spacelike based lightlike edged triangle

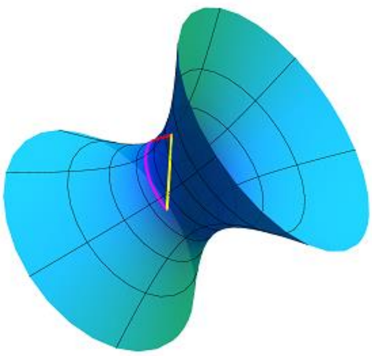

Figure 2. Spacelike based null edged triangle

c. Lightlike based timelike edged triangle

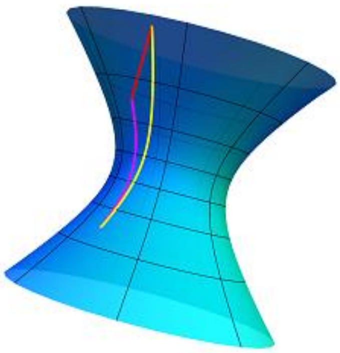

Figure 3. Null based timelike edged triangle

d. ${}_0\Delta_1^2$ Timelike based lightlike edged triangle

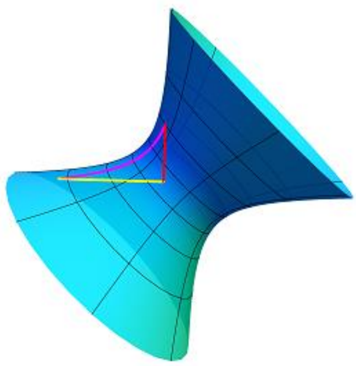

Figure 4. Timelike based null edged triangle

e. ${}_2\Delta_0^1$ Lightlike based spacelike edged triangle

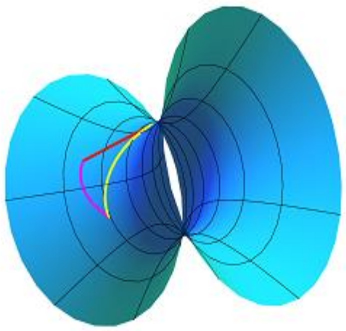

Figure 5. Null based spacelike edged triangle

f. ${}_1\Delta_1^1$ Casual Scalene Triangle

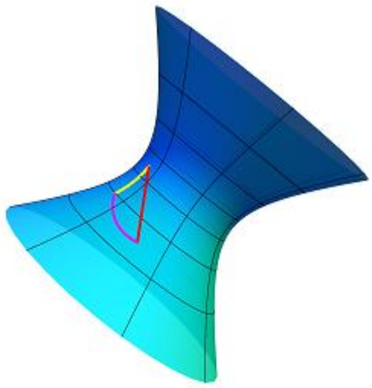

Figure 6. Casual Scalene Triangle

g. ${}_3\Delta_0^0$ Spacelike edged Triangle

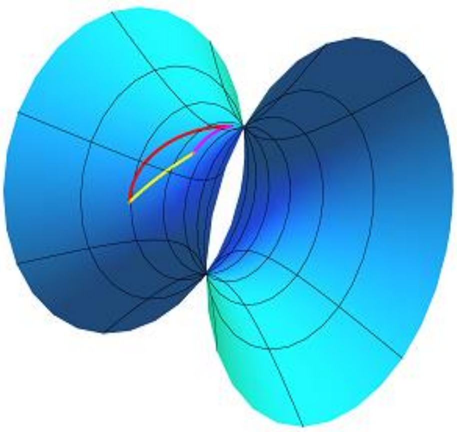

Figure 7. Spacelike edged triangle

h. ${}_0\Delta_3^0$ Timelike edged triangle

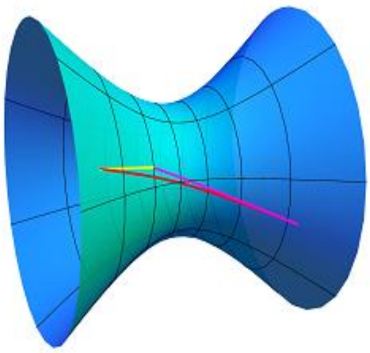

Figure 8. Timelike edged triangle

ı. ${}_2\Delta_1^0$ Timelike based spacelike edged triangle

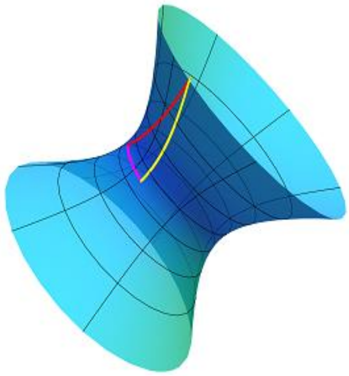

Figure 9. Timelike based spacelike edged triangle

i. ${}_1\Delta_2^0$ Spacelike based timelike edged triangle

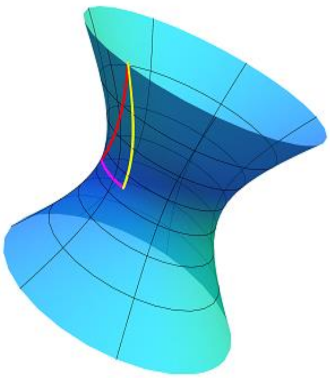

Figure 10. Timelike edged spacelike based triangle

1.2. Some Properties of the Rows Passing from Two Points in the de-Sitter Space

a. The need and sufficient condition that $V = Sp\{P_i, P_j\}$ is timelike $\langle P_i, P_j\rangle > 1$ points are on the same part of $V \cap S_1^n$ hyperbola (from ([2] Proposition 38 situation 2).

b. The need and sufficient condition that $V = Sp\{P_i, P_j\}$ is timelike $\langle P_i, P_j\rangle < -1$ points are on the same part of $V \cap S_1^n$ hyperbola (from ([2] Proposition 38 situation 2).

c. The need and sufficient condition that $V = Sp\{P_i, P_j\}$ is spacelike $|\langle P_i, P_j\rangle| < 1$ (from [2] Proposition 38 situation 1).

d. The need and sufficient condition that $V = Sp\{P_i, P_j\}$ is null $|\langle P_i, P_j\rangle| = 1$ (from [2] Proposition 38 situation 3).

Limited with P_i, P_j geodetic part is l_{ij} ,

e. The need and sufficient condition that l_{ij} is hyperbola $\langle P_i, P_j\rangle > 1$ [2].

f. The need and sufficient condition that l_{ij} is elips $|\langle P_i, P_j\rangle| < 1$ [2].

g. The need and sufficient condition that l_{ij} is null line segment $\langle P_i, P_j\rangle = 1$ [2].

2. THE AREAS OF NON-DEGENERATED EDGE TRIANGLES IN de-SITTER SPACE

Definition 1

The angle between these vectors, including

Two non-null vectors in N_1, N_2 IR_1^2 and the $\theta(N_1, N_2)$ angles between these vectors are identifed as following [3]

(i) If $\langle N_1, N_1\rangle\langle N_2, N_2\rangle > 0$ *ve* $\langle N_1, N_2\rangle < 0$; $\theta(N_1, N_2) = \operatorname{arccos} h(-\langle N_1, N_2\rangle)$,

(ii) If $\langle N_1, N_1\rangle\langle N_2, N_2\rangle > 0$ *ve* $\langle N_1, N_2\rangle > 0$; $\theta(N_1, N_2) = -\operatorname{arccos} h(\langle N_1, N_2\rangle)$,

(iii) If $\langle N_1, N_1\rangle\langle N_2, N_2\rangle < 0$; $\theta(N_1, N_2) = -\operatorname{arcsin} h(\langle N_1, N_2\rangle)$

Definition 2

As N_1, N_2 IR_1^{n+1} are two vectors which are non-null and stretching Lorentz plane;

The θ_{12} dihedral angle in edge of $\{v \in IR_1^{n+1} : \langle v, N_1\rangle \geq 0, \langle v, N_2\rangle \geq 0\}$ dihedron is identified as [3]:

$\theta_{12} = -\theta(N_1, N_2)$

In the case of $S_1^2(1)$, 2- dimensional face are their peak points on the triangle on S_1^2 is non-degenerate since $\langle P_i, P_i \rangle = 1$. Therefore, Schülei differential formula is applicable to triangles whose edges are non-null . There are four different triangles as ${}_3\Delta_0^0$, ${}_2\Delta_1^0$, ${}_1\Delta_2^0$, ${}_0\Delta_3^0$ on S_1^2 whose edges are non-null.

Theorem 3 (Schlafli Differential Formula)

The centrifugal hyperquadiene-linked component, ε marked in $S_q^n(\varepsilon)$, IR_q^{n+1} space, the differential of the volume of $V_n(\Omega)$ non-degenerated volume on $S_q^n(\varepsilon)$ 1 and 2 coherent faces as

$$dV_n(\Omega) = \frac{\varepsilon}{n-1} \sum_F V_{n-2}(F)\, d\alpha_F$$

Here, $V_{n-2}(F)$, it is the volume of Ω 's *n-2* face and α_F is the dihedral angle on the *F* face [3].

$n = 2$ as special condition, $\{i, j, k\}$ set is a permutation of $\{1, 2, 3\}$ and as the angle at θ_{ij}, P_k edge is;

$$dV_2(\Omega) = \varepsilon\left(d\theta_{12} + d\theta_{13} + d\theta_{23}\right) \tag{2.1}$$

2.1. Area of triangle from ${}_2\Delta_1^0$ type

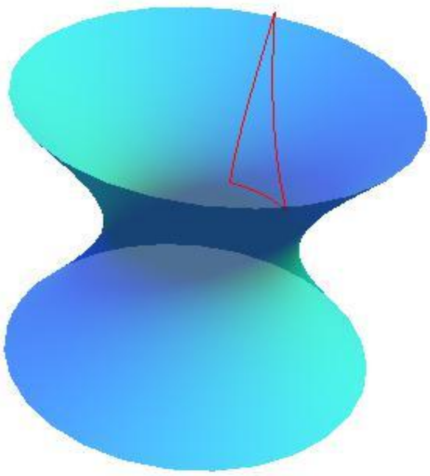

Figure 11. Triangle from ${}_2\Delta_1^0$ type

The edges of de-Sitter triangle from ${}_2\Delta_1^0$ type with $P_1=\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right), P_2=(0,0,1), P_3=\left(\frac{1}{2},0,\frac{\sqrt{5}}{2}\right)$ corner point are:

$$\alpha(t)=\left(0,\frac{\cos t-\sin t}{\sqrt{2}},\frac{\cos t+\sin t}{\sqrt{2}}\right), \qquad t\in\left[0,\frac{\pi}{4}\right]$$

$$\beta(s)=(\sinh s,0,\cosh s), \qquad s\in\left[0,\operatorname{arctan}h\left(\frac{1}{\sqrt{5}}\right)\right]$$

$$\gamma(u)=\left(\frac{\sqrt{3}\cos u-\sqrt{5}\sin u}{2\sqrt{3}},\frac{2\sin u}{\sqrt{3}},\frac{\sqrt{5}}{2}\cos u-\frac{\sin u}{2\sqrt{3}}\right), \quad u\in\left[0,\arccos\left(\frac{\sqrt{5}}{2\sqrt{2}}\right)\right]$$

The projection of these on $z=0$ plane is as:

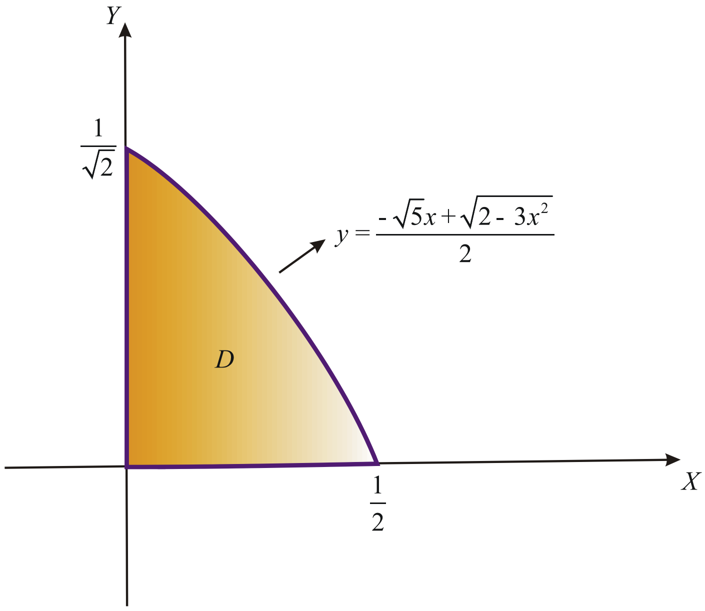

Figure 12. D zone

Since $\quad V_2(\Delta)=\iint\limits_D\sqrt{1+z_x^2+z_y^2}\,dydx$

and

$$z=\sqrt{1+x^2-y^2} \quad \text{is;}$$

$$V_2(\Delta) = \iint_D \sqrt{\frac{1+2x^2}{1+x^2-y^2}}\,dydx$$

$$= \int_0^{1/2} \left(\int_0^{\left(-\sqrt{5}x+\sqrt{2-3x^2}\right)/2} \sqrt{\frac{1+2x^2}{1+x^2-y^2}}\,dy \right) dx \tag{2.2}$$

On the other hand, the interior angles of this triangle is;

$$P_1 \otimes P_2 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{e_1}{\sqrt{2}} \text{ is timelike when } N_3 = -e_1 \,,$$

$$P_2 \otimes P_3 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{\sqrt{5}}{2} \end{vmatrix} = \frac{e_2}{2} \text{ spacelike; } N_1 = e_2 \,,$$

$$P_3 \otimes P_1 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ \frac{1}{2} & 0 & \frac{\sqrt{5}}{2} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \frac{\sqrt{5}}{2\sqrt{2}} e_1 - \frac{1}{2\sqrt{2}} e_2 + \frac{1}{2\sqrt{2}} e_3 \,; N_2 = \left(\frac{\sqrt{5}}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right) ,$$

$$\langle N_1, N_2 \rangle = -\frac{1}{\sqrt{3}} < 0$$

$$\langle N_1, N_3 \rangle = 0 \leq 0$$

$$\langle N_2, N_3 \rangle = \frac{\sqrt{5}}{\sqrt{3}} > 0 \,,$$

$$N_1 = e_2$$

$$N_2 = \frac{\sqrt{5}e_1 - e_2 + e_3}{\sqrt{3}}$$

$$N_3 = -e_1$$

and

since N_1 spacelike N_2 timelike, from Definition 2

$$-\operatorname{arcsin}h\left(\langle N_1, N_2\rangle\right)=\bar{\theta}_{12} \Rightarrow \bar{\theta}_{12}=-\operatorname{arcsin}h\left(\frac{-1}{\sqrt{3}}\right),$$

Since N_1 spacelike N_3 timelike, from Definition 2

$$-\operatorname{arcsin}h\left(\langle N_1, N_3\rangle\right)=\bar{\theta}_{13} \Rightarrow \bar{\theta}_{13}=\operatorname{arcsin}h(0)=0,$$

Since N_2, N_3 timelike, from Definition 2

$$-\operatorname{arccos}h\left(\langle N_2, N_3\rangle\right)=\bar{\theta}_{23} \Rightarrow \bar{\theta}_{23}=-\operatorname{arccos}h\left(\frac{\sqrt{5}}{\sqrt{3}}\right)$$

are found. From Theorem 3

$$V_2(\Delta)=\bar{\theta}_{12}+\bar{\theta}_{13}+\bar{\theta}_{23}+c. \tag{2.3}$$

By writing $V_2(\Delta)$ *and* $\bar{\theta}_{12}, \bar{\theta}_{13}, \bar{\theta}_{23}$ values on (2.2) to the place on (2.3)

$$c=V_2(\Delta)+\operatorname{arcsin}h\left(\frac{-1}{\sqrt{3}}\right)+\operatorname{arccos}h\left(\frac{\sqrt{5}}{\sqrt{3}}\right) \tag{2.4}$$

is obtained.

Theorem 4

As $\theta_{12}, \theta_{13}, \theta_{23}$ angles are the interior angles of ${}_2\Delta_1^0$

$$V_2\left({}_2\Delta_1^0\right)=\theta_{12}+\theta_{13}+\theta_{23}+V_2(\Delta)+\operatorname{arcsin}h\left(-\frac{1}{\sqrt{3}}\right)+\operatorname{arccos}h\left(\frac{\sqrt{5}}{\sqrt{3}}\right)$$

Proof

It is seen by writing (2.4) as in $\varepsilon=1$ form on the place (2.1)

2.2. Area of triangle from ${}_1\Delta_2^0$ type

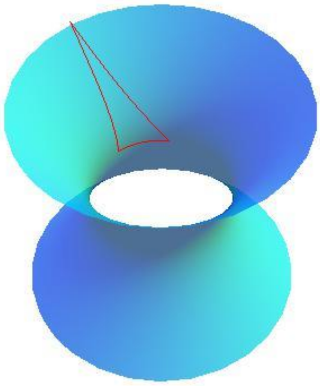

Figure 13. Triangle from ${}_1\Delta_2^0$ type

${}_1\Delta_2^0$ de-Sitter triangle is $\bar{\Delta}$, whose edges are

$$\alpha(t)=\left(0,\frac{\cos t-\sin t}{\sqrt{2}},\frac{\cos t+\sin t}{\sqrt{2}}\right),\qquad t\in\left[0,\frac{\pi}{4}\right]$$

$$\beta(s)=\left(\sqrt{3}\left(\cosh s-\sqrt{2}\sinh s\right),\frac{\sinh s}{\sqrt{2}},\frac{\left(2\sqrt{2}\cosh s-3\sinh s\right)}{\sqrt{2}}\right),\quad s\in\left[0,\log\left(1+\sqrt{2}\right)\right]$$

$$\gamma(u)=(\sinh u,0,\cosh u),\quad u\in\left[0,\log\left(2+\sqrt{3}\right)\right]$$

whose corner point is

$$P_1=\left(0,\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right),P_2=(0,0,1),P_3=\left(\sqrt{3},0,2\right)$$

Let us show the simply closed area which is $\bar{D}$ in Figure 14 obtained from the projection of $\bar{\Delta}$ edges to $z=0$ plane:

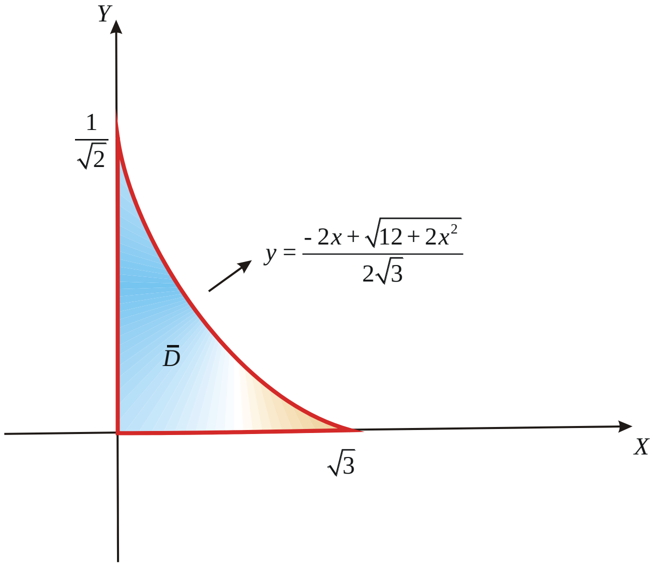

Figure 14. $\bar{\Delta}$ area

Since $V_2(\bar{\Delta}) = \iint_D \sqrt{1+z_x^2+z_y^2}\,dydx$

and

$$z = \sqrt{1+x^2-y^2}$$

$$V_2(\bar{\Delta}) = \iint_D \sqrt{\frac{1+2x^2}{1+x^2-y^2}}\,dydx$$

$$= \int_0^{\sqrt{3}} \left(\int_0^{\left(-2x+\sqrt{12+2x^2}\right)/2\sqrt{3}} \sqrt{\frac{1+2x^2}{1+x^2-y^2}}\,dy \right) dx \tag{2.5}$$

On the other hand, the interior angles of this triangle is;

$$P_1 \otimes P_2 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & 0 & 1 \end{vmatrix} = -\frac{e_1}{\sqrt{2}} \text{ timelike; } N_3 = -e_1 ,$$

$$P_2 \otimes P_3 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ 0 & 0 & 1 \\ \sqrt{3} & 0 & 2 \end{vmatrix} = \sqrt{3}e_2 \text{ timelike; } N_1 = e_2 ,$$

$$P_3 \otimes P_1 = \begin{vmatrix} -e_1 & e_2 & e_3 \\ \sqrt{3} & 0 & 2 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = \sqrt{2}e_1 - \frac{\sqrt{3}}{\sqrt{2}}e_2 + \frac{\sqrt{3}}{\sqrt{2}}e_3 ; \; N_2 = \left(\sqrt{2}, \frac{-\sqrt{3}}{\sqrt{2}}, \frac{\sqrt{3}}{\sqrt{2}} \right) ,$$

$$\langle N_1, N_2 \rangle = -\frac{\sqrt{3}}{\sqrt{2}} < 0$$

$$\langle N_1, N_3 \rangle = 0 \leq 0$$

$$\langle N_2, N_3 \rangle = \sqrt{2} > 0 ,$$

$$N_1 = e_2$$

$$N_2 = \frac{2e_1 - \sqrt{3}e_2 + \sqrt{3}e_3}{\sqrt{2}}$$

$$N_3 = -e_1$$

and

Since N_1 , N_2 is spacelike, from Definition 2

$$\operatorname{arccos} h\left(\langle N_1, N_2 \rangle\right) = \theta'_{12} \Rightarrow \theta'_{12} = \operatorname{arccos} h\left(\sqrt{\frac{3}{2}}\right) ,$$

Since N_1 is spacelike N_3 is timelike, from Definition 2

$$-\arcsin h\left(\langle N_1, N_3 \rangle\right) = \theta'_{13} \Rightarrow \theta'_{13} = \arcsin h(0) = 0$$

Since N_2 is spacelike and N_3 is timelike, from Definition 2

$$-\arcsin h\left(\langle N_2, N_3 \rangle\right) = \theta'_{23} \Rightarrow \theta'_{23} = -\arcsin h\left(\sqrt{2}\right)$$

are found. From Theorem 3

$$V_2\left(\overline{\Delta}\right) = \theta'_{12} + \theta'_{13} + \theta'_{23} + c . \tag{2.6}$$

If we write the values on (2.5) $V_2\left(\overline{\Delta}\right)$ *ve bu* $\theta'_{12}, \theta'_{13}, \theta'_{23}$ to the place on (2.6);

it is

$$c = V_2\left(\overline{\Delta}\right) - \operatorname{arccos} h\left(\sqrt{\frac{3}{2}}\right) + \operatorname{arcsin} h\left(\sqrt{2}\right) \tag{2.7}$$

Theorem 5

$\theta_{12}, \theta_{13}, \theta_{23}$ are the interior angles of ${}_1\Delta_2^0$ triangle;

$$V_2\left({}_1\Delta_2^0\right) = \theta_{12} + \theta_{13} + \theta_{23} + V_2\left(\overline{\Delta}\right) - \operatorname{arccos} h\left(\sqrt{\frac{3}{2}}\right) + \operatorname{arcsin} h\left(\sqrt{2}\right)$$

Proof

It is seen by writing (2.7) as in $\varepsilon = 1$ form on the place (2.1)

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