Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-2 (2019) 457-470

Truncated Truncated Dodecahedron and Truncated Truncated Icosahedron Spaces

Ümit Ziya SAVCI[1]

[1]*Kütahya Dumlupınar University, Educational Faculty, Department of Mathematics Education, Kütahya, Turkey*

Received: 01.03.2019; Accepted: 22.05.2019 *http://dx.doi.org/10.17776/csj.534616*

Abstract. The theory of convex sets is a vibrant and classical field of modern mathematics with rich applications. The more geometric aspects of convex sets are developed introducing some notions, but primarily polyhedra. A polyhedra, when it is convex, is an extremely important special solid in $\mathbb{R}^n$. Some examples of convex subsets of Euclidean 3-dimensional space are Platonic Solids, Archimedean Solids and Archimedean Duals or Catalan Solids. There are some relations between metrics and polyhedra. For example, it has been shown that cube, octahedron, deltoidal icositetrahedron are maximum, taxicab, Chinese Checker's unit sphere, respectively. In this study, we give two new metrics to be their spheres truncated truncated dodecahedron and truncated truncated icosahedron.

Keywords: Polyhedron, Metric, Truncated Truncated Dodecahedron, Truncated Truncated Icosahedron.

Truncated Truncated Dodecahedron ve Truncated Truncated Icosahedron Uzayları

Özet. Konveks kümeler teorisi, zengin uygulamalara sahip modern matematiğin canlı ve klasik bir alanıdır. Konveks kümelerin geometrik yönleri, bazı kavramlar, fakat öncelikle çokyüzlülerin tanıtılmasıyla geliştirilmiştir. Konveks olduğunda bir çokyüzlü, $\mathbb{R}^n$ de çok önemli bir özel cisimdir. Öklid 3 boyutlu uzayın konveks alt kümelerinin bazı örnekleri Platonik cisimler, Arşimet cisimleri ve Arşimet dualleri veya Katalan cisimleridir. Metriklerle çokyüzlüler arasında bazı ilişkiler vardır. Örneğin, küp, sekizyüzlü, deltoidal icositetrahedron'un sırasıyla, maksimum, Taksi, Çin dama uzaylarının birim küresi olduğu görülmektedir. Bu çalışmada, kürelerinin truncated truncated dodecahedron ve truncated truncated icosahedron olan iki yeni metrik tanıtıldı.

Anahtar Kelimeler: Çokyüzlü, Metrik, Truncated Truncated Dodecahedron, Truncated Truncated Icosahedron.

1. INTRODUCTION

The word polyhedron has slightly different meanings in geometry and algebraic topology. In geometry, a polyhedron is simply a three-dimensional solid which consists of a collection of polygons, usually joined at their edges. The term "polyhedron" is used somewhat differently in algebraic topology, where it is defined as a space that can be built from such "building blocks" as line segments, triangles, tetrahedra, and their higher dimensional analogs by "gluing them together" along their faces [1]. The word derives from the Greek poly(many) plus the Indo-European hedron(seat). A polyhedron is the three-dimensional version of the more general polytope which can be defined in arbitrary dimension. The plural of polyhedron is "polyhedra" (or sometimes "polyhedrons").Polyhedra have worked by people since ancient time. Early civilizations worked out mathematics as problems and their solutions. Polyhedrons

* Corresponding author. *Email address: ziyasavci@hotmail.com*

have been studied by mathematicians, geometers during many years, because of their symmetries. Recently, polyhedra and their symmetries have been cast in a new light by mathematicians.

A polyhedron is said to be regular if all its faces are equal regular polygons and the same number of faces meet at every vertex. Platonic solids are regular and convex polyhedra. Nowadays, some mathematicians are working platonic solid's metric [2,3]. A polyhedron is called semi-regular if all its faces are regular polygons and all its vertices are equal. Archimedian soldis are semi-regular and convex polyhedra.

Minkowski geometry is non-Euclidean geometry in a finite number of dimensions. Here the linear structure is the same as the Euclidean one but distance is not uniform in all directions. That is, the points, lines and planes are the same, and the angles are measured in the same way, but the distance function is different. Instead of the usual sphere in Euclidean space, the unit ball is a general symmetric convex set [4]. Some mathematicians have been studied and improved metric space geometry. According to mentioned researches it is found that unit spheres of these metrics are associated with convex solids. For example, unit sphere of maximum metric is a cube which is a Platonic Solid. Taxicab metric's unit sphere is an octahedron, another Platonic Solid. In [1,2,5,6,7,8,9,10,11,12] the authors give some metrics which the spheres of the 3-dimensional analytical space furnished by these metrics are some of Platonic solids, Archimedian solids and Catalan solids. So there are some metrics which unit spheres are convex polyhedrons. That is, convex polyhedrons are associated with some metrics. When a metric is given, we can find its unit sphere in related space geometry. This enforce us to the question "Are there some metrics whose unit sphere is a convex polyhedron?". For this goal, firstly, the related polyhedra are placed in the 3-dimensional space in such a way that they are symmetric with respect to the origin. And then the coordinates of vertices are found. Later one can obtain metric which always supply plane equation related with solid's surface.

In this study, we introduce two new metrics, and show that the spheres of the 3-dimensional analytical space furnished by these metrics are truncated truncated dodecahedron and truncated truncated icosahedron. Also we give some properties about these metrics.

2. TRUNCATED TRUNCATED DODECAHEDRON METRIC AND SOME PROPERTIES

It has been stated in [13], there are many variations on the theme of the regular polyhedra. First one can meet the eleven which can be made by cutting off (truncating) the corners, and in some cases the edges, of the regular polyhedra so that all the faces of the faceted polyhedra obtained in this way are regular polygons. These polyhedra were first discovered by Archimedes (287-212 B.C.E.) and so they are often called Archimedean solids. Notice that vertices of the Archimedean polyhedra are all alike, but their faces, which are regular polygons, are of two or more different kinds. For this reason they are often called semiregular. Archimedes also showed that in addition to the eleven obtained by truncation, there are two more semiregular polyhedra: the snub cube and the snub dodecahedron.

Five Archimedean solids are derived from the Platonic solids by truncating (cutting off the corners) a percentage less than 1/2 [14,15]. Two of them are the truncated dodecahedron and the truncated icosahedron.

One of the solids which is obtained by truncating from another solid is the truncated truncated dodecahedron. It has 12 pent-symmetric 20-gonal faces, 20 regular hexagonal faces, 60 isosceles triangular faces, 180 vertices and 270 edges. The truncated truncated dodecahedron can be obtained by two times truncating operation from dodecahedron. Truncated dodecahedron appears with first truncation operation. Using second truncation to truncated dodecahedron gives the truncated truncated dodecahedron. Figure 1 shows the dodecahedron, the truncated dodecahedron, the truncated truncated dodecahedron and the transparent truncated truncated dodecahedron, respectively.

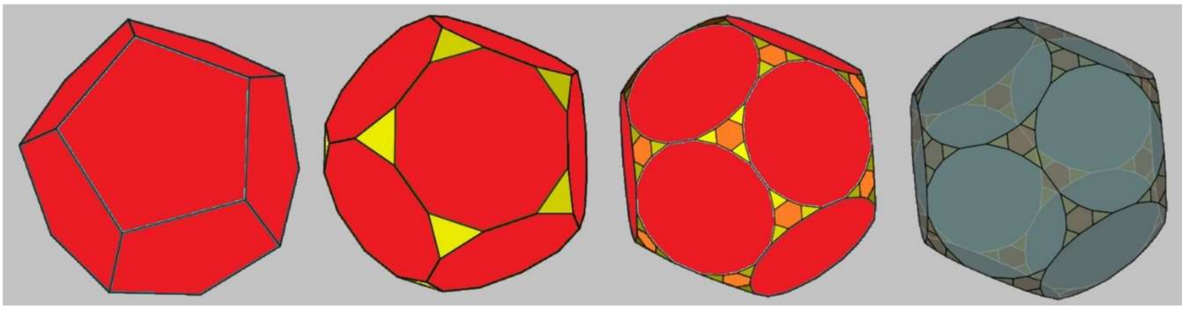

Figure 1: Dodecahedron, Truncated dodecahedron, Truncated truncated dodecahedron

Before we give a description of the truncated truncated dodecahedron distance function, we must agree on some impressions. Therefore U denote the maximum of quantities $\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$ for $P_1=(x_1,y_1,z_1), P_2=(x_2,y_2,z_2)\in\mathbb{R}^3$.Also, $X-Y-Z-X$ orientation and $Z-Y-X-Z$ orientation are called positive $(+)$ direction and negative $(-)$ directions, respectively. Accordingly, U^+ and U^- will display the next term in the respective direction according to U. For example, if $U=|y_1-y_2|$, then $U^+=|z_1-z_2|$ and $U^-=|x_1-x_2|$.

The metric that unit sphere is truncated truncated dodecahedron is described as following:

Let $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$ be two points in $\mathbb{R}^3$. The distance function

$d_{TTD}:\mathbb{R}^3\times\mathbb{R}^3\to[0,\infty)$ truncated truncated dodecahedron distance between P_1 and P_2 is defined by

$$d_{TTD}(P_1,P_2)=\max\left\{\begin{matrix}U+k_1U^+,k_2U+k_3U^-,k_4(U+U^-+U^+),k_5U+k_6U^-,\\ k_7U+k_8U^-+k_9U^+,k_{10}U+k_{11}U^-+k_{12}U^+\end{matrix}\right\}$$

where $k_1=\frac{\sqrt{5}-1}{2}, k_2=\frac{7\sqrt{5}+5}{22}, k_3=\frac{4\sqrt{5}-5}{11}, k_4=\frac{15\ \ \sqrt{5}}{22}, k_5=\frac{17\sqrt{5}+159}{202}, k_6=\frac{24\sqrt{5}-31}{101}$,

$k_7=\frac{34\sqrt{5}+15}{101}, k_8=\frac{18\sqrt{5}+2}{101}, k_9=\frac{39\sqrt{5}-63}{202}, k_{10}=\frac{10\sqrt{5}+46}{101}, k_{11}=\frac{-19\sqrt{5}+155}{202}$ and

$k_{12}=\frac{16\sqrt{5}+13}{101}$.

According to truncated truncated dodecahedron distance, there are six different paths from P_1 to P_2. These paths are

i) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{1}{2}\right)$ angle with another coordinate axis,

ii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{\sqrt{5}}{2}\right)$ angle with another coordinate axis.

iii) union of three line segments each of which is parallel to a coordinate axis.

iv) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{2691+860\sqrt{5}}{2242}\right)$ angle with another coordinate axis.

v) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{71\sqrt{5}+585}{880})$ and $\arctan\left(\frac{575+2\ \ \sqrt{5}}{330}\right)$ angles with one of the other coordinate axes .

vi) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan(\frac{213\sqrt{5}-390}{440})$ and $\arctan\left(\frac{27+1\ \ \sqrt{5}}{176}\right)$ angles with one of the other coordinate axes .

Thus truncated truncated dodecahedron distance between P_1 and P_2 is for (i) sum of Euclidean lengths of mentioned two line segments, for (ii) k_2 times the sum of Euclidean lengths of mentioned two line segments, for (iii) k_4 times the sum of Euclidean lengths of mentioned three line segments, for (iv) k_5 times the sum of Euclidean lengths of mentioned two line segments, for (v) k_7 times the sum of Euclidean lengths of mentioned three line segments, and for (vi) k_{10} times the sum of Euclidean lengths of mentioned three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 2 illustrates some of truncated truncated dodecahedron way from P_1 to P_2

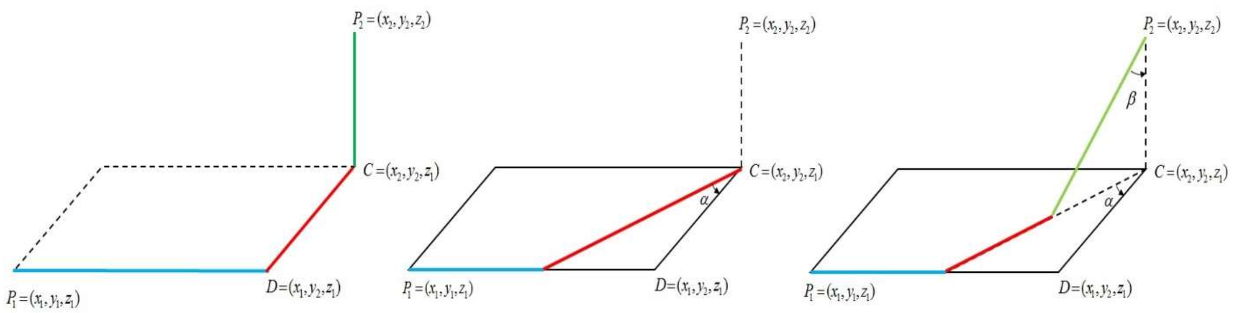

Figure 2: Some TTD way from P_1 to P_2

In [16] and [17], the authours introduce a metric and show that spheres of 3-dimensional analytical space furnished by these metric are the icosahedron and the truncated icosahedron. These metrics for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_D(P_1, P_2) = U + k_1 U^+$$

$$d_{TD}(P_1, P_2) = \max\{U + k_1 U^+, k_2 U + k_3 U^-, k_4(U + U^- + U^+)\},$$

where k_i for $i = 1,2,3,4$ are the same with definition 1.

Lemma 1: Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Then

$$d_{TTD}(P_1,P_2) \geq U_{12} + k_1 U_{12}^{+}$$
$$d_{TTD}(P_1,P_2) \geq k_2 U_{12} + k_3 U_{12}^{-}$$
$$d_{TTD}(P_1,P_2) \geq k_4 (U_{12} + U_{12}^{-} + U_{12}^{+})$$
$$d_{TTD}(P_1,P_2) \geq k_5 U_{12} + k_6 U_{12}^{-}$$
$$d_{TTD}(P_1,P_2) \geq k_7 U_{12} + k_8 U_{12}^{-} + k_9 U_{12}^{+}$$
$$d_{TTD}(P_1,P_2) \geq k_{10} U_{12} + k_{11} U_{12}^{-} + k_{12} U_{12}^{+}.$$

Proof. Proof is trivial by the definition of maximum function.

Theorem 1 The distance function d_{TTD} is a metric. Also according to d_{TTD}, the unit sphere is a truncated truncated dodecahedron in $\mathbb{R}^3$.

Proof. Let $d_{TTD}: \mathbb{R}^3 \times \mathbb{R}^3 \to [0,\infty)$ be the truncated truncated dodecahedron distance function and $P_1=(x_1,y_1,z_1)$, $P_2=(x_2,y_2,z_2)$ and $P_3=(x_3,y_3,z_3)$ are distinct three points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$. To show that d_{TTD} is a metric in $\mathbb{R}^3$, the following axioms hold true for all P_1, P_2 and $P_3 \in \mathbb{R}^3$.

M1) $d_{TTD}(P_1,P_2) \geq 0$ and $d_{TTD}(P_1,P_2) = 0$ iff $P_1 = P_2$

M2) $d_{TTD}(P_1,P_2) = d_{TTD}(P_2,P_1)$

M3) $d_{TTD}(P_1,P_3) \leq d_{TTD}(P_1,P_2) + d_{TTD}(P_2,P_3)$.

Since absolute values is always nonnegative value $d_{TTD}(P_1,P_2) \geq 0$. If $d_{TTD}(P_1,P_2) = 0$ then

$$\max\left\{\begin{matrix} U + k_1U^{+}, k_2U + k_3U^{-}, k_4(U + U^{-} + U^{+}), k_5U + k_6U^{-}, \\ k_7U + k_8U^{-} + k_9U^{+}, k_{10}U + k_{11}U^{-} + k_{12}U^{+} \end{matrix}\right\} = 0,$$

where U are the maximum of quantities $\{|x_1-x_2|,|y_1-y_2|,|z_1-z_2|\}$. Therefore, $U + k_1U^{+} = 0, k_2U + k_3U^{-} = 0, k_4(U + U^{-} + U^{+}) = 0, k_5U + k_6U^{-} = 0, k_7U + k_8U^{-} + k_9U^{+} = 0$ and

$k_{10}U + k_{11}U^{-} + k_{12}U^{+} = 0$. Hence, it is clearly obtained by $x_1 = x_2$,

$y_1 = y_2$, $z_1 = z_2$. That is, $P_1 = P_2$. Thus it is obtained that $d_{TTD}(P_1,P_2) = 0$ iff $P_1 = P_2$.

Since $|x_1-x_2| = |x_2-x_1|$, $|y_1-y_2|$=$|y_2-y_1|$ and $|z_1-z_2| = |z_2-z_1|$, obviously $d_{TTD}(P_1,P_2) = d_{TTD}(P_2,P_1)$. That is, d_{TTD} is symmetric.

U_{13}, and U_{23} denote the maximum of quantities of $\{|x_1-x_3|,|y_1-y_3|,|z_1-z_3|\}$ and $\{|x_2-x_3|,|y_2-y_3|,|z_2-z_3|\}$, respectively.

$$d_{TTD}(P_1,P_3) = \max\left\{\begin{matrix} U_{13}+k_1U_{13}^+, k_2U_{13}+k_3U_{13}^-, k_4(U_{13}+U_{13}^-+U_{13}^+), k_5U_{13}+k_6U_{13}^-, \\ k_7U_{13}+k_8U_{13}^-+k_9U_{13}^+, k_{10}U_{13}+k_{11}U_{13}^-+k_{12}U_{13}^+ \end{matrix}\right\}$$

$$\leq \max\left\{\begin{matrix} U_{12}+U_{23}+k_1(U_{12}^++U_{23}^+), k_2(U_{12}+U_{23})+k_3(U_{12}^-+U_{23}^-), \\ k_4(U_{12}+U_{23}+U_{12}^-+U_{23}^-+U_{12}^++U_{23}^+), \\ k_5(U_{12}+U_{23})+k_6(U_{12}^-+U_{23}^-), \\ k_7(U_{12}+U_{23})+k_8(U_{12}^-+U_{23}^-)+k_9(U_{12}^++U_{23}^+), \\ k_{10}(U_{12}+U_{23})+k_{11}(U_{12}^-+U_{23}^-)+k_{12}(U_{12}^++U_{23}^+) \end{matrix}\right\}$$

$$= I.$$

Therefore one can easily find that $I \leq d_{TTD}(P_1,P_2) + d_{TTD}(P_2,P_3)$ from Lemma 1.

So $d_{TTD}(P_1,P_3) \leq d_{TTD}(P_1,P_2) + d_{TTD}(P_2,P_3)$. Consequently, truncated truncated dodecahedron distance is a metric in 3-dimensional analytical space.

Finally, the set of all points $X = (x,y,z) \in \mathbb{R}^3$ that truncated truncated dodecahedron distance is 1 from $O = (0,0,0)$ is

$$S_{TTD} = \left\{(x,y,z) : \max\left\{\begin{matrix} U+k_1U^+, k_2U+k_3U^-, k_4(U+U^-+U^+), k_5U+k_6U^-, \\ k_7U+k_8U^-+k_9U^+, k_{10}U+k_{11}U^-+k_{12}U^+ \end{matrix}\right\} = 1\right\}.$$

Thus the graph of S_{TTD} is as in the figure 3:

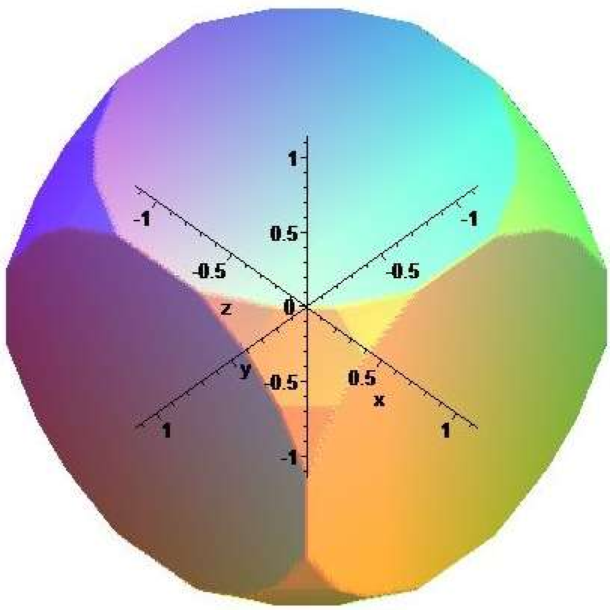

Figure 3. The unit sphere in terms of d_{TTD}: Truncated Truncated Dodecahedron

Corollary 1 The equation of the truncated truncated dodecahedron with center (x_0, y_0, z_0) and radius r is

$$\max\left\{\begin{matrix} U_0 + k_1 U_0^+, k_2 U_0 + k_3 U_0^-, k_4(U_0 + U_0^- + U_0^+), k_5 U_0 + k_6 U_0^-, \\ k_7 U_0 + k_8 U_0^- + k_9 U_0^+, k_{10} U_0 + k_{11} U_0^- + k_{12} U_0^+ \end{matrix}\right\} = r,$$

which is a polyhedron which has 92 faces and 180 vertices, where U_0 are the maximum of quantities $\{|x - x_0|, |y - y_0|, |z - z_0|\}$. Coordinates of the vertices are translation to (x_0, y_0, z_0) all circular shift of the three axis components and all possible +/- sign changes of each axis component of $(0, C_1 r, r), (C_0 r, C_4 r, C_{20} r), (C_2 r, C_5 r, C_{19} r), (C_0 r, C_7 r, C_{18} r), (C_3 r, C_8 r, C_{17} r),$ $(C_6 r, C_{11} r, C_{16} r), (C_{12} r, C_{10} r, C_{15} r)$ and $(C_9 r, C_{13} r, C_{14} r)$, where

$$C_0 = \frac{3\sqrt{5} - 5}{30}, C_1 = \frac{5\sqrt{5} - 7}{38}, C_2 = \frac{3\sqrt{5} - 5}{15}, C_3 = \frac{5\sqrt{5} - 7}{19},$$

$$C_4 = \frac{4\sqrt{5} - 5}{15}, C_5 = \frac{7\sqrt{5} - 5}{30}, C_6 = \frac{9\sqrt{5} - 5}{38},$$

$$C_7 = \frac{\sqrt{5}}{5}, C_8 = \frac{3\sqrt{5} + 11}{38}, C_9 = \frac{\sqrt{5} + 5}{15},$$

$$C_{10} = \frac{10 - \sqrt{5}}{15}, C_{11} = \frac{27 - 3\sqrt{5}}{38}, C_{12} = \frac{\sqrt{5} + 1}{6},$$

$$C_{13} = \frac{25 - 3\sqrt{5}}{30}, C_{14} = \frac{5 + 2\sqrt{5}}{15}, C_{15} = \frac{2}{3},$$

$$C_{16} \frac{7\sqrt{5} + 13}{38}, \quad C_{17} = \frac{6\sqrt{5} + 3}{19}, \quad C_{18} = \frac{2\sqrt{5}}{5},$$

$$C_{19} = \frac{4\sqrt{5}+5}{15} \text{ and } C_{20} = \frac{2\sqrt{5}+10}{15}.$$

Lemma *2* Let l be the line through the points P_1=(x_1, y_1, z_1) and P_2=(x_2, y_2, z_2)in the analytical 3-dimensional space and $\mathrm{d_E}$ denote the Euclidean metric. If l has direction vector $(\mathrm{p}, \mathrm{q}, \mathrm{r})$, then

$$d_{TTD}(P_1, P_2) = \mu(P_1 P_2) d_E(P_1, P_2)$$

where

$$\mu(P_1 P_2) = \frac{\max\left\{\begin{matrix} U_d + k_1 U_d^+, k_2 U_d + k_3 U_d^-, k_4(U_d + U_d^- + U_d^+), k_5 U_d + k_6 U_d^-, \\ k_7 U_d + k_8 U_d^- + k_9 U_d^+, k_{10} U_d + k_{11} U_d^- + k_{12} U_d^+ \end{matrix}\right\}}{\sqrt{p^2 + q^2 + r^2}},$$

U_d are the maximum of quantities $\{|p|, |q|, |r|\}$.

Proof. Equation of l gives us $x_1 - x_2 = \lambda p, y_1 - y_2 = \lambda q, z_1 - z_2 = \lambda r, \lambda \in \mathbb{R}$. Thus,

$$d_{TTD}(P_1, P_2) = |\lambda| \left(\max \left\{ \begin{matrix} U_d + k_1 U_d^+, k_2 U_d + k_3 U_d^-, k_4 (U_d + U_d^- + U_d^+), k_5 U_d + k_6 U_d^-, \\ k_7 U_d + k_8 U_d^- + k_9 U_d^+, k_{10} U_d + k_{11} U_d^- + k_{12} U_d^+ \end{matrix} \right\} \right),$$

where U_d are the maximum of quantities $\{|p|, |q|, |r|\}$, and $d_E(A, B) = |\lambda|\sqrt{p^2 + q^2 + r^2}$ which implies the desired result.

The above lemma says that d_{TTD}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 2 If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then

$d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TTD}(P_1, X) = d_{TTD}(P_2, X)$.

Corollary 3 If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then

$$d_{TTD}(X, P_1)/d_{TTD}(X, P_2) = d_E(X, P_1)/d_E(X, P_2).$$

That is, the ratios of the Euclidean and $\mathrm{d_{TTD}} -$ distances along a line are the same.

3. TRUNCATED TRUNCATED ICOSAHEDRON METRIC AND SOME PROPERTIES

The truncated truncated icosahedron can be obtained by two times truncating operation from icosahedron. Truncated icosahedron appears with first truncation operation. Using second truncation to truncated icosahedron gives the truncated truncated icosahedron. The truncated truncated icosahedron has 20 tri-symmetric dodecagonal faces, 12 regular decagonal faces, 60 equilateral triangular faces, 180 vertices and 270 edges. The truncated truncated icosahedron and the truncated truncated dodecahedron have the same number of faces, vertices and edges. Figure 4 show the icosahedron, the truncated icosahedron and the truncated truncated icosahedron, and the transparent truncated truncated icosahedron, respectively.

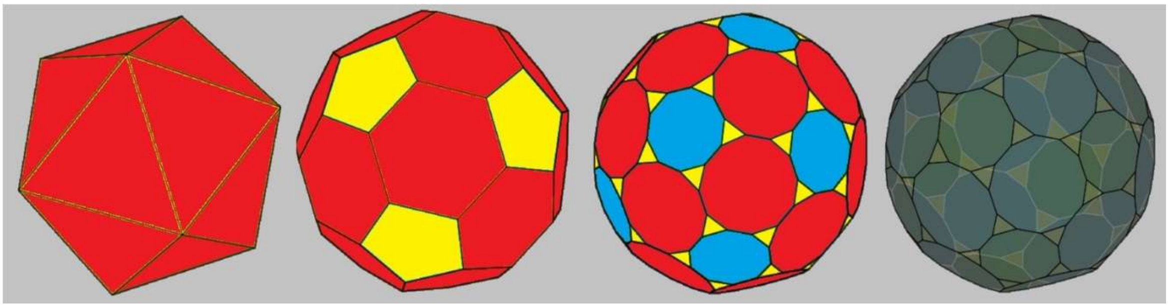

Figure 4. Icosahedron, Truncated icosahedron and Truncated truncated icosahedron

The notations U, U^+, U^- shall be used as defined in the previous section. The metric that unit sphere is the truncated truncated icosahedron is described as following:

Defnition 2 Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be two points in $\mathbb{R}^3$. The distance function $d_{TTI}: \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)$ truncated truncated icosahedron distance between P_1 and P_2 is defined by

$$d_{TTI}(P_1, P_2) = \max \left\{ \begin{matrix} U + k_1 U^+, k_2(U + U^- + U^+), k_3 U + k_4 U^+, k_5 U + k_6 U^+, \\ k_7 U + k_8 U^- + k_9 U^+, k_{10} U + k_{11} U^- + k_{12} U^+ \end{matrix} \right\}$$

where $k_1 = \frac{3-\sqrt{5}}{2}, k_2 = \frac{\sqrt{5}-1}{2}, k_3 = \frac{325+129\sqrt{5}+(26\sqrt{5}-90)\sqrt{3+2\sqrt{5}}}{590}, k_4 = \frac{80+49\sqrt{5}+(55-29\sqrt{5})\sqrt{3+2\sqrt{5}}}{295},$

$$k_5 = \frac{7162 + 910\sqrt{5} + (303\sqrt{5} - 789)\sqrt{3 + 2\sqrt{5}}}{9082}, k_6 = \frac{576 - 273\sqrt{5} + (1599 - 545\sqrt{5})\sqrt{3 + 2\sqrt{5}}}{4541},$$

$$k_7 = \frac{3915 + 4885\sqrt{5} + (1901\sqrt{5} - 3961)\sqrt{3 + 2\sqrt{5}}}{18164}, k_8 = \frac{7951 + 607\sqrt{5} + (337 - 751\sqrt{5})\sqrt{3 + 2\sqrt{5}}}{18164},$$

$$k_9 = \frac{1290\sqrt{5} - 77 + (2175 - 818\sqrt{5})\sqrt{3 + 2\sqrt{5}}}{9082}, k_{10} = \frac{7797 + 3187\sqrt{5} + (4687 - 2387\sqrt{5})\sqrt{3 + 2\sqrt{5}}}{18164},$$

$$k_{11} = \frac{6373+1213\sqrt{5}+(1357\sqrt{5}-1915)\sqrt{3+2\sqrt{5}}}{18164} \text{ and } k_{12} = \frac{1836\sqrt{5}-1229+(272\sqrt{5}-1003)\sqrt{3+2\sqrt{5}}}{9082}.$$

According to truncated truncated icosahedron distance, there are six different paths from P_1 to P_2. These paths are

i) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{\sqrt{5}}{2}\right)$ angle with another coordinate axis,

ii) union of three line segments each of which is parallel to a coordinate axis.

iii) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{1}{2}\right)$ angle with another coordinate axis..

iv) union of two line segments which one is parallel to a coordinate axis and other line segment makes $\arctan\left(\frac{(8\sqrt{5}-1)\sqrt{3+2\sqrt{5}}}{22}\right)$ angle with another coordinate axis..

v) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan\left(\frac{3\sqrt{5}+1+(2\sqrt{5}-2)\sqrt{3+2\sqrt{5}}}{12}\right)$ and $\arctan\left(\frac{15+7\sqrt{5}+(7-5\sqrt{5})\sqrt{3+2\sqrt{5}}}{24}\right)$ angles with one of the other coordinate axes .

vi) union of three line segments one of which is parallel to a coordinate axis and the others line segments makes one of $\arctan\left(\frac{3\sqrt{5}+1+(2-2\sqrt{5})\sqrt{3+2\sqrt{5}}}{12}\right)$ and $\arctan\left(\frac{15+7\sqrt{5}+(5\sqrt{5}-7)\sqrt{3+2\sqrt{5}}}{24}\right)$ angles with one of the other coordinate axes .

Thus truncated truncated icosahedron distance between P_1 and P_2 is for (i) sum of Euclidean lengths of mentioned two line segments, for (ii) k_2 times the sum of Euclidean lengths of mentioned two line segments, for (iii) k_3 times the sum of Euclidean lengths of mentioned three line segments, for (iv) k_5 times the sum of Euclidean lengths of mentioned two line segments, for (v) k_7 times the sum of Euclidean lengths of mentioned three line segments, and for (vi) k_{10} times the sum of Euclidean lengths of mentioned three line segments. In case of $|y_1 - y_2| \geq |x_1 - x_2| \geq |z_1 - z_2|$, Figure 5 shows that some of the TTI $-$path between P_1 and P_2.

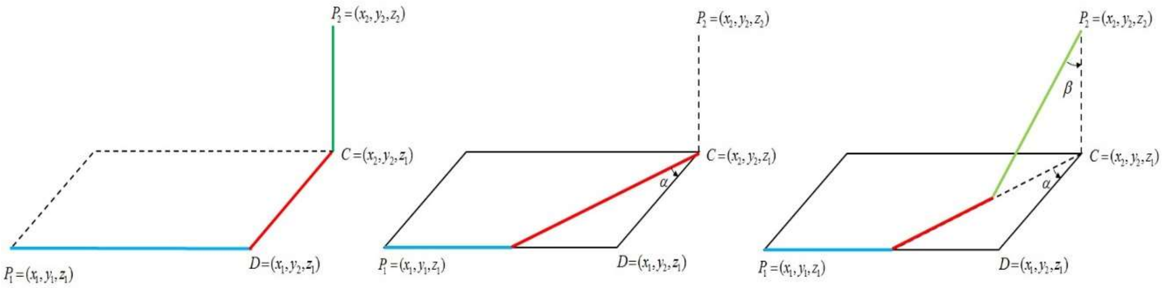

Figure 5. TTI way from P_1 to P_2

In [5] and [12], the authours introduce a metric and show that spheres of 3-dimensional analytical space furnished by these metric are the icosahedron and the truncated icosahedron. These metrics for $P_1 = (x_1, y_1, z_1)$, $P_2 = (x_2, y_2, z_2) \in \mathbb{R}^3$ are defined as follows:

$$d_I(P_1, P_2) = \max\{k_2(U + k_1 U^-), k_2(U + U^- + U^+)\}$$

$$d_{TI}(P_1, P_2) = \max\{k_2(U + U^- + U^+), U + k_1 U^-, \tfrac{3\sqrt{5}+27}{38} U + \tfrac{6\sqrt{5}-3}{19} U^+\},$$

where k_i for $i = 1,2$ are the same with definition 2.

Lemma 3 Let $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ be distinct two points in $\mathbb{R}^3$. U_{12} denote the maximum of quantities of $\{|x_1 - x_2|, |y_1 - y_2|, |z_1 - z_2|\}$. Then

$$\begin{aligned}
d_{TTI}(P_1, P_2) &\geq U + k_1 U^+, \\
d_{TTI}(P_1, P_2) &\geq k_2(U + U^- + U^+), \\
d_{TTI}(P_1, P_2) &\geq k_3 U + k_4 U^+, \\
d_{TTI}(P_1, P_2) &\geq k_5 U + k_6 U^+, \\
d_{TTI}(P_1, P_2) &\geq k_7 U + k_8 U^- + k_9 U^+, \\
d_{TTI}(P_1, P_2) &\geq k_{10} U + k_{11} U^- + k_{12} U^+.
\end{aligned}$$

Proof. Proof is trivial by the definition of maximum function.

Theorem 2 The distance function d_{TTI} is a metric. Also according to d_{TTI}, unit sphere is a truncated truncated icosahedron in $\mathbb{R}^3$.

Proof. One can easily show that the truncated truncated icosahedron distance function satisfies the metric axioms by similar way in Theorem 1.

Consequently, the set of all points $X=(x,y,z)\in\mathbb{R}^3$ that truncated truncated icosahedron distance is 1 from $O=(0,0,0)$ is

$$S_{TTI}=\left\{(x,y,z): \max\left\{\begin{matrix} U+k_1U^+, k_2(U+U^-+U^+), k_3U+k_4U^+, k_5U+k_6U^+, \\ k_7U+k_8U^-+k_9U^+, k_{10}U+k_{11}U^-+k_{12}U^+ \end{matrix}\right\}=1\right\},$$

where U are the maximum of quantities $\{|x|,|y|,|z|\}$. Thus the graph of S_{TTI} is as in the figure 6:

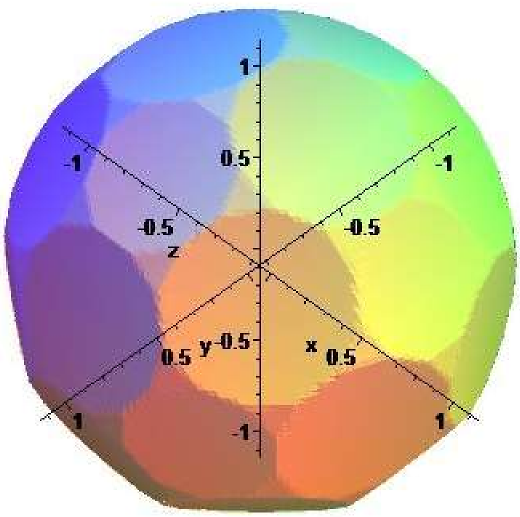

Figure 6. The unit sphere in terms of d_{TTI}: Truncated Truncated Icosahedron

Corollary 4 The equation of the truncated truncated icosahedron with center (x_0,y_0,z_0) and radius r is

$$\max\left\{\begin{matrix} U_0+k_1U_0^+, k_2(U_0+U_0^-+U_0^+), k_3U_0+k_4U_0^+, k_5U_0+k_6U_0^+, \\ k_7U_0+k_8U_0^-+k_9U_0^+, k_{10}U_0+k_{11}U_0^-+k_{12}U_0^+ \end{matrix}\right\}=r,$$

which is a polyhedron which has 92 faces and 180 vertices, where U_0 are the maximum of quantities $\{|x-x_0|,|y-y_0|,|z-z_0|\}$. Coordinates of the vertices are translation to (x_0,y_0,z_0) all circular shift of the three axis components and all possible +/- sign changes of each axis component of $(C_{19}r,0,r)$, $(C_1r,C_{19}r,C_{17}r)$, $(C_4r,C_0r,C_{16}r)$, $(C_5r,C_6r,C_{15}r)$, $(C_7r,C_3r,C_{14}r)$, $(C_2r,C_8r,C_{13}r)$, $(C_{19}r,C_{10}r,C_{12}r)$ and $(C_9r,C_0r,C_{11}r)$, where $C_0=\frac{81+1\sqrt{5}+(6\sqrt{5}-28)\sqrt{3+2\sqrt{5}}}{302}$,

$$C_1=\frac{17\sqrt{5}-29+(37\sqrt{5}-72)\sqrt{3+2\sqrt{5}}}{151}, C_2=\frac{169\sqrt{5}-235+(57-23\sqrt{5})\sqrt{3+2\sqrt{5}}}{604},$$

$$C_3=\frac{33\sqrt{5}-3+(29-17\sqrt{5})\sqrt{3+2\sqrt{5}}}{151}, C_4=\frac{26+16\sqrt{5}+(97\sqrt{5}-201)\sqrt{3+2\sqrt{5}}}{302},$$

$$C_5=\frac{133\sqrt{5}-67+(23\sqrt{5}-57)\sqrt{3+2\sqrt{5}}}{604},C_6=\frac{305-33\sqrt{5}+(17\sqrt{5}-29)\sqrt{3+2\sqrt{5}}}{604},$$

$$C_7=\frac{23+49\sqrt{5}+(80\sqrt{5}-172)\sqrt{3+2\sqrt{5}}}{302},C_8=\frac{299+33\sqrt{5}+(29-17\sqrt{5})\sqrt{3+2\sqrt{5}}}{604},$$

$$C_9=\frac{20+82\sqrt{5}+(63\sqrt{5}-143)\sqrt{3+2\sqrt{5}}}{302},C_{10}=\frac{137+3\sqrt{5}+(85-29\sqrt{5})\sqrt{3+2\sqrt{5}}}{302},$$

$$C_{11}=\frac{50+54\sqrt{5}+(171-69\sqrt{5})\sqrt{3+2\sqrt{5}}}{302},C_{12}=\frac{52+32\sqrt{5}+(43\sqrt{5}-100)\sqrt{3+2\sqrt{5}}}{151},$$

$$C_{13}=\frac{119+50\sqrt{5}+(20\sqrt{5}-43)\sqrt{3+2\sqrt{5}}}{302},C_{14}=\frac{67+18\sqrt{5}+(57-23\sqrt{5})\sqrt{3+2\sqrt{5}}}{151},$$

$$C_{15}=\frac{32+101\sqrt{5}+(43-20\sqrt{5})\sqrt{3+2\sqrt{5}}}{302},C_{16}=\frac{218+18\sqrt{5}+(57-23\sqrt{5})\sqrt{3+2\sqrt{5}}}{302},$$

$$C_{17}=\frac{131+69\sqrt{5}+(143-63\sqrt{5})\sqrt{3+2\sqrt{5}}}{302},C_{18}=1,\text{ and }C_{19}=\frac{84-18\sqrt{5}+(23\sqrt{5}-57)\sqrt{3+2\sqrt{5}}}{302}.$$

Lemma 4 Let l be the line through the points $P_1=(x_1,y_1,z_1)$ and $P_2=(x_2,y_2,z_2)$in the analytical 3-dimensional space and d_E denote the Euclidean metric. If l has direction vector (p,q,r), then

$$d_{TTI}(P_1,P_2)=\mu(P_1P_2)d_E(P_1,P_2)$$

where

$$\mu(P_1P_2)=\frac{\max\begin{Bmatrix}U_d+k_1U_d^{+},k_2(U_d+U_d^{-}+U_d^{+}),k_3U_d+k_4U_d^{+},k_5U_d+k_6U_d^{+},\\ k_7U_d+k_8U_d^{-}+k_9U_d^{+},k_{10}U_d+k_{11}U_d^{-}+k_{12}U_d^{+}\end{Bmatrix}}{\sqrt{p^2+q^2+r^2}},$$

U_d are the maximum of quantities $\{|p|,|q|,|r|\}$.

Proof. Equation of l gives us$x_1-x_2=\lambda p,y_1-y_2=\lambda q,z_1-z_2=\lambda r,\lambda\in\mathbb{R}$. Thus,

$$d_{TTI}(P_1,P_2)=|\lambda|\left(\max\begin{Bmatrix}U_d+k_1U_d^{+},k_2(U_d+U_d^{-}+U_d^{+}),k_3U_d+k_4U_d^{+},k_5U_d+k_6U_d^{+},\\ k_7U_d+k_8U_d^{-}+k_9U_d^{+},k_{10}U_d+k_{11}U_d^{-}+k_{12}U_d^{+}\end{Bmatrix}\right)$$

where U_d are the maximum of quantities $\{|p|,|q|,|r|\}$, and $d_E(A,B)=|\lambda|\sqrt{p^2+q^2+r^2}$ which implies the desired result.

The above lemma says that d_{TTI}-distance along any line is some positive constant multiple of Euclidean distance along same line. Thus, one can immediately state the following corollaries:

Corollary 5 If P_1, P_2 and X are any three collinear points in $\mathbb{R}^3$, then

$d_E(P_1, X) = d_E(P_2, X)$ if and only if $d_{TTI}(P_1, X) = d_{TTI}(P_2, X)$.

Corollary 6 If P_1, P_2 and X are any three distinct collinear points in the real 3-dimensional space, then

$$d_{TTI}(X, P_1)/d_{TTI}(X, P_2) = d_E(X, P_1)/d_E(X, P_2).$$

That is, the ratios of the Euclidean and$d_{TTI} -$ distances along a line are the same.

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