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Generalized Derivations of Hyperlattices

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Abstract. In this paper the notion of generalized derivation for a hyperlattice is introduced and some basic properties of them are derived.
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Hiperlatisler Üzerinde Genelleştirilmiş Türevler

Özet. Bu makalede hiperlatisler üzerinde genelleştirilmiş türev kavramı tanıtıldı ve bunların bazı temel özellikleri elde edildi.

Anahtar Kelimeler: Genelleştirilmiş türevler, Hiperlatis.

1. INTRODUCTION AND PRELIMINARIES

Firstly, Marty introduced the notion of hyperstructure in [1] at 8th Congress of Scandinavian Mathematicians. Normally, the composition of two elements is an element in classical algebraic structures, but the composition of two elements is a set in algebraic hyperstructures. After this study, many authors studied this subject. The many concepts in pure and applied mathematics were applied to hyperstructures [2,3]. There come out many kinds of hyperalgebras such as hypergroups in [4, 5], hyperrings in [6,7] etc. In [8], Konstantinidou and Mittas introduced hyperlattices and in [9] superlattices in [9] (for more details see [10] and [11]). In particular some interesting results of the theory of hyperlattices studied by Rasouli and Davvaz in [12, 13].

Derivations in rings and near-rings have been studied by many mathematicians in several ways [14, 15]. Bresar [16] introduced the generalized derivation in rings and many mathematicians studied on this concept. N. O. Alshehri applied the notion of generalized derivation in ring theory to lattices [17]. Now, we define the notion of derivation on hyperlattice. In this paper we aim to generalize some results given in [17] and [18] to generalized derivations of hyperlattices. In this way, we define generalized derivation on hyperlattice and give an example.

In this section, we first recall some definitions and basic results (for more detailed information see [10,12,13]).

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Definition 1.1 ([13]) *Let L be a nonempty set and $\vee: L \times L \to P^*(L)$ be a hyperoperation, where $P(L)$ is a power set of L and $P^*(L) = P(L) - \emptyset$ and $\wedge: L \times L \to L$ be an operation. Then $(L, \vee, \wedge)$ is a hyperlattice if*

(1) for all $a \in L, a \in a \vee a, a \wedge a = a$;

(2) for all $a, b \in L, a \vee b = b \vee a,\ a \wedge b = b \wedge a$;

(3) for all $a, b, c \in L, (a \vee b) \vee c = a \vee (b \vee c);\ (a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(4) for all $a, b \in L, a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$;

(5) if $a \in a \vee b$ for all $a, b \in L$, then $a \wedge b = b$.

Let A and B be nonempty subsets of L then, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$, $A \vee B = \{a \vee b \mid a \in A, b \in B\}$.

Let L be a hyperlattice. For each $x, y \in L$, we define two relations on L as follows: $(x, y) \in \leq$ if and only if $x = x \wedge y$, $(x, y) \in \preccurlyeq$ if and only if $y \in x \vee y$. For all nonempty subsets A and B of L, we define $A \leq B$ if there exist $a \in A$ and $b \in B$ such that $a \leq b$.

A zero of a hyperlattice L is an element 0 with $0 \leq x$ for all $x \in L$. A unit 1, satisfies $x \leq 1$ for all $x \in L$, so it can be seen that there are at most one zero and at most one unit. A bounded hyperlattice is one that has both 0 and 1. In a bounded hyperlattice L, y is a complement of x if $x \wedge y = 0$ and $1 \in x \vee y$. The set of complement elements of x is denoted by x^c. A complemented hyperlattice is a bounded hyperlattice in which every element has at least one complement.

Definition 1.2 ([3]) *An element $a \in L$ is called a scalar element if the set $a \vee x$ for all $x \in L$ has only one element.*

Proposition 1.3 ([13]) *Let $(L, \vee, \wedge)$ be a hyperlattice. Then the following hold:*

(1) $\leq = \preccurlyeq$ and $(L, \leq)$ is a poset. Also we can replace Definition 1.1 (4) by $x \in x \wedge (x \vee y)$ for all $x, y \in L$;

(2) $x \wedge y \leq x,\ y \leq x \vee y$ for all $x, y \in L$;

(3) $X \subseteq (X \vee X) \cap (X \wedge X)$ for a nonempty subset X of L;

(4) $X \vee (Y \vee Z) = (X \vee Y) \vee Z$ and $X \wedge (Y \wedge Z) = (X \wedge Y) \wedge Z$ for all nonempty subsets X, Y, Z of L;

(5) If $x \leq y$, then $x \wedge z \leq y \wedge z$ for all $x, y, z \in L$;

(6) If $x, y \in x \vee y$, then $x = y$, so $x \vee y = L$ implies that $x = y$ for all $x, y \in L$;

(7) If $x \vee y = \{0\}$, then $x = y = 0$ for all $x, y \in L$;

(8) If 0 is a scalar element of L, then $0 \vee 0 = 0, x \vee 0 = \{x\}$ for all $x \in L$.

Definition 1.4 ([13]) *A subhyperlattice of a hyperlattice L is a nonempty subset of L which is closed under the hyperoperation $\vee$ and operation $\wedge$ as defined in L.*

Definition 1.5 ([13]) *A hyperlattice $(L, \wedge, \vee, 0, 1)$ is said to be a distributive if for every $a, b, c \in L, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is hold.*

Definition 1.6 ([13]) *Let $(L,\wedge,\vee)$ be a hyperlattice and I be a nonempty subset of L. Then I is called a hyperideal of L when:*

(i) I is a subhyperlattice;

(ii) $x \in I$ and $y \in L$ imply $x \wedge y \in I$.

Definition 1.7 *Let L be a hyperlattice. A mapping $d: L \to L$ such that, for all $x, y \in L$, we have*

***(1)** $d(x \vee y) \subseteq d(x) \vee d(y)$, **(2)** $d(x \wedge y) \in (d(x) \wedge y) \vee (x \wedge d(y))$*

is said to be a derivation on L, and the pair (L, d) is said to be a differential hyperlattice, or more precisely, a hyperlattice with a derivation. If the map d such that $d(x \vee y) = d(x) \vee d(y)$ for all $x, y \in L$ and satisfies the condition (2), then d is called a strong derivation of L. In this case, the pair (L, d) is called a strongly differential hyperlattice

2. GENERALIZED DERIVATION OF HYPERLATTICES

In this section we define generalized derivation and strong generalized derivation of hyperlattice and give examples. Through this section L will denote a bounded hyperlattice and 0 be scalar element of L unless otherwise specified.

Definition 2.1 *A mapping $D: L \to L$ is called a generalized derivation on L if there exists a derivation $d: L \to L$ such that*

(1) $D(x \vee y) \subseteq D(x) \vee D(y)$

(2) $D(x \wedge y) \in (D(x) \wedge y) \vee (x \wedge d(y))$

for all $x, y \in L$. The pair (L, D) is said to be a differential hyperlattice or is said to be hyperlattice with generalized derivation.The map D is called strong generalized derivation if $D(x \vee y) = D(x) \vee D(y)$ and satisfies the condition (2). Then the pair (L, D) is called a strongly differential hyperlattice.

***Example 2.2**. Let $L = \{0, a, b, 1\}$ and define $\wedge$ and $\vee$ by the following Cayley tables*

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0,a\}$	$\{1\}$	$\{b,1\}$
b	$\{b\}$	$\{1\}$	$\{0,b\}$	$\{a,1\}$
1	$\{1\}$	$\{b,1\}$	$\{a,1\}$	L

Then $(L,\wedge,\vee)$ is a hyperlattice. Define the maps $d: L \to L$ by $dx = \begin{cases} 0, & x = 0, a \\ b, & x = b, 1 \end{cases}$ and $D: L \to L$ by $Dx = x$. Then we can see that D is a generalized derivation on L.

Definition 2.3 *Let D be a generalized derivation on L. If $x \leq y$ implies $Dx \leq Dy$ for all $x, y \in L$, D is called an isotone generalized derivation.*

Example 2.4 *Let L be a hyperlattice as in Example 2.2. It is easy to check that D is an isotone generalized derivation of L.*

Definition 2.5 *A generalized derivation D is said to be contractive if $Dx \leq x$ for all $x \in L$.*

Proposition 2.6 *Let D be a contractive generalized derivation and d be contractive derivation on L. Then the following hold for all $x, y \in L$*

a) *If L is distributive hyperlattice then $dx \leq Dx$,*

b) *If I is a hyperideal of L then $DI \subseteq I$,*

c) $D0 = 0$

d) $Dx \in Dx \vee (x \wedge d1)$

e) $D1 \in D1 \vee D1$

Proof. a) *For all $x \in L$, we have*

$Dx \wedge dx = D(x \wedge x) \wedge dx \in \big((Dx \wedge x) \vee (x \wedge dx)\big) \wedge dx = (Dx \vee dx) \wedge dx.$

Since L is distributive hyperlattice,we obtain

$Dx \wedge dx \in (Dx \wedge dx) \vee (dx \wedge dx)$. *By using Definition 1.1 (1) we have $Dx \wedge dx \in (Dx \wedge dx) \vee dx$. Also from Definition 1.1 (5) we get $(Dx \wedge dx) \wedge dx = dx$. Then it is a routine matter to show that $Dx \wedge dx = dx$. Consequently we have $dx \leq Dx$.*

b) *Let $y \in DI$. Then there exist an $x \in I$ such that $y = Dx \leq x$. Since I is a hyperideal of L, we have $y \in I$.*

c) *It is a routine matter to show that $D0 = D(0 \wedge 0) \in (D0 \wedge 0) \vee (0 \wedge d0)$. By using [18] we have $D0 \in 0$. Hence the result.*

d) $Dx = D(x \wedge 1) \in (Dx \wedge 1) \vee (x \wedge d1) = Dx \vee (x \wedge d1)$

e) *It is clear from Definition 2.1.*

Proposition 2.7 *Let D be a contractive generalized derivation, then we have*

a) *If $d1 \leq x$, then $d1 \leq Dx$,*

b) *If $x \leq d1$, then $Dx = x$.*

Proof. a) *Let $x \in L$ such that $d1 \leq x$, by using Proposition 2.6 d) we have $Dx \in Dx \vee (x \wedge d1)$ hence $Dx \in Dx \vee d1$. Therefore we obtain $d1 \leq Dx$.*

b) *Let $x \in L$ such that $x \leq d1$, by using Proposition 2.6 d) we have $Dx \in Dx \vee (x \wedge d1) = Dx \vee x$, then $x \leq Dx$. On the other hand D is a contractive generalized derivation, therefore $Dx = x$.*

Theorem 2.8 *Let D be a contractive generalized derivation on L. Then the following conditions are equivalent:*

1) $Dx = x$ *for all* $x \in L$,

2) $D(x \vee y) = (x \vee Dy) \wedge (Dx \vee y)$.

***Proof.* (1)⇒(2)** *Since* $D(x \vee y) = x \vee y$ *and* $(x \vee Dy) \wedge (Dx \vee y) = x \vee y$, *we get* $D(x \vee y) = (x \vee Dy) \wedge (Dx \vee y)$.

(2)⇒(1) *By putting* $x = y$ *in (2) we have* $Dx = x$ *for all* $x \in L$ *since* D *is contractive generalized derivation.*

Theorem 2.9 *Let* D *be a generalized derivation on* L, *then the following conditions are hold.*

1) D *is an isotone generalized derivation.*

2) $Dx \vee Dy \leq D(x \vee y)$.

Proof. 1) *If* $x \leq y$, *then we get* $y \in x \vee y$. *Therefore* $Dy \in D(x \vee y)$. *By using Definition 2.1 we have* $Dy \in D(x \vee y) \subseteq Dx \vee Dy$. *Hence we conclude* $Dx \leq Dy$.

2) *Since* D *is isotone generalized derivation, we have* $Dx \leq D(x \vee y)$ *and* $Dy \leq D(x \vee y)$. *Hence we conclude* $Dx \vee Dy \leq D(x \vee y)$.

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