



Analysis of a Random Zeeman Heartbeat Model with Differential Transformation Method

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Abstract. In this paper, the differential transformation method is used to examine the random Zeeman Heartbeat Model. Some of the parameters and the initial conditions of the model are taken as random variables with Beta and Normal distributions, respectively. The approximate analytical solution of the random Zeeman Model is obtained and used to find the expectation and variance of the model components. The results from the random models including Beta and normal distributed random effects are compared and the approximate numerical characteristics are obtained for these cases. The approximate formulas are also modified by using Laplace-Padé Method to increase the convergence interval of the approximations.

Keywords: Zeeman Heartbeat Model, Random differential equation, Expected value, Variance, Padé approximation.

Diferansiyel Dönüşüm Yöntemi İle Rastgele Kalp Atış Modelinin Analizi

Özet. Bu çalışmada, rastgele Zeeman Kalp Atış Modelinin incelenmesi için diferansiyel dönüşüm yöntemi kullanılmıştır. Modelin bazı parametreleri ve başlangıç koşulları sırasıyla Beta ve Normal dağılımlara sahip rastgele değişkenler olarak alınmıştır. Rastgele Zeeman Modelinin yaklaşık analitik çözümü elde edilmiş ve model bileşenlerinin beklenen değer ve varyansı elde edilmiştir. Beta ve normal olarak dağılmış rastgele etkiler altında, rastgele modellerin sonuçları karşılaştırılmış ve bu durumlar için elde edilen yaklaşık sayısal karakteristikler karşılaştırılmıştır. Elde edilen yaklaşık formüllere, yaklaşımların yakınsama aralığını artırmak için Laplace-Padé Metodu uygulanarak iyileştirilmiş çözümler bulunmuştur.

Anahtar Kelimeler: Zeeman Kalp Atış Modeli, Rastgele diferansiyel denklem, Beklenen değer, Varyans, Padé yaklaşımı

1. INTRODUCTION

The majority of the investigations of mathematical models in various fields of science is performed by using deterministic components, ignoring the randomness of the natural course of events. This fact is a setback for an accurate description of the events through equation systems since there is always an unignorable amount of uncertainty in the natural realizations of all phenomena. The random effects approach of Merdan et al. aims to handle this issue swiftly by introducing random noise terms in the deterministic parameters of compartmental mathematical models [1,2]. Compartmental models are widely used in the modeling of infectious diseases and such a random modeling approach provides a straightforward modification of these models to effectively describe the random behavior of diseases.

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The model used in this study has been introduced by E.C. Zeeman in 1972 [3-5]. It consists of differential equations that describe the heart activity under the control of electrical impulses produced by the body. The heart regulates the circulation of blood in the body through the contraction of heart muscles, an event which is triggered by an electrochemical activity. The equation systems of E.C. Zeeman represent the dynamics of the muscle fiber activity and stimulus which controls muscle fiber contraction over time. The parameters of the differential equation system will be added random effects with Gaussian (normal) and beta distribution to represent the natural fluctuations in the dynamics of heartbeat. Similar modeling studies have been made with random effects for bacterial resistance and biochemical reactions [2, 6].

In this study, Differential Transformation Method (DTM) will be used to investigate the Zeeman's heartbeat models with random components. DTM [7,8] and similar methods related to DTM have been used for analyzing various mathematical models [17,18]. While DTM has many advantages for analyzing models such its adaptability for use in fractional, delay and partial differential equation systems, its power series form approximate-analytical solution which enables a detailed analysis of the problem of interest and its usability in many problems in applied mathematics. However, the series solution brings along a serious disadvantage as well, which arises when the problems are analyzed in wide time intervals, causing issues on the convergence of the approximate solution. One of the techniques for tackling this problem is the Laplace-Pade method, which proposes the use of a ratio two polynomial functions of orders p and q , where the orders are arbitrary positive integers. Applications have shown that every different selection of p, q results in a new approximate solution and generally the selection for $p = q$ provides the best convergence. Modifying the approximate analytical solution through Pade approximants provides a new approximate solution which is convergent to the exact solution on a wider time interval. Details of Laplace-Pade modification of DTM have been given in the literature [19-20].

Our study concentrates on the improvement of the approximate solutions of the Zeeman heartbeat model with random components obtained with DTM through Laplace-Pade method [18, 21]. The paper is organized as follows: Section 2 includes an analysis of the heartbeat model with two equations including an introduction of the Differential Transformation Method and the distributions used. Section 3 contains the analysis on the model with three equations. The last section contains the conclusions.

2. TWO DIMENSIONAL ZEEMAN MODEL

The Zeeman model is used for the modeling of heartbeat and the two dimensional version consists of the following differential equations, where x_1 represents the heart muscle fiber length and x_2 represents the stimulus:

$$\begin{aligned}\frac{dx_1}{dt} &= -\frac{1}{\epsilon}(x_1^3 - Tx_1 + x_2), \\ \frac{dx_2}{dt} &= x_1 - x_d.\end{aligned}\tag{1}$$

The parameters of this system, ϵ, T, x_d are defined as follows: T represents the tension ($T > 0$), ϵ is a constant that depends on the timescale and x_d is the typical relaxed fiber length. The deterministic values of the parameters have been given as $\epsilon = 0.2; T = 0.5$ and $x_d = 0$ or 0.41 whereas the initial values of (1) are given as $x_1(0) = 0.5$ and $x_2(0) = 0$ [3-5].

2.1. Differential Transformation Method

An outline of the Differential Transformation Method (DTM) can be given as follows. Let $x(t)$ be an analytical function in the domain D and $t = t_0$ be a point in D . The function $x(t)$ can be represented by using a t_0 -centered power series. The k -th derivative of $x(t)$ is defined as:

$$X(k) = \frac{1}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=t_0}, \quad \forall t \in D. \tag{2}$$

In (2), $x(t)$ is the original function and $X(k)$ is the transformed function. The inverse differential transformation of $X(k)$ is given as [7,8]:

$$x(t) = \sum_{k=0}^{\infty} X(k)(t - t_0)^k, \quad \forall t \in D. \tag{3}$$

Using (2) and (3), we obtain

$$x(t) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} \left[\frac{d^k x(t)}{dt^k} \right]_{t=t_0}, \quad \forall t \in D. \tag{4}$$

The basic theorems for one-dimensional differential transformation are given as below.

Theorem 1. If $z(t) = x(t) \pm y(t)$, then $Z(k) = X(k) \pm Y(k)$.

Theorem 2. If $z(t) = cy(t)$, then $Z(k) = cY(k)$.

Theorem 3. If $z(t) = \frac{dy(t)}{dt}$, then $Z(k) = (k + 1)Y(k + 1)$.

Theorem 4. If $z(t) = \frac{d^n y(t)}{dt^n}$, then $Z(k) = \frac{(k+n)!}{k!} Y(k + n)$.

Theorem 5. If $z(t) = x(t)y(t)$, $Z(k) = \sum_{k_1=0}^k X(k_1)Y(k - k_1)$.

Theorem 6. If $z(t) = t^n$, then $Z(k) = \delta(k - n) = \begin{cases} 1 & k = n \\ 0 & k \neq n \end{cases}$

In applications, the function $x(t)$ is represented by an infinite series and by using (4) we can write:

$$x(t) = \sum_{k=0}^N X(k)(t - t_0)^k, \quad \forall t \in D. \tag{5}$$

The equation (5) states that $\sum_{k=N+1}^{\infty} X(k)(t - t_0)^k$ has a negligibly small value.

Consider a system of ordinary differential equations

$$\begin{aligned} \frac{dx_1}{dt} + h_1(t, x_1, x_2, \dots, x_m) &= g_1(t), \\ \frac{dx_2}{dt} + h_2(t, x_1, x_2, \dots, x_m) &= g_2(t), \\ \frac{dx_m}{dt} + h_m(t, x_1, x_2, \dots, x_m) &= g_m(t), \end{aligned} \tag{6}$$

with the initial values

$$x_1(t_0) = d_1, \quad x_2(t_0) = d_2, \dots, x_m(t_0) = d_m \tag{7}$$

Using DTM, the derivative of both sides in the equation system (6)-(7) is obtained as

$$\begin{aligned}(k+1)X_1(k+1) + H_1(k) &= G_1(k), \\ (k+1)X_2(k+1) + H_2(k) &= G_2(k), \\ &\vdots \\ (k+1)X_m(k+1) + H_m(k) &= G_m(k).\end{aligned}\tag{8}$$

with

$$X_1(0) = d_1, \quad X_2(0) = d_2, \dots, X_m(0) = d_m.\tag{9}$$

Similarly, the n-th term is obtained for (5) as

$$\begin{aligned}\varphi_{1,n}(t) = x_1(t) &= \sum_{k=1}^N X_1(k)t^k, \\ \varphi_{2,n}(t) = x_2(t) &= \sum_{k=1}^N X_2(k)t^k, \\ &\vdots \\ \varphi_{m,n}(t) = x_m(t) &= \sum_{k=1}^N X_m(k)t^k.\end{aligned}\tag{10}$$

Using DTM for equation (1), we obtain:

$$(k+1)X_1(k+1) = -\frac{1}{\varepsilon} \left[\sum_{k_2=0}^k \sum_{k_1=0}^{k_2} X_1(k_1)X_1(k_2-k_1)X_1(k-k_2) - TX_1(k) + X_2(k) \right],\tag{11}$$

$$(k+1)X_2(k+1) = X_1(k) - x_d\delta(k).$$

We will be choosing random parameters and initial values for the two dimensional Zeeman model to obtain a random model. In the next section, we introduce the beta and normal distributions which we will be using for these random components.

2.2. Beta Distribution

The Beta function $B(\alpha, \beta)$ for the parameters $\alpha > 0, \beta > 0$ is defined as

$$B(\alpha, \beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx.\tag{12}$$

Dividing both sides of (12) by $B(\alpha, \beta)$, we obtain

$$1 = \frac{1}{B(\alpha, \beta)} \int_0^1 x^{\alpha-1}(1-x)^{\beta-1} dx.\tag{13}$$

Using the definition of a probability density function, we obtain [9]:

$$f(x) = \begin{cases} \frac{x^{\alpha-1}(1-x)^{\beta-1}}{B(\alpha, \beta)}, & 0 < x < 1. \\ 0, & elsewhere \end{cases} \tag{14}$$

where

$$B(\alpha, \beta) = \frac{\Gamma(\alpha) \cdot \Gamma(\beta)}{\Gamma(\alpha + \beta)}. \tag{15}$$

If $X \sim B(\alpha, \beta)$ is a Beta distributed random variable then,

$$E[X] = \frac{\alpha}{\alpha + \beta}, \quad Var[X] = \frac{\alpha\beta}{(\alpha + \beta)^2 \cdot (\alpha + \beta + 1)},$$

$$E[X^2] = \frac{\alpha(\alpha + 1)}{(\alpha + \beta + 1)(\alpha + \beta)}, \quad E[X^3] = \frac{\alpha(\alpha + 1)(\alpha + 2)}{(\alpha + \beta + 2)(\alpha + \beta + 1)(\alpha + \beta)}, \dots$$

$$E[X^N] = \frac{\alpha(\alpha + 1)(\alpha + 2) \dots (\alpha + (N - 1))}{(\alpha + \beta + N - 1)(\alpha + \beta + N - 2) \dots (\alpha + \beta + 1)(\alpha + \beta)}.$$

2.3. Expected Value of the Two Dimensional Zeeman Model

Using Differential Transformation Method [7,8] and its random counterpart [10-14], we can find the approximate solutions of a random process in the form of a power series by using the formula

$$x_N(t) = \sum_{k=0}^N X(k)t^k.$$

The variance and covariance for the approximate solution $x_N(t)$ are given as

$$E[x_N(t)] = \sum_{k=0}^N E[X(k)]t^k, \tag{16}$$

$$Var[x_N(t)] = \sum_{j=0}^N \sum_{i=0}^N cov(X(i), X(j))t^{i+j}, \tag{17}$$

where [11]

$$cov(X(i), X(j)) = E(X(i)X(j)) - E[X(i)]E[X(j)], \quad \forall i, j = 0, 1, \dots, N. \tag{18}$$

Let the parameter T in (1) be a Beta distributed random variable such that $T \sim Beta(\alpha = 2, \beta = 1)$. In addition, assume that the initial conditions are assumed to be random variables Y_0 and Y_1 where Y_0 and Y_1 are independent random variables with normal distribution, i.e. $Y_0, Y_1 \in N(\mu, \sigma^2)$. Let $\mu = 1, \sigma^2 = 1, \sigma = 1$. Then,

$$\begin{aligned} E[Y_0] &= \mu = 1, E[Y_1] = \mu = 1, \\ E[Y_0^2] &= \mu^2 + \sigma^2 = 1 + 1 = 2, \\ E[Y_1^2] &= \mu^2 + \sigma^2 = 1 + 1 = 2, \\ E[Y_0^3] &= E[Y_1^3] = \mu^3 + 3\mu\sigma^2 = 1 + 3 \cdot 1 \cdot 1 = 4, \\ E[Y_0^4] &= E[Y_1^4] = \mu^4 + 6\mu^2\sigma^2 + 3\sigma^4 = 1 + 6 \cdot 1 \cdot 1 + 3 \cdot 1 = 10, \\ E[Y_0^5] &= E[Y_1^5] = \mu^5 + 10\mu^3\sigma^2 + 15\mu\sigma^4 = 1 + 10 \cdot 1 \cdot 1 + 15 \cdot 1 = 26, \\ E[Y_0^6] &= E[Y_1^6] = \mu^6 + 15\mu^4\sigma^2 + 45\mu^2\sigma^4 + 15\sigma^6 = 1 + 15 \cdot 1 \cdot 1 + 45 \cdot 1 \cdot 1 + 15 \cdot 1 = 76, \\ E[Y_0^7] &= E[Y_1^7] = \mu^7 + 21\mu^5\sigma^2 + 105\mu^3\sigma^4 + 105\mu\sigma^6 = 232. \end{aligned} \tag{19}$$

Since $T > 0$ is Beta distributed such that $T \sim B(\alpha = 2, \beta = 1)$ [9],

$$\begin{aligned} E[T] &= \frac{\alpha}{\alpha + \beta} = \frac{2}{3}, \\ E[T^2] &= \frac{\alpha(\alpha + 1)}{(\alpha + \beta + 1)(\alpha + \beta)} = \frac{2.3}{3.4} = \frac{1}{2}, \\ E[T^3] &= \frac{\alpha(\alpha + 1)(\alpha + 2)}{(\alpha + \beta + 2)(\alpha + \beta + 1)(\alpha + \beta)} = \frac{2}{5}, \\ E[T^4] &= \frac{\alpha(\alpha + 1)(\alpha + 2)(\alpha + 3)}{(\alpha + \beta)(\alpha + \beta + 1)(\alpha + \beta + 2)(\alpha + \beta + 3)} = \frac{1}{3}. \end{aligned} \quad (20)$$

Using (11), the approximate analytical solution of $x_1(t)$ in (1) using 4 terms is obtained as (denoted by X_1)

$$\begin{aligned} X_1 &= Y_0 + (-5Y_1 + 5TY_0 - 5Y_0^3)t + \left(\frac{75}{2}Y_0^5 - 50Y_0^3T + \frac{75}{2}Y_0^2Y_1 + \frac{25}{2}Y_0T^2 - \frac{25}{2}Y_1T - \frac{5}{2}Y_0 \right)t^2 \\ &\quad + \left(\frac{25}{6}Y_1 + \frac{50}{3}Y_0^3 - \frac{625}{2}Y_0^7 + \frac{125}{6}Y_0T^3 - \frac{125}{6}Y_1T^2 + \frac{1125}{2}Y_0^5T - \frac{875}{2}Y_0^4Y_1 \right. \\ &\quad \left. - \frac{1625}{6}Y_0^3T^2 - 125Y_0Y_1^2 + 375Y_0^2Y_1T - \frac{25}{3}Y_0T \right)t^3 + \dots \end{aligned}$$

Hence, the approximate expected value of the random variable X_1 is obtained as

$$\begin{aligned} E[X_1] &= E \left[Y_0 + (-5Y_1 + 5TY_0 - 5Y_0^3)t \right. \\ &\quad + \left(\frac{75}{2}Y_0^5 - 50Y_0^3T + \frac{75}{2}Y_0^2Y_1 + \frac{25}{2}Y_0T^2 - \frac{25}{2}Y_1T - \frac{5}{2}Y_0 \right)t^2 \\ &\quad + \left(\frac{25}{6}Y_1 + \frac{50}{3}Y_0^3 - \frac{625}{2}Y_0^7 + \frac{125}{6}Y_0T^3 - \frac{125}{6}Y_1T^2 + \frac{1125}{2}Y_0^5T - \frac{875}{2}Y_0^4Y_1 \right. \\ &\quad \left. - \frac{1625}{6}Y_0^3T^2 - 125Y_0Y_1^2 + 375Y_0^2Y_1T - \frac{25}{3}Y_0T \right)t^3 \left. \right] + \dots \end{aligned}$$

Using the independence of the random variables, we find

$$E[X_1] = 1 - \frac{65}{3}t + \frac{10945}{12}t^2 - 67353.472t^3 + \dots \quad (21)$$

The expected value of X_1 is shown in Figure 1.

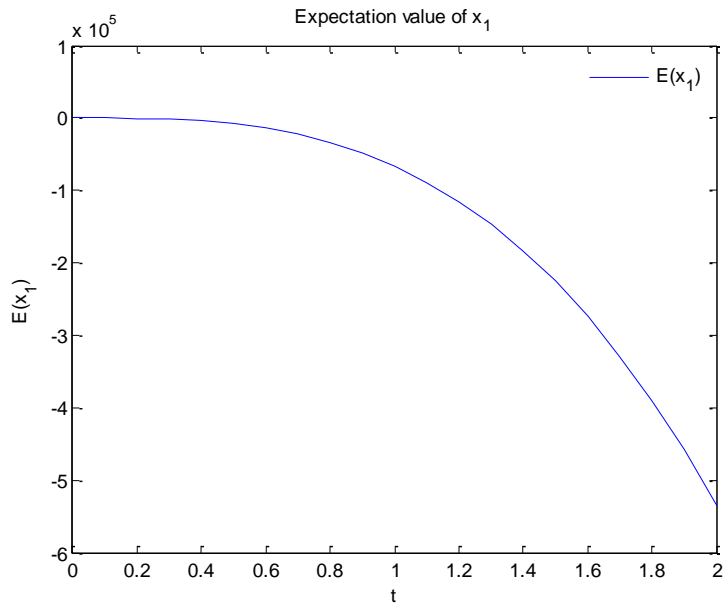


Figure 1. The expected value of X₁.

$$X_2 = Y_1 + Y_0t + \left(-\frac{5}{2}Y_1 + \frac{5}{2}TY_0 - \frac{5}{2}Y_0^3\right)t^2 + \left(\frac{25}{2}Y_0^5 - \frac{50}{3}Y_0^3T + \frac{25}{2}Y_0^2Y_1 + \frac{25}{6}Y_0T^2 - \frac{25}{6}Y_1T - \frac{5}{6}Y_0\right)t^3 + \dots$$

Hence, its expectation becomes

$$E[X_2] = E \left[Y_1 + Y_0t + \left(-\frac{5}{2}Y_1 + \frac{5}{2}TY_0 - \frac{5}{2}Y_0^3\right)t^2 + \left(\frac{25}{2}Y_0^5 - \frac{50}{3}Y_0^3T + \frac{25}{2}Y_0^2Y_1 + \frac{25}{6}Y_0T^2 - \frac{25}{6}Y_1T - \frac{5}{6}Y_0\right)t^3 \right] + \dots$$

$$E[X_2] = 1 + t - \frac{65}{6}t^2 + \frac{10945}{36}t^3 + \dots \tag{22}$$

The approximate expected value of X₂ is shown in Figure 2. It should be noted that more terms are needed for accurate approximate expectations E[X₁], E[X₂]. The first three terms have been given to present the calculation method and to underline the improvements obtained by the modification.

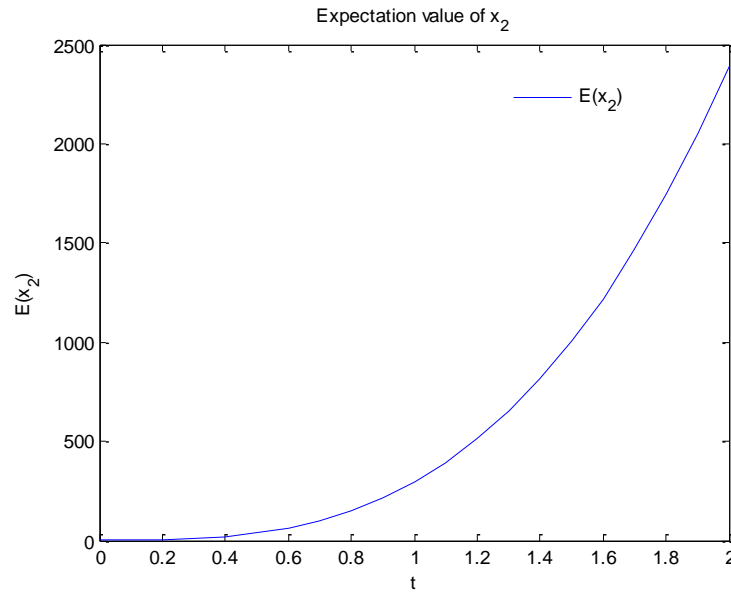


Figure 2. The expected value of X_2 .

2.4. Laplace-Padé Method

If Laplace-Padé method [15,16] is used to improve the results, i.e. to obtain convergent results in a wider interval, the following result is obtained for (21):

$$\begin{aligned}
 M_1 &= \frac{1}{7424222958497731096191780362649627} e^{-\frac{56373593243647997}{418931110051840}t} \\
 &\times \left(7424222958497731096191780362649627 \right. \\
 &\times \cosh\left(\frac{1}{418931110051840} t \sqrt{2474740986165910365397260120883209}\right) \\
 &+ 141890257577574391 \sqrt{2474740986165910365397260120883209} \\
 &\left. \times \sinh\left(\frac{1}{418931110051840} t \sqrt{2474740986165910365397260120883209}\right) \right)
 \end{aligned}$$

This new expected value of X_1 , denoted by M_1 , is shown in Figure 3.

Similarly, Laplace-Padé method for (22) gives:

$$\begin{aligned}
 M_2 &= \frac{1}{381232315} e^{-\frac{11075}{272}t} \left(381232315 \cosh\left(\frac{1}{816} t \sqrt{1143696945}\right) \right. \\
 &\quad \left. + 11347 \sqrt{1143696945} \sinh\left(\frac{1}{816} t \sqrt{1143696945}\right) \right)
 \end{aligned}$$

The modified expected value for X_2 , denoted by M_2 , is shown in Figure 4.

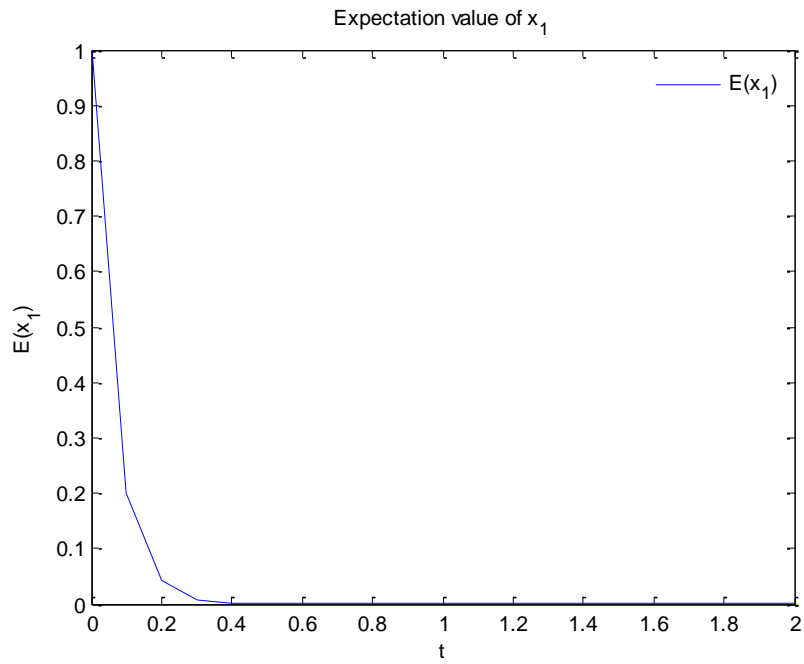


Figure 3. The expected value of X_1 , obtained by modified DTM.

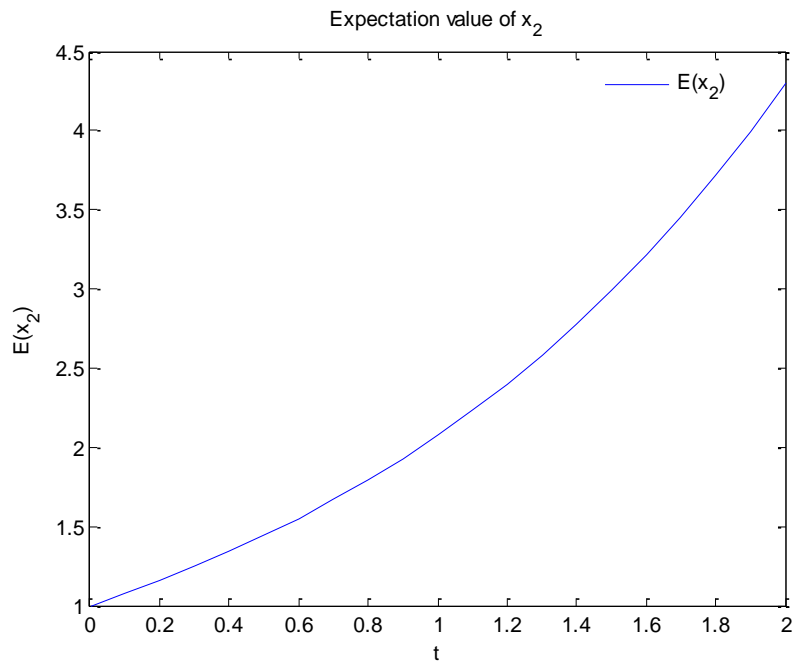


Figure 4. The expected value of X_2 , obtained by modified DTM.

2.5. Variances for Two Dimensional Zeeman Model

Using the formulas (17) and (18), the variances for the two dimensional model is obtained as follows.

For the variance of x_1 (using a three term truncated approximation):

$$\text{Var}[X_1(t)] = \sum_{j=0}^2 \sum_{i=0}^2 \text{cov}(X_1(i), X_1(j))t^{i+j} = 1 - \frac{160}{3}t + \frac{87235}{18}t^2 + \dots \quad (23)$$

Laplace-Padé method for (25) gives

$$M_3 = -\frac{1}{56738917631} e^{\left(\frac{1395760}{36981}t\right)} \left(-56738917631 \cosh\left(\frac{73}{36981}t\sqrt{3886227235}\right) + 673616\sqrt{3886227235} \sinh\left(\frac{73}{36981}t\sqrt{3886227235}\right) \right)$$

Similarly, the variance for X_2 is:

$$\text{Var}[X_2(t)] = \sum_{j=0}^2 \sum_{i=0}^2 \text{cov}(X_1(i), X_1(j))t^{i+j} = 1 - 4t^2 + \dots \quad (24)$$

and the modified variance becomes

$$M_4 = \cos(2\sqrt{2}t).$$

2.6. Expected Values for Initial Conditions with Normal Distribution

Assume that the initial conditions Y_0 and Y_1 of (1) are independent random variables with normal distribution. Using (19), the expected value of the (truncated) approximate solution X_1 becomes (note that unlike Section 2.3, the parameter T is not a random variable here):

$$E[X_1] = 1 - \frac{45}{2}t + \frac{7555}{8}t^2 - 69644.271t^3 + \dots \quad (25)$$

If Laplace-Padé method is applied to (25), we obtain

$$M_5 = 0.1 \times 10^{-9} (9999999999 \cosh(119.2044503t) + 9501091052 \sinh(119.2044503t)) e^{-135.7572336t}.$$

Similarly, the expected value of the approximate solution X_2 is

$$E[X_2] = 1 + t - \frac{45}{4}t^2 + \frac{7555}{24}t^3 + \dots \quad (26)$$

Its modification is obtained as

$$M_6 = \frac{1}{60525305} e^{\left(-\frac{7645}{188}t\right)} \left(60525305 \cosh\left(\frac{1}{188}t\sqrt{60525305}\right) + 7833\sqrt{60525305} \sinh\left(\frac{1}{188}t\sqrt{60525305}\right) \right).$$

2.7. Expected Values for Random T Parameter with Beta Distribution

If the parameter T in (1) is assumed to be Beta distributed, the approximate expected value of X_1 up to t^3 can be obtained by using (20) as follows (note that unlike Section 2.3, the initial values are not random variables here):

$$E[X_1] = \frac{1}{2} + \frac{25}{24}t - \frac{215}{192}t^2 - \frac{9625}{2304}t^3 + \dots \tag{27}$$

Using Laplace-Padé on (27) gives

$$M_7 = -\frac{1}{1357422} e^{\left(\frac{1175}{508}t\right)} \left(-678711 \cos\left(\frac{1}{508}\sqrt{2262370}t\right) + 35\sqrt{2262370} \sin\left(\frac{1}{508}\sqrt{2262370}t\right) \right).$$

Similarly, the expected value of the analytical solution X_2 becomes

$$E[X_2] = \frac{1}{2}t + \frac{25}{48}t^2 - \frac{215}{576}t^3 + \dots$$

Its modification is obtained as

$$M_8 = \frac{4}{165}\sqrt{55}e^{\left(\frac{25t}{24}\right)} \sin\left(\frac{3}{8}\sqrt{55}t\right).$$

3. THREE DIMENSIONAL ZEEMAN MODEL

Let the initial conditions Y_1, Y_2, Y_3 be normal distributed and the model parameters ξ, μ, θ be (standard) Beta distributed random variables. Consider the three dimensional model (Zeeman, 1977)

$$\begin{aligned} \frac{dx_1}{dt} &= -x_1^3 - x_2x_1 - x_3 \\ \frac{dx_2}{dt} &= -\xi x_1 - \mu x_2 \\ \frac{dx_3}{dt} &= -x_2 - \theta \end{aligned} \tag{28}$$

$$x_1(0) = Y_1, x_2(0) = Y_2, x_3(0) = Y_3$$

Since the initial conditions are independent and identically distributed, $E[Y_1] = E[Y_2] = E[Y_3]$ as in (19) and (20). The deterministic values are given as $\xi = 2, \mu = 2, \theta = 1$ (Zeeman, 1977). We use $\xi, \mu, \theta \sim \text{Beta}(\alpha, \beta)$ where $\alpha = 1$ and $\beta = 1$. Using the properties of Beta distribution and (20), we get

$$\begin{aligned} E[\xi] = E[\mu] = E[\theta] &= \frac{\alpha}{\alpha + \beta} = \frac{1}{2} \\ E[\xi^2] = E[\mu^2] = E[\theta^2] &= \frac{\alpha(\alpha + 1)}{(\alpha + \beta + 1)(\alpha + \beta)} = \frac{1}{3} \end{aligned} \tag{29}$$

$$E[\xi^3] = E[\mu^3] = E[\theta^3] = \frac{\alpha(\alpha+1)(\alpha+2)}{(\alpha+\beta+2)(\alpha+\beta+1)(\alpha+\beta)} = \frac{1}{4}$$

$$E[\xi^4] = E[\mu^4] = E[\theta^4] = \frac{\alpha(\alpha+1)(\alpha+2)(\alpha+3)}{(\alpha+\beta)(\alpha+\beta+1)(\alpha+\beta+2)(\alpha+\beta+3)} = \frac{1}{5}$$

Using this information, the expected values of X_1, X_2 and X_3 are obtained as below:

$$\begin{aligned} E[X_1] = E & \left[Y_1 + (-Y_1 Y_2 - Y_1^3 - Y_3) t \right. \\ & + \left(\frac{1}{2} Y_1 Y_2 \mu + 2 Y_1^3 Y_2 + \frac{3}{2} Y_1^2 Y_3 + \frac{1}{2} Y_1 Y_2^2 + \frac{1}{2} Y_2 Y_3 + \frac{1}{2} Y_1^2 \xi + \frac{3}{2} Y_1^5 + \frac{1}{2} Y_2 \right. \\ & \left. + \frac{1}{2} \theta \right) t^2 \\ & + \left(-3 Y_1^2 Y_2 Y_3 - \frac{5}{6} Y_1^3 Y_2 \mu - \frac{2}{3} Y_1^2 Y_2 \xi - \frac{1}{2} Y_1 Y_2^2 \mu - \frac{1}{2} Y_1 Y_3 \xi - \frac{1}{6} Y_1^2 \mu \xi \right. \\ & - \frac{1}{6} Y_1 Y_2 \mu^2 - \frac{1}{3} Y_2 Y_3 \mu - \frac{1}{6} Y_1 Y_2^3 - \frac{1}{6} Y_2^2 Y_3 - \frac{1}{6} Y_2 \theta - Y_1 Y_3^2 - \frac{9}{2} Y_1^5 Y_2 \\ & - \frac{7}{2} Y_1^4 Y_3 - \frac{13}{6} Y_1^3 Y_2^2 - Y_1^4 \xi - \frac{1}{2} Y_1^2 Y_2 - \frac{1}{2} Y_1^2 \theta - \frac{1}{6} Y_1 \xi - \frac{1}{6} Y_2 \mu - \frac{5}{7} Y_1^7 \\ & \left. - \frac{1}{6} Y_2^2 \right) t^3 \Big] + \dots = 1 - 6t + 53t^2 - \frac{27677}{36} t^3 + \dots \end{aligned} \quad (30)$$

Using Laplace-Padé for the truncated approximation (30) gives

$$\begin{aligned} M_9 = \frac{1}{403622041} e^{\left(-\frac{23861}{840}t\right)} & \left(403622041 \cosh\left(\frac{1}{840} t \sqrt{403622041}\right) \right. \\ & \left. + 18821 \sqrt{403622041} \sinh\left(\frac{1}{840} t \sqrt{403622041}\right) \right). \end{aligned}$$

Similarly, the expected value of the approximate analytical solution X_2 is found as

$$\begin{aligned} E[X_2] = E & \left[Y_2 + (-Y_1 \xi - Y_2 \mu) t + \left(\frac{1}{2} Y_1^3 \xi + \frac{1}{2} Y_1 Y_2 \xi + \frac{1}{2} Y_3 \xi + \frac{1}{2} Y_1 \mu \xi + \frac{1}{2} Y_2 \mu^2 \right) t^2 \right. \\ & + \left(-\frac{1}{2} Y_1^5 \xi - \frac{2}{3} Y_1^3 Y_2 \xi - \frac{1}{2} Y_1^2 Y_3 \xi - \frac{1}{6} Y_1 Y_2^2 \xi - \frac{1}{6} Y_2 Y_3 \xi - \frac{1}{6} Y_1^2 \xi^2 \right. \\ & \left. - \frac{1}{3} Y_1 Y_2 \mu \xi - \frac{1}{6} Y_2 \xi - \frac{1}{6} \xi \theta - \frac{1}{6} Y_1^3 \xi \mu - \frac{1}{6} Y_3 \xi \mu - \frac{1}{6} Y_1 \xi \mu^2 - \frac{1}{6} Y_2 \mu^3 \right) t^3 \Big] \\ & + \dots \end{aligned} \quad (31)$$

$$\Rightarrow E[X_2] = 1 - t + \frac{43}{24} t^2 - \frac{661}{72} t^3 + \dots$$

Using Laplace-Padé method, the modification is obtained as

$$\begin{aligned} M_{10} = \frac{1}{957199} e^{\left(-\frac{309}{31}t\right)} & \left(957199 \cosh\left(\frac{1}{186} t \sqrt{2871597}\right) \right. \\ & \left. + 556 \sqrt{2871597} \sinh\left(\frac{1}{186} t \sqrt{2871597}\right) \right). \end{aligned}$$

The expected value of X_3 is found as

$$\begin{aligned}
 E[X_3] &= E \left[Y_3 + (-Y_2 - \theta)t + \left(\frac{1}{2}Y_1\xi + \frac{1}{2}Y_2\mu \right) t^2 \right. \\
 &\quad \left. + \left(-\frac{1}{6}Y_1^3\xi - \frac{1}{6}Y_1Y_2\xi - \frac{1}{6}Y_3\xi - \frac{1}{6}Y_1\mu\xi - \frac{1}{6}Y_2\mu^2 \right) t^3 \right] + \dots \\
 \Rightarrow E[X_3] &= 1 - \frac{3}{2}t + \frac{1}{2}t^2 - \frac{43}{72}t^3 + \dots
 \end{aligned}
 \tag{32}$$

The Laplace-Padé modification of (32) becomes

$$M_{11} = -\frac{1}{151}e^{\left(\frac{5}{6}t\right)} \left(-151\cosh\left(\frac{1}{6}t\sqrt{151}\right) + 14\sqrt{151}\sinh\left(\frac{1}{6}t\sqrt{151}\right) \right).$$

4. CONCLUSION

In this study, we have used the random version of the Differential Transformation Method to investigate the approximate solutions of the two and three dimensional Zeeman models. The initial values Y_0, Y_1 and the parameter T were assumed to be normal and standard Beta distributed random variables, respectively. Using these random values, the expected values and variances were found for the two dimensional model. Similarly, the same approximate characteristics were investigated for the three dimensional model too. Finally, Laplace-Padé method was used to modify these approximations and obtain new functions for the approximate numerical characteristics. It is known from the literature that the modified approximations obtained by Laplace-Padé method are generally convergent to the exact solutions in a wider time interval. As the results show, it is seen that the modified approximate results are convergent to the exact solution, which are also consistent with the deterministic results in the literature. Results from Zeeman’s studies [3-5] and other studies which do not contain random components show that the systems (1), (28) successfully model the contraction and relaxation of the heart muscles and the electrical signal transmission process. In this study, we have shown the use of random components in these models shows that similar results for the biological significance of the equation systems can be achieved with additional results for the variance and standard deviations of the results that imply deviations in the muscle and electrical signal transmission behavior. This study underlines the adaptability of DTM and Laplace-Pade methods to heartbeat models with random components and the applications can also be generalized to other mathematical models.

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