Cumhuriyet Science Journal
CSJ

e-ISSN: 2587-246X
ISSN: 2587-2680

Cumhuriyet Sci. J., Vol.40-2 (2019) 340-346

# On The Synchronization of Van Der Pol-Duffing Oscillator

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Received: 18.07.2018; Accepted: 21.04.2019 http://dx.doi.org/10.17776/csj.445260

**Abstract.** The most well known property of chaotic systems is their sensitivity to initial conditions. In this work the criterion presented in literature for synchronizing two chaotic systems is applied to a system consisting of two Van der Pol-Duffing oscillators. First, the route to chaos is investigated for the Duffing oscillator. Furthermore, the Lyapunov function approach is used to design a high dimensional chaotic system. Then certain subsystems of a nonlinear chaotic system are synchronized by linking them with a common signal. Synchronization has been observed when there exists an asymptotic stability and an appropriate Lyapunov function, also by computing all the Lyapunov exponents and Kolmogorov entropy.

***Keywords*:** Chaotic systems, Lyapunov exponent, Kolmogorov entropy.

# Van Der Pol-Duffing Osilatöründe Senkronizasyon

**Özet.** Kaotik sistemlerin bilinen en önemli özelliği başlangıç koşullarına duyarlılıktır. Bu çalışmada, literatürde sunulmuş iki kaotik sistemin senkronizasyon kriteri, iki Van Der Pol-Duffing osilatörüne uygulandı. İlk olarak Duffing osilatörü için kaos yolu araştırıldı. Sonra Lyapunov fonksiyon yaklaşımı yüksek boyutlu kaotik bir sistemin oluşturulmasında kullanıldı. Daha sonra doğrusal olmayan bir kaotik sistemin belirli alt sistemleri ortak bir sinyal ile bağlanarak senkronize edildi. Senkronizasyon, uygun Lyapunov fonksiyonu ve asimptotik kararlığının varlığı ile gözlendi.

## 1. INTRODUCTION

Two identical autonomous chaotic systems started at virtually identical initial conditions would be observed to quickly diverge from one another [1]. That is, their trajectories become uncorrelated, even if each maps out the same attractor in phase space. It is thus impossible to build up two identical, chaotic, synchronized and separated systems. Pecora and Carroll showed that two chaotic systems can be synchronized by dividing each of them into two subsystems, namely, a drive subsystem and a response subsystem and by keeping the variable values of the drive subsystems the same [2]. When the Lyapunov exponents are all negative for the response subsystem, synchronization is achieved [12].

A possible application of synchronization of chaotic signal is to implement a secure communication system. Since chaotic signals are usually broadband, noise like, and difficult to predict, they can be used for masking information bearing waveforms. A chaotic masking signal is added at the transmitter to a message, and at the receiver the masking is reproduced and removed from the received signal [3]. Also, the Van der Pol-Duffing oscillator can be used as model in

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physics, engineering electronics, biology, neurology and many other disciplines [13].

Pecora and Carroll have extended the synchronization of chaotic circuits to periodically forced circuits [4]. When one of the subsystem is periodically forced, all the Lyapunov exponents for the subsystem are not negative, the periodical forcing introduces a zero exponent. To make the zero Lyapunov exponent in the response system negative, they use a phase correction circuit to match the phase in a response circuit to the phase in a drive circuit. They also show that the chaotic behavior is a good candidate to keep period-multiplied systems in phase (synchronized) [5]. There is a need for this when many devices are coupled into arrays to increase the sensitivity or power output beyond what one device would provide.

A method of controlling nonlinear and chaotic systems which can synchronize the phase space trajectory to a desired unstable orbit is discussed in [6]. This method utilizes the principles of adaptive control and time dependent changes in the system parameters. The system parameter values are changed according to the deviations of the system variables from desired orbit and the deviations of the controlled parameters from their values for the desired orbit.

A method of transmitting signals in a secure way through chaos synchronization in a physical model has been discussed in [7]. A criterion for synchronization of chaos based on the asymptotic stability has been created and the model developed has been proved to be useful in analog signal transmission.

## 2. THEORETICAL FRAMEWORK

### 2.1. Asymptotic Stability

Asymptotic stability commonly appears in linear damped forced systems. When the transient part of the system response is completed and only the forced part remains, the system response no longer depends on the initial conditions, namely, it has converged to stable point or a desired orbit. This state of forgetting the initial conditions, steady state response in forced systems, is known as asymptotic stability.

There is a very close relationship between synchronization and asymptotic stability. The term synchronization denotes an eventual coincidence of two different systems starting with different initial conditions. Asymtotic, however, indicates a case where both systems converge, after sufficient time, to the same eventual state without respect to the initial conditions.

Since chaotic systems very much remember the initial conditions, asymptotic stability for the total chaotic system would be almost impossible. But, it is reasonable that a subsystem of a total chaotic system can exhibit a characteristic of asymptotic stability. Such a system can be considered to be constructed from two parts, namely, master and slave. Now this master-slave system can operate synchronously.

An outline of the Differential Transformation Method (DTM) can be given as follows. Let $x(t)$ be an analytical function in the domain $D$ and $t = t_0$ be a point in $D$. The function $x(t)$ can be represented by using a $t_0$-centered power series. The k-th derivative of $x(t)$ is defined as:

### 2.2. Synchronization of Systems

The phenomena of synchronization is that the slave system knows which state (attractor) to go to when driven (stimulated) by a parameter signal. A dynamical system may be described by the following ordinary differential equation

$$\dot{x} = f(x, \alpha). \quad (1)$$

Where $x$ and $f$ are n-dimensional vectors of the form $x = (x_1,\ldots,x_n)$ and $f(x, \alpha) = \{f_1(x,\alpha), \ldots, f_n(x,\alpha)\}$ respectively where α is a set of parameters such that the system lies in the chaotic regime. The desired orbit may be chaotic or periodic. The system is then divided into two subsystem a drive subsystem $x_d = (x_1,\ldots,x_m)$ and a response subsystem $x_r = (x_{m+1}, \ldots, x_n)$ such that $x = (x_d, x_r)$.

A master system $u = (u_d, u_r)$ may be governed by

$$\dot{u_d} = f_d(u_d, u_r, \alpha), \quad (2)$$

$$\dot{u_r} = f_r(u_d, u_r, \alpha). \quad (3)$$

Furthermore, let the slave system $u' = (u'_d, u'_r)$ be governed by

$$\dot{u'_d} = f_d(u'_d, u'_r, \alpha'), \quad (4)$$

$$\dot{u'_r} = f_r(u'_d, u'_r, \alpha') \quad (5)$$

If the derive subsystem of the master is allowed to drive that of the slave (see Fig. 1), then

$$u'_d = u_d \quad (6)$$

In order to lock the given system onto a given unstable orbit, start the evolution of the system with an initial condition $u'_o = (u'_{do}, u'_{ro})$ which slightly deviates from the desired orbit such that $u'_d = u_d$ but $u'_r = u_r + \delta u_r$. The drive variable $u'_d$ now evolves according to Eq. (2) and the response variable $u'_r$ evolves according to the following equation.

$$\dot{u'_r} = f_r(u_d, u'_r, \alpha') \quad (7)$$

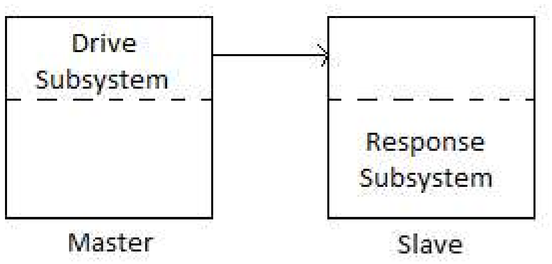


**Figure 1.** Block diagram of a master-slave system.

Thus the drive variables of the slave are continuously set to those of the desired orbit while the response variables are allowed to evolve freely. The total system will settle down onto the desired orbit, when the drive variables are such that the Lyapunov exponents of the response system are all negative. In this case the difference $u'_r - u_r = \Delta u_r$ goes to zero as $t \rightarrow \infty$.

### 2.3. A Model System-Duffing Oscillator

The Duffing oscillator with a double-well potential can be described by a nonlinear Langevin equation of the form

$$m\ddot{x} + \gamma\dot{x} + \frac{d\phi}{dx} = f(t), \quad (8)$$

where $\gamma$ is the damping constant, φ is a double-well potential, and $f$ is a random force or white noise [8,9].

The physical realization of the Duffing oscillator circuit is shown in Fig. 2. The circuit element denoted by N represents a nonlinear negative resistor and can be constructed by using a set of diodes and an operational amplifier. The unfolding parameter is represented by the parameter $\gamma$ which is controlled by the offset votage of the amplifier. Such a nonlinear element can be described as

$$I_N(V) = \gamma + aV + bV^3, \quad (9)$$

where $a < 0$ and b> 0. The circuit equations are easily obtained by Kirchoff's laws to the various branches of the circuit as follows

$$\dot{x} = -m(x^3 - \alpha x + \mu - y), \quad (10a)$$

$$\dot{y} = x - y - z, \quad (10b)$$

$$\dot{z} = \beta y, \quad (10c)$$

where differentiation is with respect to time. Here x, y and z correspond to the rescaled form of the voltage across capacitor $C_1$, the voltage across capacitor $C_2$, and the current through $L$, respectively. The rescaled circuit parameters m, $\alpha$, β, and μ are given as

$$m = \frac{C_2}{C_1}, \quad \alpha = -(1+\alpha r), \quad \beta = \frac{C_2 r^2}{L},$$

$$\mu = (br^3)^{1/2} V \quad (11)$$

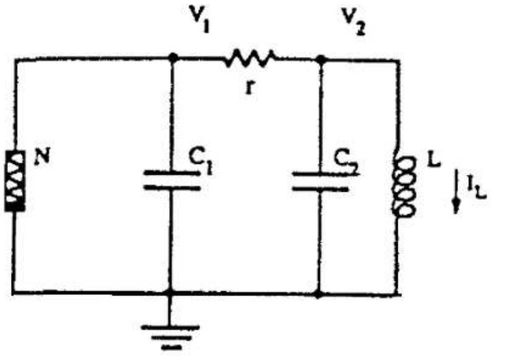


**Figure 2.** Equivalent circuit of the Duffing oscillator.

Here μ unfolds the double well, if μ = 0 then the wells have equal probability and the system becomes

$$\dot{x}=-\mathrm{m}(x^3-\alpha x-y), \quad (12a)$$

$$\dot{y}=\mathrm{x-y-z}, \quad (12b)$$

$$\dot{z}=\beta \mathrm{y}. \quad (12c)$$

There are several types of synchronization. One of them holds for systems which are not chaotic, but follow periodic limit cycles. Here a chaotic synchronization approach is introduced together with the necessary and sufficient condition for synchronization of linear or nonlinear systems. This approach exploits an appropriate Lyapunov function to globally establish the asymptotic stability of subsystem. The Lyapunov function can be further used to create a high–dimensional chaotic system, with a nonlinear subsystem. Here we obtained Lyapunov exponents ($\lambda_1$ = 2.13, $\lambda_2$ = 0.44, $\lambda_3$ = 0.00, $\lambda_4$= 0.00, $\lambda_5$ = -75.52). Also we obtained Kolmogorov entropy and Lyapunov dimension ( $h_K$=2.48 and $D_L$ = 4.04) [13].

For a slave system governed by the subsystem (12a), the following set of equations can be written

$$\dot{x}'=\mathrm{x}, \quad (13a)$$

$$\dot{y}'=\mathrm{x}-y'-z', \quad (13b)$$

$$\dot{z}'=\beta y'. \quad (13c)$$

Considering the differences between the unprimed and primed quantities and starring them,

$$\dot{y}^*=-y^*-z^*, \quad (14a)$$

$$\dot{z}^*=\beta \mathrm{y}. \quad (14b)$$

If the Lyapunov function is chosen as in [7], i.e., as

$$\mathrm{E}=\tfrac{1}{2}[(\beta y^*+z^*)^2+\beta y^{*2}+(1+\beta)z^{*2}], \quad (15)$$

then

$$\dot{E}=-\beta(y^{*2}+z^{*2})\leq 0. \quad (16)$$

The equality sign applies only at the origin, therefore the subsystem [(12b) and (12c)] is globally asymptotically stable. Thus the master and slave systems eventually synchronize.

## 3. CALCULATIONS

The choice of system form should be based on the fact that master and slave systems would be synchronous. Thus a 5-dimensional Duffing system (17a-e) which is derived from the system (12a-c) has been taken.

$$\dot{x}_1=-\mathrm{m}x_1^3+\alpha\mathrm{m}x_1+\mathrm{m}x_2+x_5, \quad (17a)$$

$$\dot{x}_2=x_1-x_2-x_3, \quad (17b)$$

$$\dot{x}_3=\beta x_2, \quad (17c)$$

$$\dot{x}_4=-x_4^3+x_5 \quad (17d)$$

$$\dot{x}_5=-x_1-x_4-x_5, \quad (17e)$$

where $x_i$ = $x_i$(t), i=1,2,3,4,5. The slave has an identical set of equations with the master expect the signal $x_3$ which is common. Notice that the subsystems are nonlinear, although this is not always necessary. For dissipative system it must be ensured that the divergence of the system is negative. This condition is readily satisfied by the system (17a-e). If all the Lyapunov exponents for the slave system are less than zero, then after initial transients decay, $x_1'$, $x_2'$, $x_4'$, and $x_5'$ will be equal to $x_1'$, $x_2'$, $x_4'$, and, $x_5'$; that is, the subsystems synchronize. Thus, the first step is to calculate the rest points of the system (17), and the corresponding Jacobian eigenvalues. The Jacobian matrix can be formed as

| | | | | |
|---|---|---|---|---|
| $-3m\boldsymbol{x_1^2}$ +αm | m | 0 | 0 | 1 |
| 1 | −1 | −1 | 0 | 0 |
| 0 | $\beta$ | 0 | 0 | 0 |
| 0 | 0 | 0 | $-3x_4^2$ | 1 |
| −1 | 0 | 0 | −1 | −1 |

where $x_1$ , $x_2$, $x_3$, $x_4$, $x_5$ denote the rest points. When m = 100, $\alpha$ = .35 and β = 300, the system has been found chaotic because it has at least one positive Lyapunov exponent. At the same time, this situation has been observed from the phase portrait of the dynamical variables $x_1$ and $x_2$. For m = 100, α =0.11 and β = 300, the solution of the system is found to be periodic, Fig.3b. For the system (17) the rest points and the corresponding Jacobian eigenvalues are shown in Table 1.

**Table 1.** Rest points and Jacobian eigenvalues of Eq. (17).

| **Rest Points** | | | | | **Jacobian Eigenvalues** | | |
|---|---|---|---|---|---|---|---|
| $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | | | |
| 0.00 | 0.00 | 0.00 | 0.00 | 0.00 | 37.1376 | $-1.5828 \pm 16.7355i$ | $-48.60 \pm 0.8743i$ |
| 0.59 | 0.00 | 0.59 | 0.479 | 0.11 | −70.7669 | $-0.8524 \pm 0.9868i$ | $0.1757 \pm 17.154i$ |
| −0.59 | 0.00 | −0.59 | 0.479 | 0.11 | −70.7669 | $-0.8524 \pm 0.9868i$ | $0.1757 \pm 17.154i$ |

The slave system driven by $x_3$ is given as

$$\dot{x}_1' = -\, m x_1'^3 + \alpha m x_1' + m x_2' + x_5', \tag{18a}$$

$$\dot{x}_2' = x_1' - x_2' - x_3, \tag{18b}$$

$$\dot{x}_4' = -\, x_4'^3 + x_5', \tag{18c}$$

$$\dot{x}_5' = -\, x_1' - x_4' - x_5' \,. \tag{18d}$$

Because Eqs.(17a-e) and (18a-d) establish a unique dynamical system it is possible to observe chaos in the slave system even though all the Lyapunov exponents are negative. If the differences between the corresponding dynamic variables of the master and slave are established and starred, then

$$\dot{x}_1^* = m(\alpha - k_1)x_1^* + m x_2^* + x_5^*, \tag{19a}$$

$$\dot{x}_2^* = x_1^* - x_2^*, \tag{19b}$$

$$\dot{x}_4^* = -\, k_2\, x_4^* + x_5^*, \tag{19c}$$

$$\dot{x}_5^* = -\, x_1^* - x_4^* - x_5^* \,. \tag{19d}$$

where $k_1 = ( x_1^2 + x_1\, x_1' + x_1'^2 ) \geq 0$ and $k_2 = ( x_4^2 + x_4 x_4' + x_4'^2 ) \geq 0$.

Consider the Lyapunov function given by

$$E = \frac{1}{2}( x_1^{*2} + x_2^{*2} + x_4^{*2} + x_5^{*2} ). \tag{20}$$

The derivative of Eq. (20) with respect to time is given by

$$\dot{E} = (m(\alpha - k_1)\, x_1^{*2} + 101\, x_1^*\, x_2^*$$

$$- x_2^{*2} - k_2\, x_4^{*2} - x_5^{*2} ) \,. \tag{21}$$

It is clear that $\dot{E} \leq 0$ if $x_1^*$ and $x_2^*$ are of opposite sign and α≤ $k_1$ and the equality sign holds only at the origin ( $x_1^* = x_2^* = x_4^* = x_5^* = 0$ ). Therefore the slave system is glabally asymptotically stable [10, 11, a]. The master (Eqs. (17a-e)) and slave (Eqs.(18a-d)) systems will eventually synchronize as shown in Fig. (3a-d). Furthermore, Lyapunov exponents of the slave are also not positive ($\lambda_1$ = −1.05, $\lambda_2$ =−504.6, $\lambda_3$ =−2.39, $\lambda_4$ = 0.00).

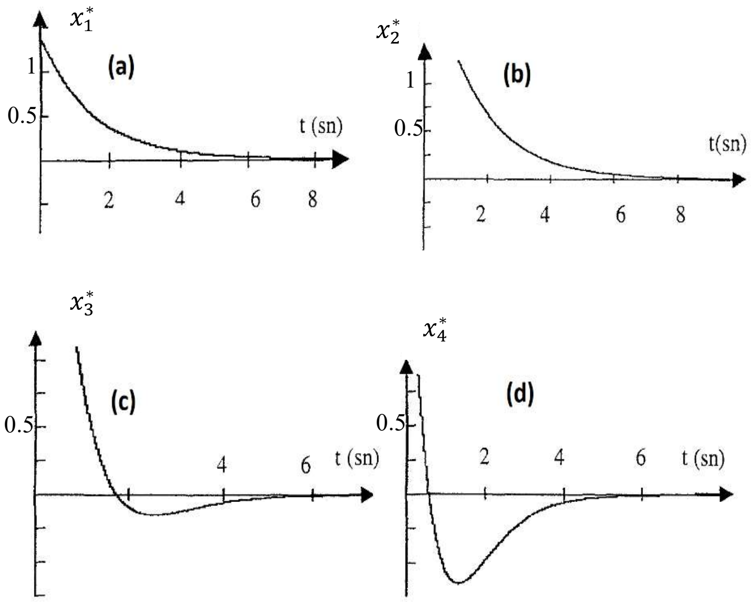


**Figure 3(a-d). Synchronization of the dynamics variables of the master and slave systems for β=300 when t→ ∞.**

## 4. CONCLUSION

In this work a criterion for synchronization of chaos, based on the asymptotic stability has been investigated for the Duffing oscillator. This criterion holds for only if the appropriate Lyapunov function is available. This criterion makes it possible to create a high dimensional chaotic system with a nonlinear subsystem. Such a chaotic system has exhibited synchronization in the case of both periodic limits cycles and chaos, Fig. 3a-b. The ability to be able to design a synchronous system consisting of nonlinear and especially chaotic systems has created new opportunities for modelling complex systems and applications of chaos to communications.

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