Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-2 (2019) 369-377

On The Inverse Sum In Degree Index and Co Index

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Received: 30.11.2018; Accepted: 15.05.2019   http://dx.doi.org/10.17776/csj.490918

Abstract. The inverse sum in degree index of G is specified the degrees d_i and d_j. Some bounds are found for inverse sum in degree index in this study. Also, some definitions and relations are obtained in terms of degrees.

Keywords: İnverse sum in degree index, co index.

Derece Endeksinde ve Ko Endeksinde Ters Toplam

Özet. G' nin derece endeksinde ters toplam d_i ve d_j dereceleri ile belirtilir. Bu çalışmada, derece endeksinde toplam için bazı sınırlar bulunur. Ayrıca, dereceler açısından bazı tanımlar ve bağıntılar elde edilir.

Anahtar Kelimeler: Derece endeksinde ters toplam, Eş endeks.

1. INTRODUCTION

Let G be a simple, connected graph on the vertex set V(G) and the edge set E(G). For $v_i \in V(G)$, the degree of vertex v_i denoted by d_i, the maximum degree is denoted by Δ and the minimum degree is denoted by δ.

The inverse sum in degree matrix $[ISI](G)$ of graphs is defined as

$$[ISI]_{ij} = \begin{cases} \dfrac{d_i + d_j}{d_i d_j} & if\ i\ adjacent\ to\ j \\ 0 & otherwise. \end{cases}$$

The eigenvalues of $[ISI](G)$ are denoted by δ^+. New bounds for these eigenvalues are reported in terms of the degrees.

The Inverse Sum In Degree Index (ISI) index of G is defined as

$$\sum_{v_i v_j \in E(G)} \frac{d_i + d_j}{d_i d_j}.$$

(See [2] for details.)

In this study, different bounds are set using the Estrada index and Zagreb index for ISI index.

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The Estrada index of graph G is explained as

$$EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$$

where λ is the eigenvalue of adjacency matrix of G. ([1], [10])

The Zagreb co index of G is described in [5], [7] as

$$\overline{Z_1}(G) = \sum_{v_iv_j \notin E(G),} (d_G(i) + d_G(j)),$$

$$\overline{Z_2}(G) = \sum_{v_iv_j \notin E(G),} (d_G(i) d_G(j)).$$

The Harmonic index of G is specified in [8] as

$$H(G) = \sum_{v_iv_j \in E(G)} \frac{2}{d_i + d_j}.$$

Considering these topological indices, Estrada inverse sum in degree index and inverse sum in degree co index are defined. Indeed, some inequalities are obtained concerned with these indices.

(See [6] for more details deal with this topic.)

2. PRELIMINARIES

In this section, some lemmas and theorems that are needed in main results will be given.

Lemma 2.1. [9]

Let $M = (m_{ij})$ be an nxn irreducible nonnegative matrix and $\lambda_1(M)$ be the greatest eigenvalue with $R_i(M) = \sum_{j=1}^{m} m_{ij}$. Then,

$$(minR_i(M): 1 \leq i \leq n) \leq \lambda_1(M) \leq (maxR_i(M): 1 \leq i \leq n)$$

Lemma 2.2. [4]

If G is a simple connected graph and $\lambda_1(G)$ is the spectral radius , then

$$\lambda_1(G) \leq max\left(\sqrt{m_i m_j}: 1 \leq i\,, j \leq n\,, v_{i\,,} v_j \in E\right)$$

Theorem 2.1. [3]

If $a_i, b_i \in R^+$, $1 \le i \le n$, then

$$\sum_{i=1}^{n} a_i^{2} \sum_{i=1}^{n} b_i^{2} - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{4}(M_1M_2 - m_1m_2)^2$$

where $M_1 = \max_{1\le i \le n} a_i$, $M_2 = \max_{1\le i \le n} b_i$, $m_1 = \min_{1\le i \le n} a_i$, $m_2 = \min_{1\le i \le n} b_i$.

Theorem 2.2. [5]

Let *G* be a graph with *n* vertices and *m* edges. Then,

$$Z_1(\bar{G}) = Z_1(G) + n(n-1)^2 - 4m(n-1)$$

$$\overline{Z_1}(G) = 2m(n-1) - Z_1(G) = \overline{Z_1}(\bar{G}).$$

Lemma 2.3. [7]

If *G* is a regular graph, then

$$Z_1(G) \ge \frac{4m^2}{n},$$

$$Z_2(G) \ge \frac{4m^3}{n^2},$$

$$\overline{Z_1}(G) \le \frac{-4m^2}{n} + 2m(n-1),$$

$$\overline{Z_2}(G) \le 2m^2\left(1 - \frac{2m}{n^2} - \frac{1}{n}\right).$$

3. MAIN RESULTS

3.1. Inverse Sum In Degree Index and Estrada Inverse Sum In Degree Index

Firstly, a relation is given for the largest eigenvalue of *ISI* matrix including degrees in this subsection. After, an inequality is obtained for *ISI* index using this relation. In addition, Estrada inverse sum in degree index is defined and some relations are found in terms of degrees and vertices.

Theorem 3.1.1. Let *G* be graph on *n* vertices and *m* edges. Then,

$$ISI(G) \ge \sqrt{\left(Z_2(G)\,H(G)\right)^2 - \frac{n^2}{4}\left(\frac{\Delta^3 - \delta^3}{2\Delta\delta}\right)}.$$

Proof. Let choose $a_k = d_i d_j$, $b_k = \frac{1}{d_i + d_j}$, $M_1 = \Delta^2$, $m_1 = \delta^2$, $M_2 = \frac{1}{2\delta}$, $m_2 = \frac{1}{2\Delta}$.

By Theorem 2.1, it is seen that

$$\sum_{v_iv_j\in E(G)}(d_id_j)^2\sum_{v_iv_j\in E(G)}\left(\frac{1}{d_i+d_j}\right)^2-\left(\sum_{v_iv_j\in E(G)}\frac{d_id_j}{d_i+d_j}\right)^2\leq\frac{n^2}{4}\left(\frac{\Delta^2}{2\delta}-\frac{\delta^2}{2\Delta}\right).$$

If necessary organizing is applied, this inequality is obtained as follows:

$$\left(\sum_{v_iv_j\in E(G)}(d_id_j)\right)^2\left(\sum_{v_iv_j\in E(G)}\left(\frac{1}{d_i+d_j}\right)\right)^2-\frac{n^2}{4}\left(\frac{\Delta^2}{2\delta}-\frac{\delta^2}{2\Delta}\right)$$

$$\leq\left(\sum_{v_iv_j\in E(G)}\frac{d_id_j}{d_i+d_j}\right)^2.$$

Putting the definitions in the above inequality, it gets

$$\left(Z_2(G)\right)^2\left(H(G)\right)^2-\frac{n^2}{4}\left(\frac{\Delta^3-\delta^3}{2\Delta\delta}\right)\leq(ISI\ (G))^2.$$

Hence,

$$ISI(G)\geq\sqrt{\left(Z_2(G)\,H(G)\right)^2-\frac{n^2}{4}\left(\frac{\Delta^3-\delta^3}{2\Delta\delta}\right)}.$$

Lemma 3.1.1. For a simple connected graph of $ISI\ (G)$,

$$\gamma_1^+\leq\frac{\Delta}{n^{1/n}\sqrt{(d_i^n+\Delta)(d_j^n+\Delta)}}$$

where Δ is the maximum degree of G.

Proof. Let $D(G)^{-1}ISI(G)D(G)=Q(G)$ and $X=(x_1,\ x_2,\dots,x_n)^T$ be an eigen vector of $Q(G)$ corresponding to an eigen value γ^+. Also, $x_i=1$ and $0<x_k\leq1$ for every k. Let $x_j=max_k(x_k\colon v_iv_k\in E,\ i\ is\ adjacent\ to\ k)$. It is known that $(Q(G))X=\gamma_1^+(G)X$. If i_th equation from above equation is taken, then $\gamma_1^+(G)x_i=\sum_k\left(\frac{d_id_k}{d_i+d_k}\right)x_k=\left(d_i\sum_k\frac{d_k}{d_i+d_k}\right)x_k$. By the Aritmetic-Geometric mean inequality, it gives

$$\frac{\sum_k \frac{d_k}{d_i+d_k}}{n} \geq \left(\frac{\prod_{k=1}^{n} \frac{d_k}{d_i+d_k}}{n}\right)^{1/n} \geq \frac{\left(\frac{\prod_{k=1}^{n} d_k}{\prod_{k=1}^{n}(d_i+d_k)}\right)^{1/n}}{n^{1/n}}$$

$$\geq \frac{\frac{(\Delta^n)^{1/n}}{\left(d_i^n+\prod_{k=1}^{n} d_k\right)^{1/n}}}{n^{1/n}} \geq \frac{\Delta}{n^{1/n}\left(d_i^n+(\Delta^n)^{1/n}\right)}.$$

Using the Lemma 2.1,

$$\gamma_1^+(G) \leq \frac{d_i \Delta}{n^{1/n}\left(d_i^n+\Delta\right)}$$

The *j*_th equation of same equation has

$$\gamma_1^+(G) \leq \frac{d_j \Delta}{n^{1/n}\left(d_j^n+\Delta\right)}$$

From Lemma 2.2, it is expressed that

$$\gamma_1^+(G) \leq \sqrt{\left(\frac{d_i \Delta}{n^{1/n}\left(d_i^n+\Delta\right)}\right)\left(\frac{d_j \Delta}{n^{1/n}\left(d_j^n+\Delta\right)}\right)}.$$

Hence,

$$\gamma_1^+ \leq \frac{\Delta}{n^{1/n}\sqrt{\left(d_i^n+\Delta\right)\left(d_j^n+\Delta\right)}}.$$

Since, $\Delta = d_1 \geq d_2 \geq, \dots, \geq d_n = \delta$, it is clear that

$$\gamma_1^+ \leq \frac{\Delta}{n^{1/n}\sqrt{(\Delta^n+\Delta)(\delta^n+\Delta)}}.$$

Definition 3.1.1. Let G be a graph and $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+$ be eigenvalues of inverse sum in degree matrix of G. Estrada inverse sum in degree index is defined as

$$E_{\iota s \iota} = \sum_{j=1}^{n} e^{\gamma_j^+}.$$

Theorem 3.1.2. Let G be a graph with n vertices and $E_{\iota s \iota}$ be an Estrada inverse sum in deg index. Then,

$$E_{\iota s \iota} \geq e^{K} + \frac{(n-1)}{e^{1/n-1}}$$

where $K = \frac{\Delta}{n^{1/n}\sqrt{(\Delta^n+\Delta)(\delta^n+\Delta)}}$.

Proof. $E_{\iota s\iota} = \sum_{j=1}^{n} e^{\gamma_j^+} \geq e^{\gamma_1^+} + (n-1)\left(\prod_{j=2}^{n} e^{\gamma_j^+}\right)^{1/n-1}$ using the Aritmetic-Geometric mean inequality. Since, $\sum_{i=1}^{n} e^{\gamma_j^+} = 0$ then $E_{\iota s\iota} \geq e^{\gamma_j^+} + \frac{(n-1)}{e^{1/n-1}}$. It is known that $\gamma_j^+ \leq K$. Hence,

$$E_{\iota s\iota} \geq e^K + \frac{(n-1)}{e^{1/n-1}}.$$

Theorem 3.1.3. Let G be a graph with n vertices and $E_{\iota s\iota}$ be an Estrada inverse sum in deg. index. Then,

$$E_{\iota s\iota} \leq \sqrt{-Kn\sum_{k\geq 0}^{\infty}\frac{2^k}{k!}}.$$

Proof. It is easy to see that $\frac{1}{n}\sum_{j=1}^{n}\left(e^{\gamma_j^+}\right)^2 \geq \left(\frac{1}{n}\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2$. On the other hand, $\frac{1}{n}\sum_{j=1}^{n} e^{2\gamma_j^+} \geq \frac{1}{n^2}{E_{\iota s\iota}}^2$. Hence,

$$n.\sum_{k\geq 0}^{\infty}\frac{1}{k!}\sum_{j=1}^{n}\left(2\gamma_j^+\right)^k \geq {E_{\iota s\iota}}^2$$

and thus,

$${E_{\iota s\iota}}^2 \leq n.\sum_{k\geq 0}^{\infty}\frac{2^k}{k!}\sum_{j=1}^{n}\left(\gamma_j^+\right)^k.$$

Knowing that $\gamma_1^+ \geq \cdots \geq \gamma_n^+$ and $\gamma_1^+ \leq K$, it is obtained that

$${E_{\iota s\iota}}^2 \leq n.\sum_{k\geq 0}^{\infty}\frac{2^k}{k!}.n.K^k.$$

It is clear that the equality holds

$$E_{\iota s\iota} \leq \sqrt{n^2.\sum_{k\geq 0}^{\infty}\frac{(2K)^k}{k!}}.$$

Theorem 3.1.4. Let G be a graph with n vertices and $E_{\iota s\iota}$ be an Estrada inverse sum in degree index. Then,

$$E_{\iota s\iota} \leq \sqrt{e^{2K} - 2e^K} + e^K.$$

Proof. $\left(E_{\iota s\iota} - e^{\gamma_1^+}\right)^2 = \left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 - 2\left(\sum_{j=1}^{n} e^{\gamma_j^+} e^{\gamma_1^+}\right) + e^{2\gamma_1^+}$

$$\leq \left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 - 2ne^{\gamma_1^+}\left(\prod_{j=1}^{n} e^{\gamma_j^+}\right)^{1/n} + e^{2\gamma_1^+}.$$

Since $\left(\sum_{j=1}^{n} e^{\gamma_j^+}\right)^2 = \left(\sum_{k\geq 0}^{\infty}\frac{1}{k!}\sum_{j=1}^{n}(\gamma_j^+)^k\right)^2 \leq \left(\sum_{k\geq 0}^{\infty}\frac{1}{k!}\left(\sum_{j=1}^{n}\gamma_j^+\right)^k\right)^2$ and $\sum_{j=1}^{n}\gamma_j^+ = 0$, then

$$\left(E_{\iota s\iota} - e^{\gamma_1^+}\right)^2 \leq -2ne^{\gamma_1^+} + e^{2\gamma_1^+} = e^{\gamma_1^+}\left[e^{\gamma_1^+} - 2n\right].$$

The inequality states that

$$E_{\iota s\iota} \leq \sqrt{e^{\gamma_1^+}\left(e^{\gamma_1^+} - 2n\right)} + e^{\gamma_1^+}.$$

In the sequel, Theorem 3.1.2 says that

$$E_{\iota s\iota} \leq \sqrt{e^{K}(e^{K} - 2n)} + e^{K}.$$

3.2 Inverse Sum In Degree Co Index

In this subsection, ISI co index is described and different bounds are yielded concerned with Zagreb co indices, the vertices and the edges.

Definition 3.2.1.

Let *G* be a simple, connected graph. ISI co index is defined as follows:

$$ISI(\bar{G}) = \sum_{v_i v_j \notin E(G)} \frac{d_G(i)d_G(j)}{d_G(i) + d_G(j)}.$$

Theorem 3.2.1. Let $ISI(\bar{G})$ be the complement of inverse sum in degree index. If G is regular then,

$$ISI\,(\bar{G}) \leq \frac{(n-1)^2\left[\binom{n}{2} - m\right] - (n-1).\left(-\frac{4m^2}{n} + 2m(n-1)\right) + \left(2m^2\left(1 - \frac{2m}{n^2} - \frac{1}{n}\right)\right)}{2.(n-1).\left[\binom{n}{2} - m\right] + \frac{4m^2}{n} - \left(2m(n-1)\right)}.$$

Proof. By the definition of $ISI\,(\bar{G})$; $ISI\,(\bar{G}) = \sum_{v_i v_j \in E(\bar{G})} \frac{d_{\bar{G}}(i)d_{\bar{G}}(j)}{d_{\bar{G}}(i) + d_{\bar{G}}(j)}$. Since $d_{\bar{G}}(i) = (n - 1 - d_i)$ and $d_{\bar{G}}(j) = \left(n - 1 - d_j\right)$, then

$$ISI\ (\bar{G}) = \sum_{v_iv_j \in E(\bar{G})} \frac{(n-1-d_i).(n-1-d_j)}{(n-1-d_i)+(n-1-d_j)}$$

$$= \sum_{v_iv_j \in E(\bar{G})} \frac{(n-1)^2-(n-1)(d_i+d_j)+d_id_j}{2(n-1)-(d_i+d_j)}$$

$$\leq \frac{\sum_{v_iv_j \in E(\bar{G})}(n-1)^2-(n-1)(d_i+d_j)+d_id_j}{\sum_{v_iv_j \in E(\bar{G})} 2(n-1)-(d_i+d_j)}$$

$$ISI\ (\bar{G}) \leq \frac{(n-1)^2\sum_{v_iv_j \in E(\bar{G})} 1-(n-1)\sum_{v_iv_j \in E(\bar{G})}(d_i+d_j)+\sum_{v_iv_j \in E(\bar{G})} d_id_j}{2.(n-1).\left[\binom{n}{2}-m\right]-\sum_{v_iv_j \in E(\bar{G})}(d_i+d_j)}.$$

Because, G has $\binom{n}{2}-m$ edges. By Theorem 2.2, it is stated that

$$ISI\ (\bar{G}) \leq \frac{(n-1)^2\left[\binom{n}{2}-m\right]-(n-1)Z_1\ (\bar{G})+Z_2\ (\bar{G})}{2.(n-1).\left[\binom{n}{2}-m\right]-Z_1\ (\bar{G})}$$

Since $Z_1\ (\bar{G}) = \overline{Z_1}\ (G)$, then

$$ISI\ (\bar{G}) \leq \frac{(n-1)^2\left[\binom{n}{2}-m\right]-(n-1).\overline{Z_1}\ (G)+\overline{Z_2}\ (G)}{2.(n-1).\left[\binom{n}{2}-m\right]-\overline{Z_1}\ (G)}.$$

Using the Lemma 2.3, it is concluded that

$$ISI\ (\bar{G}) \leq \frac{(n-1)^2\left[\binom{n}{2}-m\right]-(n-1).\left(-\frac{4m^2}{n}+2m(n-1)\right)+\left(2m^2\left(1-\frac{2m}{n^2}-\frac{1}{n}\right)\right)}{2.(n-1).\left[\binom{n}{2}-m\right]+\frac{4m^2}{n}-\left(2m(n-1)\right)}.$$

Corollary 3.2.1. Let $ISI(\bar{G})$ be the complement of inverse sum in degree index. If G is regular then,

$$\overline{ISI}\ (\bar{G}) \leq \frac{(n-1)m-(n-1).\left(-\frac{4m^2}{n}+2m(n-1)\right)+\left(2m^2\left(1-\frac{2m}{n^2}-\frac{1}{n}\right)\right)}{2.(n-1)m+\frac{4m^2}{n}-\left(2m(n-1)\right)}$$

Proof. Applying similar steps to the Theorem 3.2.1, it is obtained that

$$\overline{ISI}\ (\bar{G}) = \sum_{v_iv_j \notin E(\bar{G})} \frac{d_{\bar{G}}(i).d_{\bar{G}}(j)}{d_{\bar{G}}(i)+d_{\bar{G}}(j)}$$

$$= \sum_{v_iv_j \in E(G)} \frac{(n-1)^2-(n-1)(d_i+d_j)+d_id_j}{2(n-1)-(d_i+d_j)}$$

$$\leq \frac{(n-1)^2 m - (n-1).\overline{Z_1}\,(G) + \overline{Z_2}\,(G)}{2.(n-1)m - \overline{Z_1}\,(G)}.$$

By Lemma 2.3, it is resulted that

$$\overline{ISI}\,(\bar{G}) \leq \frac{(n-1)^2 m - (n-1).\left(-\frac{4m^2}{n} + 2m(n-1)\right) + \left(2m^2\left(1 - \frac{2m}{n^2} - \frac{1}{n}\right)\right)}{2.(n-1)m + \frac{4m^2}{n} - \left(2m(n-1)\right)}$$

4. CONCLUSION

In this study, the inverse sum in degree index is expanded, the Estrada inverse sum in degree index is defined and some bounds are found deal with these indices. In the sequel, inverse sum in degree co index is described and some inequalities are obtained in terms of the edges and vertices.

5. ACKNOWLEDGEMENT

The author would like thank for the valuable suggestions of referees.

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