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On Mochizuki-Trooshin Theorem for Sturm-Liouville Operators

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Abstract. In this paper, the inverse spectral problems of Sturm-Liouville operators are considered. Some new uniqueness theorems and analogies of the Mochizuki-Trooshin Theorem are proved.

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Keywords: Inverse spectral problem, Sturm-Liouville equation.

Sturm-Liouville Operatörleri için Mochizuki-Trooshin Teoremi Üzerine

Özet. Bu makalede, Sturm-Liouville operatörlerinin ters spektral problemleri ele alınmıştır. Bazı yeni teklik teoremleri ve Mochizuki-Trooshin teoreminin benzetimleri ispatlanmıştır.

Anahtar Kelimeler: Ters spektral problem, Sturm-Liouville denklemi.

1. INTRODUCTION

We consider the classical Sturm-Liouville problem $L = L(q(x), h, H)$

$$-y'' + q(x)y = \lambda y \tag{1}$$

$$y'(0) - hy(0) = 0 \tag{2}$$

$$y'(1) + Hy(1) = 0 \tag{3}$$

where $h, H \in \mathbb{R}$, λ is a spectral parameter and $q(x) \in L_1(0,1)$. The spectrum of such problems consists of countable many real eigenvalues, which have no finite limit point.

The inverse spectral problem for L is to determine the potential function $q(x)$ from some given data. The first result on this area is given by Ambarzumian [1]. Borg [2] showed that generally a single spectrum is insufficient to determine the potential. Levinson [9] showed that if the potential $q(x)$ is symmetric, $q(x) = q(1-x)$, then it is determined uniquely by a single spectrum. Later Gelfand and Levitan [3] proved that the eigenvalues and normalizing coefficients uniquely determine the potential $q(x)$. Hochstadt and Lieberman [7] proved that a single spectra and the potential on the interval $[1/2, 1]$ uniquely determine the potential $q(x)$ on the whole interval $[0,1]$.

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In 2001, Mochizuki and Trooshin [5] proved a uniqueness theorem for interior spectral data of the Sturm-Liouville operator. They used similar techniques in [7]. This kind of problems for the Sturm-Liouville operator were formulated and studied in [12-19].

Together with L, we consider a boundary value problem $\tilde{L} = L(\tilde{q}(x), h, H)$ of the same form but with a different coefficient $\tilde{q}$. We agree that if a certain symbol s denotes an object related to L, then $\tilde{s}$ will denote an analogous object related to $\tilde{L}$. The eigenvalues and the corresponding eigenfunctions of the problem L are denoted by λ_n and $\varphi_n(x) = \varphi(x, \lambda_n)$, respectively.

The statement of Mochizuki and Trooshin theorem is as following:

Theorem 1.1. [5] *If for every* $n = 0, 1, 2, \ldots$ *we have*

$$\lambda_n = \tilde{\lambda}_n, \quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)} = \frac{\tilde{\varphi}'_n(1/2)}{\tilde{\varphi}_n(1/2)} \tag{4}$$

then $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,1]$.

The purpose of the present study is to prove some analogies of this theorem and new uniqueness theorems for inverse Sturm-Liouville problems.

In the second section, we give some preliminaries. Section 3 contains new uniqueness theorems and alternative proofs for Mochizuki-Trooshin theorem and Levinson's theorem.

2. PRELIMINARIES

We shall first mention some known results which will be needed later. Let $\varphi(x, \lambda)$ be the solution of equation (1) satisfying the initial conditions,

$$\varphi(0, \lambda) = 1, \ \varphi'(0, \lambda) = h. \tag{5}$$

We need specifically to focus on the properties of $\varphi(1/2, \lambda)$. It is known that, [4,8,17,18] for each $x \in [0,1]$, $\varphi(x, \lambda)$ and $\varphi'(x, \lambda)$ are entire functions of λ and there exist some constants $c_1, c_2 > 0$ such that $\varphi(1/2, \lambda)$ and $\varphi'(1/2, \lambda)$ are all bounded by $c_1 \exp(c_2 |\lambda|^{1/2})$. For $|\lambda| \to \infty$ uniformly with respect to $x \in [0,1]$,

$$\begin{aligned} \varphi(x, \lambda) &= \cos \rho x + O\left(\frac{\exp \tau x}{\rho}\right) \\ \varphi'(x, \lambda) &= -\rho \sin \rho x + O(\exp \tau x). \end{aligned} \tag{6}$$

Here $\rho = \sqrt{\lambda}$ and $\tau = |\operatorname{Im} \rho|$. The function

$$\omega(\lambda) = \varphi'(1, \lambda) + H\varphi(1, \lambda)$$

is entire in λ and it has an at most countable set of zeros, $\{\lambda_n\}$. Denote

$$G_\delta = \{\rho : |\rho - k\pi| \geq \delta, k = 0, \pm 1, \pm 2, \ldots\}, \delta > 0.$$

We have that [8]

$$|\omega(\lambda)| \geq C_\delta |\rho| \exp \tau \tag{7}$$

for $\rho \in G_\delta$, $|\rho| \geq \rho^*$ and sufficiently large ρ^*. The Weyl $m\_$ function is defined by:

$$m\_(a,\lambda) = -\frac{\varphi(a,\lambda)}{\varphi'(a,\lambda)}$$

where $a \in [0,1]$. The following Marchenko's uniqueness theorem [6] is also necessary for our analysis.

Theorem 2.1. [6] *The Weyl* $m\_(a,\lambda)$ *function uniquely determines* h *as well as* $q(x)$ *almost everywhere on* $[0,a]$.

3. UNIQUENESS THEOREMS

Here we provide an alternative proof for Mochizuki and Trooshin theorem.

Proof of the Theorem 1.1. Consider the initial-value problems:

$$\begin{gathered} -\varphi'' + q(x)\varphi = \lambda\varphi \\ \varphi(0) = 1, \varphi'(0) = h \end{gathered} \tag{8}$$

and

$$\begin{gathered} -\tilde{\varphi}'' + \tilde{q}(x)\tilde{\varphi} = \lambda\tilde{\varphi} \\ \tilde{\varphi}(0) = 1, \tilde{\varphi}'(0) = h. \end{gathered} \tag{9}$$

The functions $\varphi(x,\lambda)$ and $\varphi'(x,\lambda)$ satisfy

$$\tilde{\varphi}(0,\lambda)\varphi'(0,\lambda) - \varphi(0,\lambda)\tilde{\varphi}'(0,\lambda) = 0.$$

Multiplying (8) by $\tilde{\varphi}(x,\lambda)$ and (9) by $\varphi(x,\lambda)$, subtracting, and integrating from 0 to $1/2$, we obtain

$$f(\lambda) = \int_0^{1/2} \left(q(x) - \tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx = \tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda) - \varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda). \tag{10}$$

The conditions of the theorem imply

$$f(\lambda_n) = 0.$$

Define $h(\lambda) = \dfrac{f(\lambda)}{\omega(\lambda)}$, which is an entire function. From the asymptotics (6) and (7) for $f(\lambda)$ and $\omega(\lambda)$, we see that

$$h(\lambda) = O\left(\frac{1}{|\rho|}\right)$$

for large $|\rho|$. Thus, by Liouville's theorem, we obtain for all λ,

$$h(\lambda)=0$$

or

$$f(\lambda)=0.$$

From (10), we have that

$$\frac{\varphi(1/2,\lambda)}{\varphi'(1/2,\lambda)}=\frac{\tilde{\varphi}(1/2,\lambda)}{\tilde{\varphi}'(1/2,\lambda)}$$

and hence

$$m_{-}(1/2,\lambda)=\widetilde{m_{-}}(1/2,\lambda).$$

By Theorem 2.1, we prove $q(x)=\tilde{q}(x)$ almost everywhere on $[0,1/2]$.

To prove that $q(x)=\tilde{q}(x)$ almost everywhere on $[1/2,1]$, we will consider the supplementary problem $\overline{L}$:

$$-y''+q(1-x)y=\lambda y$$

$$y'(0)-Hy(0)=0$$

$$y'(1)+hy(1)=0.$$

Since $\varphi_n(1-x)=\overline{\varphi}_n(x)$, the assumption conditions in Theorem 1.1 are still satisfied. If we repeat the above arguments then this yields $q(1-x)=\tilde{q}(1-x)$ on $[0,1/2]$, that is $q(x)=\tilde{q}(x)$ almost everywhere on $[1/2,1]$. This completes the proof. □

By the remark to proof of Theorem 1, we have proved the following result:

Corollary 3.1. *Let* $f(\lambda)=0$ *for all* λ. *If for every* $n=0,1,2,\dots$ *we have*

$$\lambda_n=\widetilde{\lambda_n},$$

then $q(x)=\tilde{q}(x)$ *almost everywhere on* $[0,1]$.

Let L_0:

$$-y''+q(x)y=\lambda y$$

$$y'(0)-hy(0)=0$$

$$y'(1)+hy(1)=0.$$

Here we provide an alternative proof for the following Levinson's theorem [9].

Theorem 3.2. [9] *If* $q(x)=q(1-x)$ *then the function* $q(x)$ and h *are uniquely determined by the spectrum of problem* L_0.

Proof. Applying the same arguments as that in the proof of Theorem 1.1, we can see that

$$f(\lambda)=2\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx=0$$

and hence

$$f(\lambda_n)=2\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda_n)\tilde{\varphi}(x,\lambda_n)dx=0.$$

We obtain for all λ,

$$f(\lambda)=\tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda)-\varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda)=0$$

Thus we arrive at

$$m\_(1/2,\lambda)=\widetilde{m\_}(1/2,\lambda).$$

By Theorem 2.1, the proof is complete. □

Let us consider the following Sturm-Liouville problems

$$-y''+q(x)y=\lambda y \tag{11}$$

$$y(0)=y(1/2)=0 \tag{12}$$

$$y(0)=y'(1/2)=0\,. \tag{13}$$

Let $\{\mu_n\}_{n=0}^{\infty}$ and $\{\upsilon_n\}_{n=0}^{\infty}$ be the spectra of the problems (11), (12) and (11), (13), respectively. Consider the problem: given three spectra $\{\lambda_n\}_{n=0}^{\infty}$, $\{\mu_n\}_{n=0}^{\infty}$ and $\{\upsilon_n\}_{n=0}^{\infty}$ determine $q(x)$. Knowledge of $\{\mu_n\}_{n=0}^{\infty}$ and $\{\upsilon_n\}_{n=0}^{\infty}$ is equivalent to the knowledge of $q(x)$ on $[0,1/2]$. Thus this problem is the Hochstadt-Lieberman problem in [7]. Now consider the problem: given $\{\lambda_n\}_{n=0}^{\infty}\subset\left\{\{\upsilon_n\}_{n=0}^{\infty}\cup\{\mu_n\}_{n=0}^{\infty}\right\}$ determine $q(x)$. In this case, only spectra $\{\lambda_n\}_{n=0}^{\infty}$ uniquely determine the potential $q(x)$ on the whole $[0,1]$. We can give the following uniqueness theorem.

Theorem 3.3. *Let* $\{\lambda_n\}_{n=0}^{\infty}\subset\left\{\{\upsilon_n\}_{n=0}^{\infty}\cup\{\mu_n\}_{n=0}^{\infty}\right\}$ *and* $\{\tilde{\lambda}_n\}_{n=0}^{\infty}\subset\left\{\{\tilde{\upsilon}_n\}_{n=0}^{\infty}\cup\{\tilde{\mu}_n\}_{n=0}^{\infty}\right\}$. *If for every* $n=0,1,\ldots$ *we have*

$\lambda_n=\tilde{\lambda}_n$,

then $q(x)=\tilde{q}(x)$ *almost everywhere on* $[0,1]$.

Proof. As in the proof of Theorem 1.1, we can show that

$$f(\lambda)=\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx=\tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda)-\varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda).$$

To prove, as in the Corollary 3.1, it suffices to show that $f(\lambda)=0$ for all λ. The assumptions of the theorem imply that

$$\varphi_n(1/2,\lambda_n)=0 \text{ or } \varphi'_n(1/2,\lambda_n)=0 \text{ and } \tilde{\varphi}_n(1/2,\lambda_n)=0 \text{ or } \tilde{\varphi}'_n(1/2,\lambda_n)=0.$$

Hence, we have

$$f(\lambda_n)=0.$$

Thus, repeating the proof Theorem 1.1, we arrive at

$$f(\lambda)=0,$$

which implies that

$$m_{-}(1/2,\lambda)=\widetilde{m_{-}}(1/2,\lambda)$$

and $q(x)=\tilde{q}(x)$ almost everywhere on $[0,1/2]$. The supplementary problem $\overline{L}$ in proof of Theorem 1.1 completes the proof. □

Let us define

$$g(\rho)=\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx=\tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda)-\varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda) \quad (14)$$

where $\rho=\sqrt{\lambda}$. The asymptotics (6) imply that the entire function $g(\rho)$ is a function of exponential type ≤ 1. As shown by the above discussion, let $g(\rho)=0$ then only spectra $\{\lambda_n\}_{n=0}^{\infty}$ uniquely determine the potential $q(x)$ on $[0,1]$. We now consider the problem: If the zeros of an entire function of exponential type are known to include a given sequence of positive real numbers what can be said about growth of the function. The first result of this type is given by Carlson's Theorem. This theorem [11, p.153] says, if g is entire function of exponential type $<\pi$ and vanishes on the positive integers then g vanishes everywhere. This

idea has been further developed by Rubel [10, p.422]:

Theorem 3.4. [10] *Let* $\rho=t+i\tau$ *and* $\Omega=\left\{\rho_n:\rho_{n+1}-\rho_n\geq\gamma>0,\rho_n>0,n\in\mathbb{Z}^+\right\}$. *In order to each entire function* $g(\rho)$ *satisfying*

$$g(\rho)=O(1)\exp(a|\rho|),\quad a<\infty \quad (15)$$

$$g(i\tau)=O(1)\exp(b|\tau|),\quad b<\delta \quad (16)$$

$$g(\rho_n)=0 \quad (17)$$

vanish identically, it is sufficient that

$$\inf_{\rho>1}\limsup_{k\to\infty}(\ln\rho)^{-1}\sum_{\rho_n\leq pk}\frac{1}{\rho_n}=L(\Omega)\geq\frac{\delta}{\pi}. \quad (18)$$

Here, $L(\Omega)$ is the logarithmic block density of Ω.

We turn repeat that equation (14). From asymptotics (6), the entire function

$$g(\rho)=\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx$$

satisfies (15) and (16). Also we have that

$$\rho_{n+1}-\rho_n>0$$

where $\sqrt{\lambda_n}=\rho_n$. In this case, we can give a uniqueness theorem by using Theorem 3.4.

Theorem 3.5. *Let* $\Lambda\subset\mathbb{N}\cup\{0\}$ *be a subset of nonnegative integer numbers and let* $\Omega:=\{\lambda_n\}_{n\in\Lambda}$ *be a part of the spectrum of* L *such that the numbers* $\sqrt{\lambda_n}=\rho_n$ satisfy (18) for *function* $g(\rho)$. *If for* $n\in\Lambda$, *we have*

$$\lambda_n=\widetilde{\lambda}_n,\quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)}=\frac{\widetilde{\varphi}'_n(1/2)}{\widetilde{\varphi}_n(1/2)}$$

then $q(x)=\tilde{q}(x)$ *almost everywhere on* $[0,1]$.

Proof. As in the proof of Theorem 1, we obtain

$$g(\rho)=\int_0^{1/2}\left(q(x)-\tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx=\tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda)-\varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda).$$

The assumptions of the theorem imply

$$g(\rho_n)=0,\quad n\in\Lambda.$$

By the Theorem 3.4, we have that

$$g(\rho)=0$$

on the whole ρ -plane. Thus, $\varphi(x,\lambda)$ and $\tilde{\varphi}(x,\lambda)$ satisfy

$$\tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda)-\varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda)=0$$

and hence

$$m\_(1/2,\lambda)=\widetilde{m\_}(1/2,\lambda).$$

By the Theorem 2.1, we prove $q(x)=\tilde{q}(x)$ almost everywhere on $[0,1/2]$. Repeating the supplementary problem in the last part of proof of Theorem 1.1, we can show that $g(\rho)=0$ on the whole ρ -plane, which implies that $q(x)=\tilde{q}(x)$ on $[1/2,1]$ and consequently, $q(x)=\tilde{q}(x)$ almost everywhere on $[0,1]$. This completes the proof. □

Let us consider the Sturm-Liouville problem L for $q(x)\in L_2(0,1)$. Horvath [15, 19, p.268] proved Hochstadt-Lieberman type an uniqueness theorem by using simple closedness properties of the exponential system corresponding to the known eigenvalues. We can give the following uniqueness theorem with same arguments in [15] for Mochizuki-Trooshin type theorem.

Theorem 3.6. *Let* $\Lambda \subset \mathbb{N} \cup \{0\}$ *be a subset of nonnegative integer numbers and let* $\Omega := \{\lambda_n\}_{n\in\Lambda}$ *be a part of the spectrum of* L *such that the system of functions* $\{\cos 2\rho_n x\}_{n\in\Lambda}$ *is complete in* $L_2(0,1/2)$. *If for* $n \in \Lambda$, *we have*

$$\lambda_n = \tilde{\lambda}_n, \quad \frac{\varphi'_n(1/2)}{\varphi_n(1/2)} = \frac{\tilde{\varphi}'_n(1/2)}{\tilde{\varphi}_n(1/2)}$$

then $q(x) = \tilde{q}(x)$ *almost everywhere on* $[0,1]$.

Proof. As in the proof of Theorem 1, we can show that

$$f(\lambda) = \int_0^{1/2} \left(q(x) - \tilde{q}(x)\right)\varphi(x,\lambda)\tilde{\varphi}(x,\lambda)dx = \tilde{\varphi}(1/2,\lambda)\varphi'(1/2,\lambda) - \varphi(1/2,\lambda)\tilde{\varphi}'(1/2,\lambda).$$

Hence, we have that

$$f(\lambda_n) = 0, \ \ n \in \Lambda. \tag{19}$$

The following representation holds [4,6,8]

$$\varphi(x,\lambda) = \cos\rho x + \int_0^x K(x,t)\cos\rho t dt$$

where $K(x,t)$ is a continuous function which does not depend on λ. Hence,

$$\varphi(x,\lambda)\tilde{\varphi}(x,\lambda) = \frac{1}{2}\left(1 + \cos 2\rho x + \int_0^x K_1(x,t)\cos\rho t dt\right) \tag{20}$$

where $K_1(x,t)$ is a continuous function which does not depend on λ. From (19) and (20), we have

$$\int_0^{1/2}\left[\phi(x) + \int_x^{1/2} K_1(x,t)\phi(t)dt\right]\cos 2\rho_n x dx + \int_0^{1/2}\phi(x)dx = 0, \ \ n \in \Lambda,$$

where $\phi(x) = q(x) - \tilde{q}(x)$. By the Riemann-Lebesgue lemma,

$$\int_0^{1/2}\phi(x)dx = 0.$$

By the completeness of the functions $\{\cos 2\rho_n x\}_{n\in\Lambda}$ we have

$$\phi(x) + \int_x^{1/2} K_1(x,t)\phi(t)dt = 0.$$

Since this homogeneous integral equation has only the trivial solution it follows that and $q(x)=\tilde{q}(x)$ almost everywhere on $[0,1/2]$. The supplementary problem $\overline{L}$ in proof of Theorem 1.1 completes the proof. □

REFERENCES

[1]. Ambarzumyan, V.A., Uber eine Frage der Eigenwerttheorie, Z. Phys. 53 (1929) 690-695.

[2]. Borg, G., Eine umkehrung der Sturm-Liouvillesehen eigenwertaufgabe, Acta Math. 78 (1946) 1-96.

[3]. Gelfand, L.M., Levitan, B.M., On the determination of a differential equation from its spectral function, Izv. Akad. Nauk SSR. Ser. Mat. 15 (1951) 309-360 (in Russian), English transl. in Amer. Math. Soc.. Transl. Ser. 2 (1) (1955) 253-304.

[4]. Levitan B.M., Sargsjan I.S., Sturm-Liouville and Dirac Operators. Dordrecht: Kluwer; 1991.

[5]. Mochizuki, K., Trooshin, I., Inverse problem for interior spectral data of Sturm-Liouville operator, J. Inverse Ill-posed Probl. 9 (2001) 425-433.

[6]. Marchenko V., Some questions in the theory of one-dimensional linear differential operators of the second order. I. Tr.Mosk. Mat. Obs. (1952) 1:327-420. (Russian). English transl. in Am.Math. Soc. Trans. (1973) 2:1-104.

[7]. Hochstadt, H., Lieberman, B., An inverse Sturm-Liouville problem with mixed given data, SIAM J. Appl. Math. 34 (1978) 676-680.

[8]. Freiling, G., Yurko, V.A., Inverse Sturm-Liouville Problems and Their Applications, NOVA Science Publishers, New York, 2001.

[9]. Levinson, N., The inverse Sturm-Liouville problem, Math. Tidsskr, 13 (1949), 25- 30.

[10].Rubel, L.A., Necessary and suffcient conditions for Carison's theorem on entire functions, Trans. Amer. Math. Soc. vol. 83 (1956) 417-429.

[11].Boas, R.P., Entire functions, New York, Academic Press, 1954.

[12].Mochizuki, K., Trooshin, I., Inverse problem for interior spectral data of the Dirac operator on a finite interval, Publ. RIMS, Kyoto Univ. 38 (2002) 387-395.

[13]. Sat, M., Panakhov, E., A uniqueness theorem for Bessel operator from interior spectral data. Abstr. Appl. Anal., Volume 2013, Article ID 713654, 6 pages.

[14].Ozkan, A.S., Amirov, R. Kh., An interior inverse problem for the impulsive Dirac operator, Tamkang Journal of Mathematics, 42 (2011) 259-263.

[15].Horvath, M., Inverse spectral problems and closed exponential systems, Ann. of Math. 162 (2005) 885-918.

[16].Panakhov, E., Sat, M., Inverse problem for the interior spectral data of the equation of hydrogen atom, Ukrainian Mathematical Journal, 64 (2013), no.11.

[17].Guo, Y., Wei, G., Inverse Sturm-Liouville problems with the potential known on an interior subinterval, Appl. Anal., 94 (5) (2015) 1025-1031.

[18].Gesztesy F., Simon B., Inverse spectral analysis with partial information on the potential. II. The case of discrete spectrum. Trans. Am. Math. Soc. 352 (6) (2000) 2765-2787.

[19].Shieh, C.T., Yurko, V.A., Inverse nodal and inverse spectral problems for discontinuous boundary value problems. J. Math. Anal. Appl. 347 (2008) 266-272.