Approximate Bayes Estimation for Log-Dagum Distribution

Caner TANIŞ[1*] ID *, Merve ÇOKBARLI*[2] ID *, Buğra SARAÇOĞLU*[1] ID

[1,2,3] Selçuk University Department of Statistics, Konya, TURKEY

Abstract. In this article, the approximate Bayes estimation problem for the log-Dagum distribution with three parameters is considered. Firstly, the maximum likelihood estimators and asymptotic confidence intervals based on these estimators for unknown parameters of log-Dagum distribution are constructed. In addition, approximate Bayes estimators under squared error loss function for unknown parameters of this distribution are obtained using Tierney and Kadane approximation. A Monte-Carlo simulation study is performed to compare performances of maximum likelihood and approximate Bayes estimators in terms of mean square errrors and biases. Finally, real data analysis for this distribution is performed.

Keywords: Log-Dagum Distribution Maximum Likelihood Estimation, Asymptotic Confidence Interval, Approximate Bayesian Estimation, Tierneyand Kadane Approximation.

Log-Dagum Dağılımı İçin Yaklaşık Bayes Tahmini

Özet. Bu makalede, log-Dagum dağılımı için yaklaşık Bayes tahmini problemi düşünüldü. İlk olarak, Log-Dagum dağılımının bilinmeyen parametreleri için en çok olabilirlik tahmin edicileri ve bu tahmin edicilere dayalı asimptotik güven aralıkları oluşturuldu. Ayrıca, bu dağılımın bilinmeyen parametreleri için karesel kayıp fonksiyonu altında yaklaşık Bayes tahmin edicileri Tierney and Kadane yaklaşımı kullanılarak elde edildi. Bu tahmin edicilerin performanslarını, hata kareler ortalaması ve yan bakımından karşılaştırmak için bir Monte-Carlo simülasyon çalışması gerçekleştirilmiştir. Son olarak bu dağılım için gerçek veri analizi gerçekleştirilmiştir.

Anahtar Kelimeler: Log-Dagum dağılımı, En çok olabilirlik tahmini, Asimptotik güven aralığı, Yaklaşık Bayes tahmini, Tierney and Kadane yaklaşımı.

1. INTRODUCTION

Statistical distributions are widely used for analysis of data in the real world. In literature, new statistical distributions have been obtained for modeling data in many areas such as science, engineering, medicine and economy. One of these statistical distributions is the dagum distribution suggested by Dagum [1,2] used for modelling wealth and income data . The cumulative distribution function (cdf) and probability density function (pdf) of a Y random variable having to Dagum distribution with parameters β, λ and δ are given by,

$$F_Y\left(y;\beta,\lambda,\delta\right) = \left(1 + \lambda y^{-\delta}\right)^{-\beta} \tag{1.1}$$

$$f_Y\left(y;\beta,\lambda,\delta\right) = \beta\lambda\delta y^{-\delta-1}\left(1 + \lambda y^{-\delta}\right)^{-\beta-1} \tag{1.2}$$

where $y > 0, \lambda > 0$, $\beta > 0$, $\delta > 0$. Domma [3] has introduced the log-Dagum (LDa) distribution by using logarithmic transformation, $X = \ln Y$, of a Y random variable having to Dagum distribution. The cdf and pdf of the log-Dagum (LDa) distribution with β, λ and δ parameters are

$$F_X\left(x;\beta,\lambda,\delta\right) = \left(1 + \lambda e^{-\delta x}\right)^{-\beta} \tag{1.3}$$

and

$$f_X\left(x;\beta,\lambda,\delta\right) = \beta\lambda\delta e^{-\delta x}\left(1 + \lambda e^{-\delta x}\right)^{-\beta-1} \tag{1.4}$$

respectively. Where $x \in \mathbb{R}$, $\beta > 0$, $\lambda > 0$ and $\delta > 0$. There are few studies about LDa distribution in literature. Domma [4] has proved that the kurtosis for log-dagum distribution depends only on parameter β. Domma and Perri [5] have examined some characteristic properties of this distribution and they have studied about maximum likelihood estimation (MLE) and asymptotic confidence interval for the unknown parameters of log-dagum distribution. The plots of pdf for various parameter values of $LDa\left(\beta,\lambda,\delta\right)$ distribution are given in Figure 1.
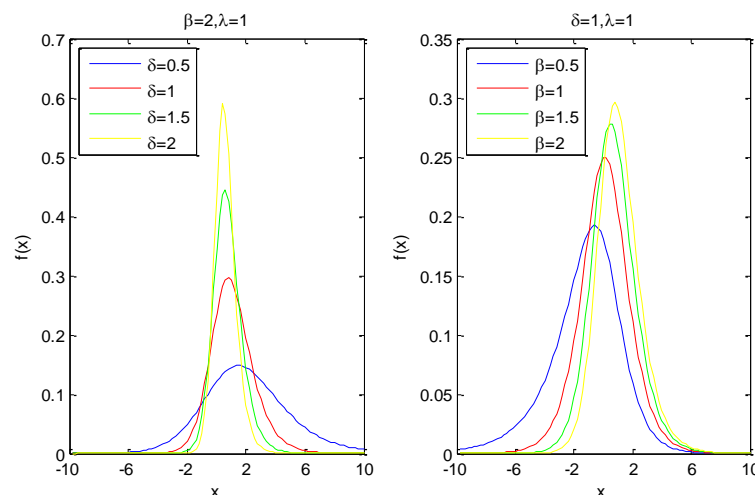

Figure 1. Density function plots of log-Dagum distribution for different parameter values

In this paper, we consider approximate Bayes estimation problem of unknown parameters $\left(\beta,\lambda,\delta\right)$ for the log-Dagum distribution. This study is organized as follows. In Section 2, maximum likelihood estimators (MLEs) for unknown parameters of the log-Dagum distribution and asymptotic confidence intervals based on these estimators are presented. In section 3, Bayes estimators with Tierney and Kadane approximation under squared loss function for unknown parameters of the log-Dagum distribution are obtained. In section 4, a Monte-Carlo simulation study is performed to compare maximum likelihood (ML) and approximate Bayes estimators in terms of mean square errors (MSEs) and biases. In addition, in this section, a simulation study based on asymptotic confidence intervals is carried out. A real data application is performed in section 5. In the last section, the conclusion of this study is given.

2. ML ESTIMATION and ASYMPTOTIC CONFIDENCE INTERVALS for LOG-DAGUM DISTRIBUTION

Let $\underline{X} = (X_1, X_2, ..., X_n)$ be a random sample with size n taken from $LDa(\beta, \lambda, \delta)$ distribution. In that case, the log-likelihood function is given by;

$$\ell(\beta, \delta, \lambda \mid \underline{x}) = n\log\beta + n\log\lambda + n\log\delta - \delta\sum_{i=1}^{n} x_i - (\beta+1)\sum_{i=1}^{n}\log\left(1 + \lambda e^{-\delta x_i}\right) \qquad (2.1)$$

In order to obtain ML estimators, the following likelihood equations should be solved.

$$\frac{\partial\ell(\beta, \delta, \lambda \mid \underline{x})}{\partial\beta} = \frac{n}{\beta} - \sum_{i=1}^{n}\log\left(1 + \lambda e^{-\delta x_i}\right) = 0 \qquad (2.2)$$

$$\frac{\partial\ell(\beta, \delta, \lambda \mid \underline{x})}{\partial\lambda} = \frac{n}{\lambda} - (\beta+1)\sum_{i=1}^{n}\frac{e^{-\delta x_i}}{1 + \lambda e^{-\delta x_i}} = 0 \qquad (2.3)$$

$$\frac{\partial\ell(\beta, \delta, \lambda \mid \underline{x})}{\partial\delta} = \frac{n}{\delta} - \sum_{i=1}^{n} x_i + (\beta+1)\sum_{i=1}^{n}\frac{\lambda x_i e^{-\delta x_i}}{1 + \lambda e^{-\delta x_i}} = 0 \qquad (2.4)$$

The solution of these non-linear equations can be obtained by using iteration methods such as Newton-Raphson method (Domma and Perri [5]).

Large-sample approach is used to obtain asymptotic confidence intervals for unknown parameters. Let $\hat{\Theta}$ is ML estimator of Θ and $I(\Theta), \Theta = (\beta, \lambda, \delta)$ is Fisher information matrix. In this case, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ and the Fisher information matrix are

$$\sqrt{n}(\hat{\Theta} - \Theta) \xrightarrow{d} N\left(0, I^{-1}(\Theta)\right)$$

$$I(\Theta) = \begin{pmatrix} -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\beta^2}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\beta\partial\lambda}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\beta\partial\delta}\right] \\[3mm] -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\lambda\partial\beta}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\lambda^2}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\lambda\partial\delta}\right] \\[3mm] -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\delta\partial\beta}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\delta\partial\lambda}\right] & -E\left[\dfrac{\ell^2(\beta, \lambda, \delta \mid \underline{x})}{\partial\delta^2}\right] \end{pmatrix}, \qquad (2.5)$$

respectively. The elements of fisher information matrix have been obtained by Domma and Perri [5]. The above approaches are used to find the approximate confidence intervals of β, λ and δ parameters. The $(1-\eta)100\%$ confidence intervals of the β, λ and δ parameters are obtained as in equations (2.6),(2.7) and (2.8).

$$P\left(\hat{\beta} - z_{\frac{\eta}{2}}\sqrt{Var(\hat{\beta})} < \beta < \hat{\beta} + z_{\frac{\eta}{2}}\sqrt{Var(\hat{\beta})}\right) = 1 - \eta \qquad (2.6)$$

$$P\left(\hat{\lambda} - z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\lambda}\right)} < \lambda < \hat{\lambda} + z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\lambda}\right)}\right) = 1 - \eta \tag{2.7}$$

$$P\left(\hat{\delta} - z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\delta}\right)} < \delta < \hat{\delta} + z_{\frac{\eta}{2}}\sqrt{Var\left(\hat{\delta}\right)}\right) = 1 - \eta \tag{2.8}$$

where diagonal elements of inverse of Fisher information matrix are variances of $\hat{\beta}$, $\hat{\lambda}$ and $\hat{\delta}$ (Domma and Perri [5]).

3. BAYES ESTIMATION for PARAMETERS of LOG-DAGUM DISTRIBUTION

Let $X_1, X_2, ..., X_n$ be a random sample with size n taken from $LDa(\beta, \lambda, \delta)$ distribution. It is needed to prior distributions for these parameters to obtain Bayesian estimation of parameters. In this study, it is taken as following gamma priors for unknown β, λ and δ parameters.

$$\pi(\beta) \propto \beta^{d_1-1} e^{-\beta e_1} \quad , \beta, e_1, d_1 > 0 \tag{2.9}$$

$$\pi(\lambda) \propto \lambda^{d_2-1} e^{-\lambda e_2} \quad , \lambda, e_2, d_2 > 0 \tag{2.10}$$

$$\pi(\delta) \propto \delta^{d_3-1} e^{-\delta e_3} \quad , \delta, e_3, d_3 > 0 \tag{2.11}$$

The joint priors and posterior distributions of β, λ and δ parameters are,

$$\pi(\beta, \lambda, \delta) = \pi(\beta)\pi(\lambda)\pi(\delta) \propto \beta^{d_1-1}\lambda^{d_2-1}\delta^{d_3-1}e^{-(\beta e_1 + \lambda e_2 + \delta e_3)}$$

$$\begin{aligned}\pi(\beta, \lambda, \delta \mid x) &= \frac{f(x \mid \beta, \lambda, \delta)\pi(\beta, \lambda, \delta)}{f_x(x)} \\ &= \frac{k(x_i; \beta, \lambda, \delta)\pi(\beta, \lambda, \delta)}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty k(x_i; \beta, \lambda, \delta)\pi(\beta, \lambda, \delta)d\beta d\lambda d\delta}, \end{aligned} \tag{2.12}$$

respectively. Where $k(x_i; \beta, \lambda, \delta) = (\beta, \lambda, \delta)^n \exp\left(-\delta\sum\limits_{i=1}^n x_i\right)\prod\limits_{i=1}^n\left(1 + \lambda\exp(-\delta x_i)\right)^{-\beta-1}$. In this case, Bayes estimator for any function of β, λ and $\delta, u(\beta, \lambda, \delta)$, under squared loss function is as follows.

$$\begin{aligned}\hat{u}_B(\beta, \lambda, \delta) &= E\left[u(\beta, \lambda, \delta) \mid x\right] \\ &= \frac{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty u(\beta, \lambda, \delta \mid x)e^{\left[\ell(\beta, \lambda, \delta \mid x) + \rho(\beta, \lambda, \delta)\right]}d\beta d\lambda d\delta}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty e^{\left[\ell(\beta, \lambda, \delta \mid x) + \rho(\beta, \lambda, \delta)\right]}d\beta d\lambda d\delta} \end{aligned} \tag{2.13}$$

Where $\ell(\beta, \lambda, \delta \mid x)$ is log-likelihood function, $\rho(\beta, \lambda, \delta)$ is logarithm of joint prior distribution. It is very difficult to the obtain solution of above Eq. (2.13) in closed form. Some approximate methods for solution of this equation are used. One of these methods is Tierney Kadane's approximation.

a. Bayes Estimation with Tierney and Kadane's Method

Tierney and Kadane's approximation introduced by Tierney and Kadane [6] to compute integral ratios in bayes analysis has been studied by many authors such as Gencer and Saraçoğlu [7], Howloader and Hossain [8], Mousa and Jaheen [9], Kınacı et al. [10], Tanış and Saraçoğlu [11]. Tierney and Kadane approximation can be summarized as follows.

$$I(\beta,\lambda,\delta) = \frac{1}{n}\left\{\rho(\beta,\lambda,\delta) + \ell(\beta,\lambda,\delta \mid x)\right\} \tag{2.14}$$

$$I^*(\beta,\lambda,\delta) = \frac{1}{n}\log u(\beta,\lambda,\delta) + I(\beta,\lambda,\delta) \tag{2.15}$$

Where, $\rho(\beta,\lambda,\delta)$ is defined as follows.

$$\rho(\beta,\lambda,\delta) = (d_1-1)\log(\beta) + (d_2-1)\log(\lambda) + (d_3-1)\log(\delta) - (\beta e_1 + \lambda e_2 + \delta e_3) \tag{2.16}$$

Bayes estimators with Tierney and Kadane approximation of $u(\beta,\lambda,\delta)$ under squared error loss function for $LDa(\beta,\lambda,\delta)$ distribution is obtained as follows

$$\hat{u}_b(\beta,\lambda,\delta) = E\left[u(\beta,\lambda,\delta) \mid x\right] = \frac{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty e^{nI^*(\beta,\lambda,\delta)}d\beta d\lambda d\delta}{\int\limits_0^\infty\int\limits_0^\infty\int\limits_0^\infty e^{nI(\beta,\lambda,\delta)}d\beta d\lambda d\delta} \tag{2.17}$$

$$= \left[\left(\frac{\det\Sigma^*}{\det\Sigma}\right)^{1/2}\exp\left[n\left(I^*\left(\hat{\beta}_{I^*},\hat{\lambda}_{I^*},\hat{\delta}_{I^*}\right) - I\left(\hat{\beta}_I,\hat{\lambda}_I,\hat{\delta}_I\right)\right)\right]\right]$$

where $\left(\hat{\beta}_{I^*},\hat{\lambda}_{I^*},\hat{\delta}_{I^*}\right)$ and $\left(\hat{\beta}_I,\hat{\lambda}_I,\hat{\delta}_I\right)$ maximize $I^*\left(\hat{\beta}_{I^*},\hat{\lambda}_{I^*},\hat{\delta}_{I^*}\right)$ and $I\left(\hat{\beta}_I,\hat{\lambda}_I,\hat{\delta}_I\right)$, respectively. Σ^* and Σ are minus the inverse Hessians of $I^*\left(\hat{\beta}_{I^*},\hat{\lambda}_{I^*},\hat{\delta}_{I^*}\right)$ and $I\left(\hat{\beta}_I,\hat{\lambda}_I,\hat{\delta}_I\right)$ at $\left(\hat{\beta}_{I^*},\hat{\lambda}_{I^*},\hat{\delta}_{I^*}\right)$ and $\left(\hat{\beta}_I,\hat{\lambda}_I,\hat{\delta}_I\right)$, respectively.

4. SIMULATION STUDY

In this section, a Monte-Carlo simulation study in order to compare the performances of ML estimators and aproximate bayesian estimators according to MSEs and biases for $LDa(\beta,\lambda,\delta)$ distribution is performed. In addition, in this section, a simulation study based on coverage probabilities (cp) and lengths of asymptotic confidence intervals based on ML estimators is carried out. Firstly, it is needed to generate random samples from $LDa(\beta,\lambda,\delta)$ distribution for simulation study.

4.1. Random Sample Generation

Inverse conversion method in order to generate random number from $LDa(\beta,\lambda,\delta)$ distribution is used. Let u state a random number generated from $Uniform(0,1)$. x generated from $LDa(\beta,\lambda,\delta)$ distribution with inverse conversion method is given as follows.

$$x = \frac{1}{\delta_0}\ln\left(\frac{\lambda_0}{u^{-1/\beta_0}-1}\right) \tag{3.1}$$

where β_0, λ_0 and δ_0 are initial values. (Domma and Perri [7]).

In simulation study, it is generated $N = 5000$ samples of sizes $n = 100, 200, 500, 1000$ from $LDa(\beta, \lambda, \delta)$ distribution with $(\beta_0 = 0.43, \lambda_0 = 0.2, \delta_0 = 0.5)$, $(\beta_0 = 0.8, \lambda_0 = 0.15, \delta_0 = 0.3)$ and $(\beta_0 = 0.5, \lambda_0 = 0.1, \delta_0 = 0.7)$. The biases and MSEs of ML and approximate bayes estimators for unknown parameters at different samples sizes as $n = 100, 200, 500, 1000$ are given in Table1. In this table, prior values for approximate bayes estimators are $d_1 = 0.01, e_1 = 0.01$, $d_2 = 0.01, e_2 = 0.01, d_3 = 0.01, e_3 = 0.01$. The results of asymptotic confidence intervals based on ML estimators for unknown parameters of $LDa(\beta, \lambda, \delta)$ distribution for different samples sizes as $n = 100, 200, 500, 1000$ are presented in Table 2.

Table 1. Biases and MSEs of MLE and Bayes estimators for $LDa(\beta, \lambda, \delta)$

$(\beta_0, \lambda_0, \delta_0)$	N	$\hat{\beta}_{MLE}$ bias	mse	$\hat{\beta}_{BAYES}$ bias	mse	$\hat{\lambda}_{MLE}$ bias	mse	$\hat{\lambda}_{BAYES}$ bias	mse	$\hat{\delta}_{MLE}$ bias	mse	$\hat{\delta}_{BAYES}$ bias	mse
(0.43, 0.2, 0.5)	100	-0.0261	0.0261	-0.0417	0.0352	-0.0018	0.0066	-0.0236	0.0105	-0.0247	0.0137	-0.0487	0.0201
	200	-0.0110	0.0102	-0.0154	0.0111	-0.0005	0.0027	-0.0101	0.0033	-0.0120	0.0053	-0.0225	0.0062
	500	-0.0032	0.0036	-0.0044	0.0037	-0.0005	0.0010	-0.0041	0.0011	-0.0052	0.0019	-0.0093	0.0020
	1000	-0.0020	0.0017	-0.0026	0.0017	-0.0001	0.0005	-0.0018	0.0005	-0.0022	0.0009	-0.0042	0.0009
(0.8, 0.15, 0.3)	100	-0.1117	2.5370	-0.3782	2.8199	0.0010	0.0037	-0.0115	0.0051	-0.0092	0.0031	-0.0159	0.0038
	200	-0.0425	0.0556	-0.0758	0.0816	0.0005	0.0016	-0.0053	0.0019	-0.0046	0.0013	-0.0079	0.0015
	500	-0.0135	0.0164	-0.0227	0.0178	-0.0001	0.0006	-0.0023	0.0007	-0.0020	0.0005	-0.0033	0.0005
	1000	-0.0076	0.0077	-0.0118	0.0080	0.0001	0.0003	-0.0010	0.0003	-0.0008	0.0002	-0.0015	0.0002
(0.5, 0.1, 0.7)	100	-0.0366	0.0392	-0.0645	0.0824	0.0026	0.0013	-0.0039	0.0016	-0.0294	0.0227	-0.0594	0.0327
	200	-0.0153	0.0146	-0.0224	0.0164	0.0013	0.0006	-0.0018	0.0007	-0.0148	0.0094	-0.0278	0.0108
	500	-0.0045	0.0050	-0.0067	0.0052	0.0004	0.0002	-0.0008	0.0002	-0.0066	0.0034	-0.0115	0.0036
	1000	-0.0028	0.0024	-0.0038	0.0024	0.0003	0.0001	-0.0003	0.0001	-0.0028	0.0016	-0.0052	0.0017

Table 2. Length and cp based on MLE for $LDa(\beta, \lambda, \delta)$

$(\beta_0, \lambda_0, \delta_0)$	n	$\hat{\beta}_{MLE}$ cp	length	$\hat{\lambda}_{MLE}$ cp	length	$\hat{\delta}_{MLE}$ cp	length
(0.43, 0.2, 0.5)	100	0.9298	0.5739	0.8994	0.2926	0.9456	0.4061
	200	0.9446	0.3788	0.9232	0.1982	0.9502	0.2704
	500	0.9470	0.2313	0.9388	0.1232	0.9490	0.1662
	1000	0.9512	0.1623	0.9444	0.0864	0.9514	0.1161
(0.8, 0.15, 0.3)	100	0.9310	1.4932	0.8988	0.2216	0.9460	0.2012
	200	0.9422	0.8406	0.9260	0.1524	0.9448	0.1370
	500	0.9488	0.4928	0.9398	0.0952	0.9538	0.0850
	1000	0.9508	0.3424	0.9460	0.0669	0.9518	0.0596
(0.5, 0.1, 0.7)	100	0.9340	0.6945	0.8974	0.1346	0.9480	0.5367
	200	0.9418	0.4516	0.9202	0.0942	0.9502	0.3610
	500	0.9494	0.2740	0.9380	0.0593	0.9500	0.2227
	1000	0.9506	0.1920	0.9420	0.0418	0.9506	0.1557

According to results of simulation study, it is seen that MSEs and biases values for ML and approximate bayes estimators of parameters are decreases when the number of samples increases. Furthermore, as

sample sizes increases, it is observed that cp approaches to 0.95 and the length of the asymptotic confidence interval decreases as expected.

5. REAL DATA APPLICATION

The data set consist of 76 observations about the life of fatigue fracture of Kevlar 373/epoxy which is considered in this section. These data are obtained by subject to constant pressure at the 90% stress level until all fatigue fracture had failed. (Kharazmi and Saatınık, [12]). This data set have been studied Andrews and Herzberg [13], Barlow et al. [14] and Merovci et. al. [15]. Let $x \in \mathbb{R}^+$ express data, we consider a transformation with $y = \ln(x)$ on Kevlar 373/epoxy data set. Thus, it is obtained $y \in \mathbb{R}$ data. Then, new data after transformation is given in Table 3. This data set has been analyzed to compare the log-Dagum distribution with other distributions such as, Normal, Logistic, Laplace, t location-Scale, Extreme Value and Generalized Extreme Value (GEV). Probability density functions of these distributions given by;

$$Normal: f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(\frac{-(x-\mu)^2}{2\sigma^2}\right), \ \sigma > 0, \ \mu, x \in \mathbb{R}$$

$$Logistic: f(x) = \frac{\exp\left(\frac{x-\mu}{\sigma}\right)}{\sigma\left(1+\exp\left(\frac{x-\mu}{\sigma}\right)\right)^2}, \ \sigma > 0, \ x, \mu \in \mathbb{R}$$

$$Laplace: f(x) = \frac{1}{2\sigma} \exp\left(-\frac{|x-\mu|}{\sigma}\right), \ \sigma > 0, \ x, \mu \in \mathbb{R}$$

$$t \ location\text{-}Scale: f(x) = \frac{\Gamma\left(\frac{\upsilon+1}{2}\right)}{\sigma\sqrt{\upsilon\pi}\Gamma\left(\frac{\upsilon}{2}\right)} \left[\frac{\upsilon+\left(\frac{x-\mu}{\sigma}\right)^2}{\upsilon}\right]^{-\left(\frac{\upsilon+1}{2}\right)}, \ \sigma, \upsilon > 0, \ \mu, x \in \mathbb{R}$$

$$Extreme \ Value: f(x) = \sigma^{-1}\exp\left(\frac{x-\mu}{\sigma}\right)\exp\left(-\exp\left(\frac{x-\mu}{\sigma}\right)\right), \ \sigma > 0, \ \mu, x \in \mathbb{R}$$

$$GEV: f(x) = \sigma^{-1}\exp\left(-\left(1+k\frac{(x-\mu)}{\sigma}\right)^{\frac{-1}{k}}\right)\left(1+k\frac{(x-\mu)}{\sigma}\right)^{-1-\frac{1}{k}}, \ \sigma > 0, \ k, \mu, x \in \mathbb{R}$$

Table 3. Kevlar 373/epoxy data set

-3.6849	-2.4236	-2.4180	-1.3859	-1.1670	-1.0639	-0.7417	-0.5709
-0.5672	-0.4207	-0.3933	-0.3929	-0.3926	-0.2619	-0.1773	-0.1754
-0.1714	-0.1456	-0.1221	-0.0929	-0.0921	-0.0165	0.0472	0.0579
0.0745	0.1598	0.2287	0.2442	0.2612	0.2785	0.3003	0.3039
0.3781	0.3974	0.4529	0.4532	0.5355	0.5460	0.5573	0.5670
0.5736	0.6029	0.6084	0.6153	0.6317	0.6354	0.6356	0.6583
0.6708	0.6955	0.7133	0.7373	0.7464	0.7575	0.7930	0.8092
0.8276	0.8417	0.8531	0.8550	0.9143	0.9266	1.0956	1.1071
1.1841	1.2251	1.2484	1.3200	1.3206	1.3646	1.5701	1.6865
1.6944	1.7101	1.8801	2.2078				

MLEs and their standard errors, AIC values for seven distributions are given In Table 5. Moreover, plots fitted to cdfs, reliability functions and pdfs are presented in Figure 4-6.

Table 4. Parameter estimates (standard errors) and AIC values for Kevlar 373/epoxy data set

Distribution	ML Estimate	-2ℓ	AIC
Log-Dagum	$\hat{\beta} = 0.4248(0.1275)$, $\hat{\lambda} = 17.7962(16.7845)$, $\hat{\delta} = 3.1766(0.5964)$	**187.0024**	**193.0024**
Normal	$\hat{\mu} = 0.3379(0.1102)$, $\hat{\sigma} = 0.9610(0.0787)$	206.64	210.64
Logistic	$\hat{\mu} = 0.4244(0.0938)$, $\hat{\sigma} = 0.4810(0.0469)$	195.7154	199.1792
Laplace	$\hat{\mu} = 0.5516(0.0754)$, $\hat{\sigma} = 0.6578(0.0754)$	191.7154	195.7154
t location-Scale	$\hat{\mu} = 0.4675(0.0860)$, $\hat{\sigma} = 0.5984(0.0919)$, $\hat{\upsilon} = 2.9338(1.1357)$	189.7942	195.7942
Extreme Value	$\hat{\mu} = 0.7574(0.0911)$, $\hat{\sigma} = 0.7543(0.0647)$	191.6836	195.6836
Generalized Extreme Value	$\hat{\mu} = 0.0888(0.1271)$, $\hat{\sigma} = 1.0401(0.0873)$, $\hat{k} = -0.4695(0.0475)$	187.7040	203.7040

Also, the approximate Bayes estimation values of the unknown parameters of Log-Dagum distribution are obtained as $\hat{\beta}_{BAYES} = 0.4613$, $\hat{\lambda}_{BAYES} = 23.2287$, $\hat{\delta}_{BAYES} = 3.1277$ with prior gamma distribution $(d_1 = 0.01, e_1 = 0.01, d_2 = 0.01, e_2 = 0.01, d_3 = 0.01, e_3 = 0.01)$.
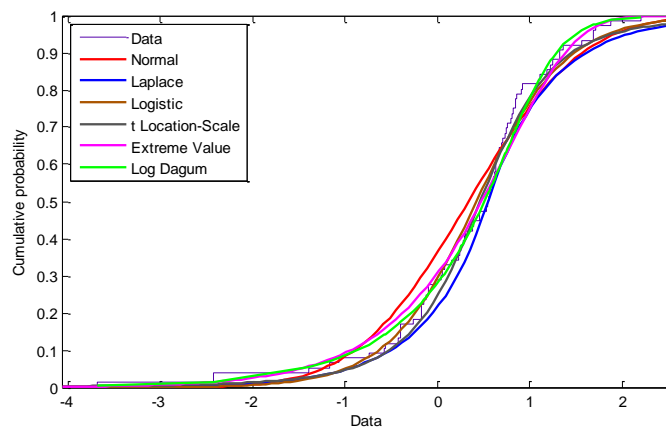

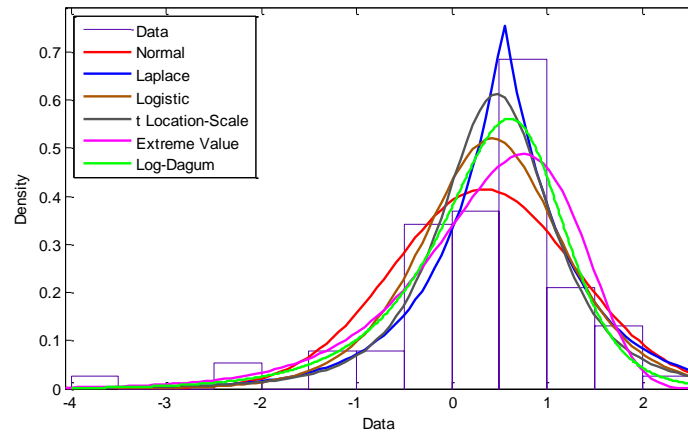**Figure 2**. Fitted cdfs plots for Kevlar 373/epoxy data set

Figure 3. Fitted pdfs plots for Kevlar 373/epoxy data set

6. CONCLUSION

We have analyzed the $LDa(\beta,\lambda,\delta)$ distribution in terms of estimation of unknown parameters. The approximate Bayesian estimators for unknown parameters of this distribution are obtained. The Bayesian estimators under squared error loss function are found using Tierney and Kadane approximation. The performances of ML and approximate Bayes estimators have been compared with the Monte Carlo simulation study according to MSE and bias criteria. A simulation study based on asymptotic confidence intervals is performed. It is seen that the biases and MSEs of ML and Bayes estimators decrease as sample size increases. It can be concluded that biases and MSEs of these two estimators are very close to each other. In interval estimation based on ML estimators of unknown parameters for the $LDa(\beta,\lambda,\delta)$ distribution, it is seen that coverage probabilities (cp) approach to 0.95 and length of asymptotic confidence intervals decreases as sample size increases. Furthermore, a real data application is performed in order to show that the $LDa(\beta,\lambda,\delta)$ distribution can be used in new areas. It is presented a real data set related to the life of fatigue fracture of Kevlar 373/epoxy. We have concluded that the $LDa(\beta,\lambda,\delta)$ distribution has to best fit between other six distributions (Normal, Logistic, Laplace, t location-Scale, Extreme Value, Generalized Extreme Value) according to AIC and -2ℓ .

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