Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-2(2019) 447-457

Some φ-Fixed Point Results in b−Metric Spaces and Applications

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Received: 08.05.2018; Accepted: 21.02.2019 *http://dx.doi.org/10.17776/csj.421975*

Abstract. The purpose of this study is to introduce the existence and uniqueness of φ-fixed point for some new contractions in complete b-metric spaces. Firstly, in this paper, we presented new definitions called $(F,\alpha,\varphi,\theta)_s$ and $(F,\alpha,\varphi,\theta)_s$ -weak contractions in complete b-metric spaces as a generalization of metric spaces. Later, we proved φ-fixed point theorems for $(F,\alpha,\varphi,\theta)_s$ and $(F,\alpha,\varphi,\theta)_s$-weak contractions in complete b-metric spaces. As applications, we derived some fixed point results in complete partial b-metric spaces as a generalization of partial metric spaces. The presented theorems extend and generalize some φ-fixed point results which are known in the literature. Also, some results in this paper generalizes many existing some fixed point results in the literature.

Keywords: b-metric space, φ-fixed point, $(F,\alpha,\varphi,\theta)_s$-contraction.

b-Metrik Uzaylarda φ-Sabit Nokta Teoremleri ve Uygulamaları

Özet. Bu çalışmanın amacı, b-metrik uzaylarda bazı yeni büzülmelerin φ-sabit noktalarının varlığını ve tekliğini göstermektir. Öncelikle, bu çalışmada, b-metrik uzaylarda $(F,\alpha,\varphi,\theta)_s$ ve $(F,\alpha,\varphi,\theta)_s$ –zayıf büzülme isimli iki tanım verilmiştir. Sonra, b-metrik uzaylarda bu tanımlar için φ-sabit nokta teoremleri ispatlanmıştır. Uygulama olarak, kısmi metrik uzayların genelleştirmesi olan tam kısmi b-metrik uzaylarda bazı sabit nokta sonuçları verilmiştir. Bu çalışmada elde edilen teoremler, literatürde bilinen φ-sabit nokta sonuçlarından daha genel ve geniş olduğu gibi, literatürde var olan bazı sabit nokta sonuçlarından da daha geneldir.

Anahtar Kelimeler: b-metrik uzay, φ-sabit nokta, $(F,\alpha,\varphi,\theta)_s$-büzülme.

1. INTRODUCTION

The Banach contraction principle is one of the most important subjects in mathematics. By using this principle, most authors have proved several fixed point theorems for various mappings in several metric spaces [1-3,5,6,8-11,16-23]. Bakhtin [12] and Czerwik [21] introduced b-metric spaces as a generalization of metric spaces and proved the contraction mapping principle in b-metric spaces that is an extension of the Banach contraction principle in metric spaces. Since then, a number of authors have investigated fixed point theorems in b-metric spaces [13, 14, 24].

On the other hand, Jleli, Samet and Vetro [13] introduced the concept of φ-fixed point and established some existence results of φ-fixed points for various classes of operators in metric spaces. Samet, Vetro C. and Vetro P. [4] introduced the notion of α−admissible mapping in metric spaces.

Later, Sintunavarat [27] introduced the concepts of α-admissible mapping type S, as some generalizations of α-admissible mapping and then he proved some fixed point theorems by using his new types of α-admissibility mapping in b-metric spaces.

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In this paper, we introduced some new mappings satisfying $(F,\alpha,\varphi,\theta)_s$-contraction and $(F,\alpha,\varphi,\theta)_s$-weak contraction and proved some new φ-fixed point theorems in b-complete metric spaces. The presented theorems extend and generalize the φ-fixed point results. As applications of the obtained results, we presented some fixed point theorems in partial b-metric spaces are derived from our main theorems.

2. PRELIMINARIES

Definition 1. [21] Let X be a nonempty set and s ≥ 1 a real number. A mapping $d_b: X \times X \to [0,\infty)$ is called a b-metric if for all $x,y,z \in X$, the following conditions are satisfied:

(i) $d_b(x,y) = 0$ if and only if $x = y$,

(ii) $d_b(x,y) = d_b(y,x)$,

(iii) $d_b(x,z) \leq s[d_b(x,y) + d_b(y,z)]$.

In this case, (X,d_b) is called a b-metric space.

Definition 2. [7] A sequence $\{x_n\}$ in a b-metric space (X,d_b) is said to be:

(i) b-convergent to a point $x \in X$ if $\lim_{n\to\infty} d_b(x_n,x) = 0$.

(ii) A sequence $\{x_n\}$ in a b-metric space (X,d_b) is called a Cauchy sequence if $\lim_{n,m\to\infty} d_b(x_n,x_m) = 0$.

(iii) A b-metric space (X,d_b) is called complete if every Cauchy sequence $\{x_n\}$ in X b-converges to a point $x \in X$.

(iv) A function $f : X \to Y$ is b-continuous at a point $x \in X$ if $\{x_n\} \subset X$ b-converges to x, then $\{fx_n\} \subset Y$ b-converges to fx, where (Y,ρ) is a b-metric space.

Definition 3. [27] Let X be a nonempty set and $s \geq 1$ a given real number. Let $\alpha : X \times X \to [0,\infty)$ and $T : X \to X$ be mappings. We say T is an α-admissible mapping type S if for all $x,y \in X, \alpha(x,y) \geq s$ leads to $\alpha(Tx,Ty) \geq s$. In particular, T is called α-admissible mapping if s = 1.

Definition 4. [26] Let $s \geq 1$ be a real number. A mapping $\phi : [0,\infty) \to [0,\infty)$ is called a (b)-comparison function if

(b1) ϕ is monotone increasing,

(b2) there exists $p_0 \in N, a \in (0,1)$ and a convergent series of nonnegative terms $\sum_{p=1}^{\infty} v_p$ such that $b^{p+1}\phi^{p+1}(t) \leq ab^{p+1}\phi^p(t) + v_p$, for $p \geq p_0$ and any $t \in [0,\infty)$.

Lemma 5. [25] If $\phi : [0,\infty) \to [0,\infty)$ is a (b)-comparison function, then;

(1) the series $\sum_{p=0}^{\infty} b^p\phi^p(t)$ converges for any $t \in R^+$,

(2) the function $s_b: [0,\infty) \to [0,\infty)$ defined by $s_b(t) = \sum_{p=0}^{\infty} b^p\phi^p(t), t \in [0,\infty)$, is increasing and continuous at 0.

Lemma 6. [15] Let $\phi : [0,\infty) \to [0,\infty)$ be (b)-comparison function with constant $s \geq 1$ and $a_n \in R^+, n \in N$ such that $a_n \to 0, as\ n \to \infty$ then $\sum_{p=0}^{n} s_{n-p}\phi^{n-p}(t) \to 0, as\ n \to \infty$.

Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function and $T : X \to X$ be an operator. The set of all fixed points of the operator T will be denoted by

$$F_T = \{x \in X : Tx = x\}.$$

The set all zeros of the function φ will be denoted by

$$Z_\varphi = \{x \in X : \varphi(x) = 0\}.$$

Definition 7. [13] An element $z \in X$ is said to be a φ-fixed point of the operator T if and only if $z \in F_T \cap Z_\varphi$.

Definition 8. [13] T is a φ-Picard operator if and only if

(i) $F_T \cap Z_\varphi = \{z\}$,

(ii) $x_n \to z$ as $n \to \infty$, for all $n \in N$.

Definition 9. [13] T is a weakly φ-Picard operator if and only if

(i) $F_T \cap Z_\varphi = \emptyset$,

(ii) the sequence $\{x_n\}$ converges for each $x \in X$ and the limit is a φ-fixed point of the operator T.

We denote by $\mathcal{F}$ the set of functions $F : [0, \infty)^3 \to [0, \infty)$ satisfying the following conditions:

(F1) $max\{a, b\} \leq F(a, b, c)$ for all $a, b, c \in [0, \infty)$,

(F2) $F(0,0,0) = 0$,

(F3) F is continuous.

The following functions are given as examples:

(i) $F(a, b, c) = a + b + c$,

(ii) $F(a, b, c) = max\{a, b\} + c$,

(iii) $F(a, b, c) = a + a^2 + b + c$.

Definition 10. [13] Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. The operator $T : X \to X$ is an(F, φ)- contraction if and only if for $x, y \in X$

$$F\big(d(Tx, Ty), \varphi(Tx)\varphi(Ty)\big) \leq kF\big(d(x, y), \varphi(x), \varphi(y)\big)$$

for some constant $k \in (0,1)$.

Definition 11. [13] Let (X, d) be a metric space, $\varphi : X \to [0, \infty)$ be a given function and $F \in \mathcal{F}$. The operator $T: X \to X$ is an (F, φ)-weak contraction if and only if for $x, y \in X$

$$F\big(d(Tx, Ty), \varphi(Tx)\varphi(Ty)\big) \leq kF(d(x, y), \varphi(x), \varphi(y))$$

$$+L(F(d(y, Tx), \varphi(y), \varphi(Tx)) - F(0, \varphi(y), \varphi(Tx))).$$

for some constant $k \in (0,1)$ and $L \geq 0$.

3. MAIN RESULTS

In this work, we use J_b to denote the class of all (b)-comparison functions
$\theta : [0,\infty) \to [0,\infty)$ such that $\theta(t) < t$ for all $t > 0$ unless and until it is stated otherwise.

Definition 12. Let (X, d_b) be a b-metric space with coefficient $s \geq 1$, $\alpha : X \times X \to [0,\infty)$ be a mapping and $\varphi : X \to [0,\infty)$ be lower semi continuous function, $\theta \in J_b$ and $\varepsilon > 1$. A mapping $T: X \to X$ is said to be an $(F, \alpha, \varphi, \theta)_s$-contraction mapping if

$x, y \in X$ with $\alpha(x,y) \geq s \Rightarrow$

$$s^{\varepsilon} F\big(d_b(Tx,Ty), \varphi(Tx)\varphi(Ty)\big) \leq \theta\Big(F\big(d_b(x,y), \varphi(x), \varphi(y)\big)\Big). \tag{3.1}$$

Theorem 13. Let (X, d_b) be a complete b-metric space with coefficient s ≥ 1 and $T: X \to X$ be α-admissible mapping type S. Suppose that the following conditions hold:

(1) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq s$,

(2) T is an $(F, \alpha, \varphi, \theta)_s$- contraction mapping,

(3) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq s$ and $x_n \to x$ then $\alpha(x_n, x) \geq s$ for all $n \in N$.

Then

(i) $F_T \subseteq Z_{\varphi}$,

(ii) T is φ- Picard operator. Moreover, if $\alpha(x,y) \geq s$ for all $x, y \in F_T$, then T has a unique φ-fixed point.

Proof. (i) Assume that $\xi \in X$ is a fixed point of T such that $\alpha(\xi,\xi) \geq s$. Applying (3.1) with $x = y = \xi$, we obtain

$$\begin{aligned} F(0,\varphi(\xi),\varphi(\xi)) &\leq s^{\varepsilon} F(0,\varphi(\xi),\varphi(\xi)) \\ &\leq \theta\Big(F\big(0,\varphi(\xi),\varphi(\xi)\big)\Big). \end{aligned} \tag{3.2}$$

Then we get

$$F\big(0,\varphi(\xi),\varphi(\xi)\big) \leq s^{\varepsilon} F\big(0,\varphi(\xi),\varphi(\xi)\big) \leq \theta(F(0,\varphi(\xi),\varphi(\xi)))$$

then we have

$$F(0,\varphi(\xi),\varphi(\xi)) \leq \theta\Big(F\big(0,\varphi(\xi),\varphi(\xi)\big)\Big).$$

From the property of θ, we have

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \tag{3.3}$$

On the other hand, from (F1), we have

$$\phi(\xi) \leq F\big(0,\varphi(\xi),\varphi(\xi)\big). \tag{3.4}$$

From (3.3) and (3.4), we obtain $\varphi(\xi) = 0$, which proves (i).

(ii) Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq s$. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in N$. Since T is an α-admissible mapping and $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq s$, we deduce that

$\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq s$. Continuing this process, we get $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in N \cup \{0\}$. If $x_n = x_{n+1}$, for some $n \in N$, then $x_n = Tx_n$. Thus, x_n is a fixed point of T.

Therefore, we assume that $x_n \neq x_{n+1}$, for all $n \in N$. Using condition (1) as $\alpha(x_{n-1}, x_n) \geq s$ for all

$n \in N$, we obtain

$$\begin{aligned} & F(d_b(Tx_{n-1}, Tx_n), \varphi(Tx_{n-1}), \varphi(Tx_n)) \\ & \leq s^{\varepsilon} F(d_b(Tx_{n-1}, Tx_n), \varphi(Tx_{n-1}), \varphi(Tx_n)) \\ & \leq \theta(F(d_b(x_{n-1}, x_n), \varphi(x_{n-1}), \varphi(x_n))) \\ & \leq \theta^n(F(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1))). \end{aligned} \tag{3.5}$$

Then from (F1), we have

$$max\{d_b(x_n, x_{n+1}), \varphi(x_n)\} \leq \theta^n\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) \tag{3.6}$$

which implies

$$d_b(x_n, x_{n+1}) \leq \theta^n\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right). \tag{3.7}$$

Now we show that $\{x_n\}$ is a Cauchy sequence. Suppose that $k \in N$ such that $k > 0$. By using the triangle inequality, we get

$$\begin{aligned} d_b(x_n, x_{n+k}) & \leq s d_b(x_n, x_{n+1}) + s^2 d_b(x_{n+1}, x_{n+2}) + \dots + s^k d_b(x_{n+k-1}, x_{n+k})) \\ & \leq s\theta^n\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) + s^2\theta^{n+1}\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) + \cdots \\ & \quad + s^k\theta^{n+k-1}\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) \\ & = \frac{1}{s^{n-1}}[s^n\theta^n\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) + s^{n+1}\theta^{n+1}\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right) + \\ & \quad \dots + s^{n+k-1}\theta^{n+k-1}F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right))]. \end{aligned}$$

We denote $S_n = \sum_{p=0}^{n} s^p\theta^p\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right)$ for $n \geq 1$, then we get

$$d_b(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}}[S_{n+k-1} - S_{n-1}], \; n \geq 1, \; k \geq 1.$$

From Lemma 5, we have $\sum_{p=1}^{n} s^p\theta^p\left(F\left(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1)\right)\right)$ is convergent. Hence, there exists S= $\lim_{n\to\infty} S_n$ and from above the inequality, it implies that $\{x_n\}$ is a Cauchy sequence. Since (X, d_b) is complete, then the sequence $\{x_n\}$ converges some $z \in X$ and

$$\lim_{n\to\infty} d_b(x_n, z) = 0. \tag{3.8}$$

Now, we shall prove that z is a φ-fixed point of T. Observe that from (3.6), we have

$$\lim_{n\to\infty} \varphi(x_n) = 0. \quad (3.9)$$

Since φ is lower semi continuous, from (3.8) and (3.9) we obtain

$$\varphi(z) = 0. \quad (3.10)$$

Using (3.1) and from condition (3), we have

$$s^{\varepsilon} F(d_b(x_{n+1}, Tz), \varphi(x_{n+1}), \varphi(z)) \leq \theta(F(d_b(x_n, z), \varphi(x_n), \varphi(z))). \quad (3.11)$$

Letting $n \to \infty$ in 3.11, using 3.8, 3.9, 3.10, (F2) and the continuity of F, we have

$$s^{\varepsilon} F(\lim_{n\to\infty} d_b(x_{n+1}, Tz), 0, \varphi(z)) \leq \theta(F(0,0,0)) = 0$$

which implies from condition (F1) that

$$\lim_{n\to\infty} d_b(x_{n+1}, Tz) = 0. \quad (3.12)$$

On the other hand, from the condition (iii) of definition b-metric space, we have

$$d_b(z, Tz) \leq s[d_b(z, x_{n+1}) + d_b(x_{n+1}, Tz)].$$

Taking the limit as $n \to \infty$ in above the inequality, using (3.8) and (3.12), we get $d_b(z, Tz) = 0$, that is, $Tz = z$. Hence z is a φ-fixed point of T. Now we show that z is the unique φ-fixed point of T. Assume that $w \in X$ is another φ-fixed point of T. From (3.1), we have

$$s^{\varepsilon} F(d_b(Tz, Tw), \varphi(Tz), \varphi(Tw)) \leq \theta(F(d_b(z, w), \varphi(z), \varphi(w)))$$

and then

$$s^{\varepsilon} F(d_b(z, w), 0, 0) \leq \theta(F(d_b(z, w), 0, 0))$$

which implies $d_b(z, w) = 0$, that is $z = w$.

Definition 14. Let (X, d_b) be a b-metric space with coefficient $s \geq 1$, $\alpha: X \times X \to [0, \infty)$ be a mapping and $\varphi : X \to [0, \infty)$ be lower semi continuous function, $\theta \in J_b$ and $\varepsilon > 1$. A mapping $T: X \to X$ is said to be an $(F, \alpha, \varphi, \theta)_s$-weak contraction mapping if

$x, y \in X$ with $\alpha(x, y) \geq s \Rightarrow$

$$s^{\varepsilon} F\big(d_b(Tx, Ty), \varphi(Tx)\varphi(Ty)\big) \leq \theta(F(d_b(x, y), \varphi(x), \varphi(y)))$$

$$+L\Big(F\big(d_b(y, Tx), \varphi(y), \varphi(Tx)\big) - F\big(0, \varphi(y), \varphi(Tx)\big)\Big) \quad (3.13)$$

Theorem 15. Let (X, d_b) be a complete b-metric space with coefficient $s \geq 1$ and $T: X \to X$ be α-admissible mapping type S. Suppose that the following conditions hold:

(1) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq s$,

(2) T is an $(F, \alpha, \varphi, \theta)_s$−weak contraction mapping,

(3)) if $\{x_n\}$ is a sequence in X such that $\alpha(x_n, x_{n+1}) \geq s$ and $x_n \to x$ then $\alpha(x_n, x) \geq s$ for all $n \in N$.

Then

(i) $F_T \subseteq Z_\varphi$,

(ii) T is φ-weakly Picard operator. Moreover, if $\alpha(x,y) \geq s$ for all $x, y \in F_T$, then T has a unique φ-fixed point.

Proof. (i) Assume that $\xi \in X$ is a fixed point of T such that $\alpha(\xi,\xi) \geq s$. Applying (3.13) with $x = y = \xi$, we obtain

$$\begin{aligned} F\big(0,\varphi(\xi),\varphi(\xi)\big) &\leq s^\varepsilon F\big(0,\varphi(\xi),\varphi(\xi)\big) \\ &\leq \theta(F(0,\varphi(\xi),\varphi(\xi))) \\ &\quad + L(F(0,\varphi(\xi),\varphi(\xi)) - F(0,\varphi(\xi),\varphi(\xi))) \\ &= \theta\Big(F\big(0,\varphi(\xi),\varphi(\xi)\big)\Big). \end{aligned} \tag{3.14}$$

$$F\big(0,\varphi(\xi),\varphi(\xi)\big) \leq s^\varepsilon F\big(0,\varphi(\xi),\varphi(\xi)\big) \leq \theta(F(0,\varphi(\xi),\varphi(\xi)))$$

then we get

$$F(0,\varphi(\xi),\varphi(\xi) \leq \theta\Big(F\big(0,\varphi(\xi),\varphi(\xi)\big)\Big).$$

From Lemma 5, we have

$$F(0,\varphi(\xi),\varphi(\xi)) = 0. \tag{3.15}$$

On the other hand, from (F1), we have

$$\phi(\xi) \leq F\big(0,\varphi(\xi),\varphi(\xi)\big). \tag{3.16}$$

From (3.15) and (3.16), we obtain $\varphi(\xi) = 0$, which proves (i).

(ii) Let $x_0 \in X$ be such that $\alpha(x_0, Tx_0) \geq s$. Define a sequence $\{x_n\}$ by $x_n = Tx_{n-1}$ for all $n \in N$. Since T is an α-admissible mapping and $\alpha(x_0, x_1) = \alpha(x_0, Tx_0) \geq s$, we deduce that $\alpha(x_1, x_2) = \alpha(Tx_0, Tx_1) \geq 1$. Continuing this process, we get $\alpha(x_n, x_{n+1}) \geq s$ for all $n \in N \cup \{0\}$. If $x_n = x_{n+1}$, for some $n \in N$, then $x_n = Tx_n$. Thus, x_n is a fixed point of T.

Therefore, We assume that $x_n \neq x_{n+1}$, for all $n \in N$. Using condition (1) as $\alpha(x_{n-1}, x_n) \geq s$ for all

$n \in N$, we obtain

$$\begin{aligned} F(d_b(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n))) &\leq s^\varepsilon F(d_b(Tx_{n-1},Tx_n),\varphi(Tx_{n-1}),\varphi(Tx_n)) \\ &\leq \theta(F(d_b(x_{n-1},x_n),\varphi(x_{n-1}),\varphi(x_n))) \\ &\quad + L(F(0,\varphi(Tx_{n-1}),\varphi(Tx_n)) \\ &\qquad - F(0,\varphi(Tx_{n-1}),\varphi(Tx_n))) \end{aligned}$$

$$\leq \theta^{n}(F(d_b(x_0, x_1), \varphi(x_0), \varphi(x_1))). \qquad (3.17)$$

The rest of the proof follows using similar argument to proof of Theorem 13.

4. APPLICATIONS

In this section, we give some fixed point results in partial b-metric spaces, using the main results in the previous section.

Firstly, let us recall some basic definitions on partial b-metric spaces.

Definition 16. [23] Let X be a nonempty set and and $s \geq 1$ be a given real number.

A function $p_b: X \times X \to R^+$ is a partial b-metric if for all $x, y, z \in X$, the following

conditions are satisfied:

(p1) $x = y \Leftrightarrow p_b(x, x) = p_b(x, y) = p_b(y, y)$,

(p2) $p_b(x, x) \leq p_b(x, y)$,

(p3) $p_b(x, y) = p_b(y, x)$,

(p4) $p_b(x, y) \leq s(p_b(x, z) + p_b(z, y) - p_b(z, z)) + (\frac{1-s}{2})(p_b(x, x) + p_b(y, y))$.

Definition 17. [28] A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is said to be:

(i) p_b-convergent to a point $x \in X$ if $\lim_{n\to\infty} p_b(x, x_n) = p_b(x, x)$.

(ii) A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is called a Cauchy sequence if $\lim_{m,n\to\infty} p_b(x_n, x_m)$ exists and is finite.

(iii) A partial b-metric space (X, p_b) is called complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$ such that, $\lim_{m,n\to\infty} p_b(x_n, x_m) = \lim_{m,n\to\infty} p_b(x_n, x) = p_b(x, x)$.

Proposition 18. [28] Every partial b-metric p_b defines a b-metric d_{p_b}, where

$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, y) - p_b(y, y)$ for all $x, y \in X$.

Lemma 19. [28] Let (X, p_b) be a partial b-metric space. Then,

(i) A sequence $\{x_n\}$ in a partial b-metric space (X, p_b) is a Cauchy sequence if and only if it is a Cauchy sequence in the b-metric space (X, d_b).

(ii) A partial b-metric space (X, p_b) is complete if and only if the b-metric space (X, d_b) is complete.

(iii) Given a sequence $\{x_n\}$ in a partial b-metric space (X, p_b) and $x \in X$, we have that

$$\lim_{n\to\infty} p_b(x, x_n) = 0 \Leftrightarrow p_b(x, x) = \lim_{n\to\infty} p_b(x, x_n) = 0 = \lim_{m,n\to\infty} p_b(x_n, x_m) = 0$$

Now, we give our some results in partial b- metric spaces.

Theorem 20. Let (X, p_b) be a complete partial b- metric space and let $T: X \to X$ is a mapping,

$\alpha: X \times X \rightarrow [0, \infty)$ and $\theta \in J_b$. Assume that the following conditions hold:

(i) $\theta(2t) = 2\theta(t)$ for all $t \in [0, \infty)$,

(ii) For all $x, y \in X$, for $s \geq 1$ and for $\varepsilon > 0$,

$$s^{\varepsilon} p_b(Tx, Ty) \leq \theta(p_b(x, y)).$$

Then

(i) T has a unique fixed point $z \in X$.

(ii) $p_b(z, z) = 0$.

Proof. Let the metric d_{p_b} on X which is defined by

$d_{p_b}(x, y) = 2p_b(x, y) - p_b(x, y) - p_b(y, y)$

for all $x, y \in X$ and $\varphi(x) = p_b(x, x)$ for all $x \in X$. Let $F: [0, \infty)^3 \rightarrow [0, \infty)$ be defined by

$F(a, b, c) = a + b + c$. From (i) and (ii), it is easy to verify

$$s^{\varepsilon}[2p_b(Tx, Ty) - p_b(Tx, Tx) - p_b(Ty, Ty) + p_b(Tx, Tx) + p_b(Ty, Ty))]$$
$$\leq \theta(2p_b(x, y) - p_b(x, x) - p_b(y, y) + p_b(x, x) + p_b(y, y)).$$

Then, from above the inequality, we have

$$s^{\varepsilon} F(d(Tx, Ty), \varphi(Tx), \varphi(Ty)) \leq \theta(F(d(x, y), \varphi(x), \varphi(y))).$$

Then the hypothesis of Theorem 13 is satisfied and then T has a unigue φ-fixed point. Hence, T has a unique fixed point $z \in X$ such that $p_b(z, z) = 0$. Therefore, the proof is completed.

Theorem 21. Let (X, p_b) be a complete partial b- metric space and let $T: X \rightarrow X$ is a mapping,

$\alpha: X \times X \rightarrow [0, \infty)$ and $\theta \in J_b$. Assume that the following conditions hold:

(a) $\theta(2t) = 2\theta(t)$ for all $t \in [0, \infty)$,

(b) For all $x, y \in X$, for $s \geq 1$ and for $\varepsilon > 0$,

$$s^{\varepsilon} p_b(Tx, Ty) \leq \theta(p_b(x, y)) + L(p_b(Ty, Tx) - \frac{p_b(y, y) + p_b(Tx, Tx)}{2})$$

Then

(i) T has a unique fixed point $z \in X$.

(ii) $p_b(z, z) = 0$

Taking $\theta(t) = kt$, where $k \in [0,1)$ in Theorem 20 and 21, we obtain the following corollaries.

Corollary 22. Let (X, p_b) be a complete partial b- metric space and let $T: X \rightarrow X$ is a mapping such that for all $x, y \in X$ and for some constant $k \in [0,1)$

$$s^{\varepsilon}p_b(Tx,Ty) \leq kp_b(x,y).$$

Then T has a unique fixed point $z \in X$. Morever $p_b(z,z) = 0$.

Corollary 23. Let (X, p_b) be a complete partial b- metric space and let $T: X \to X$ is a mapping such that for all $x, y \in X$ and for some constant $k \in [0,1)$

$$s^{\varepsilon}p_b(Tx,Ty) \leq \theta(p_b(x,y)) + L(p_b(Ty,Tx) - \frac{p_b(y,y) + p_b(Tx,Tx)}{2})$$

Then T has a unique fixed point $z \in X$. Morever p_b(z,z) = 0.

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